Scalable Analysis and Control of Positive Systems

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Wind Farms Need Control

Picture from http://www.hochtief.com/hochtief_en/9164.jhtml

Most wind farms today are paid to maximize power production.
Future farms will have to curtial power at contracted levels.

New control objective:
Minimize fatigue loads subject to fixed total production.

Minimizing Fatigue Loads

Single turbine control:
Minimize tower pressure variance subject to linearized dynamics with measurements of pitch angle and rotor speed.

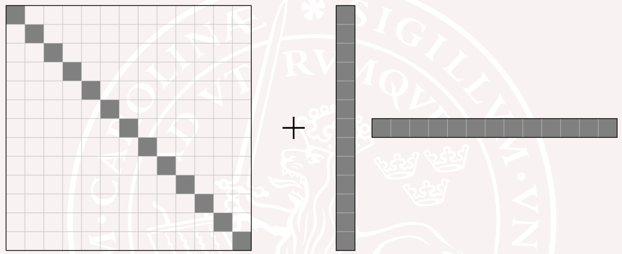
Optimal controller: $u_i^{\text{loc}}(t)$

Wind farm control:
Minimize sum of all tower pressure variances subject to *fixed total production of the farm:* $\sum_{i=1}^m u_i = 0$

Optimal controller: $u_i(t) = u_i^{\text{loc}}(t) - \frac{1}{m}\sum_{j=1}^m u_j^{\text{loc}}(t)$.

[PhD thesis by Daria Madjidian, Lund University, June 2014]

Controller Structure



Linear quadratic control of m identical systems and a constraint $\sum_{i=1}^m u_i = 0$ gives an optimal feedback matrix with two parts:

- One is localized (diagonal).
- The other has rank one (control of the average state).

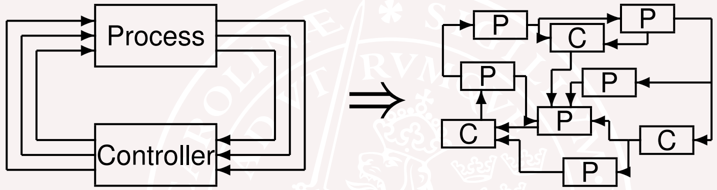
Server Farms Need Control

Picture from http://www.dawn.com/news/1017980

Single server control:
Assign resources (processor speed, memory, etc.) to minimize variance in completion time.

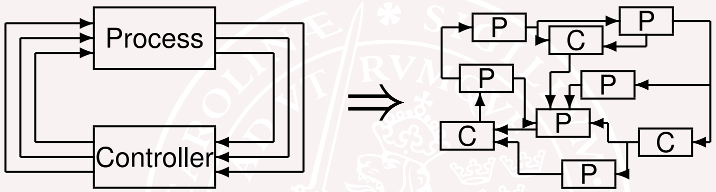
Server farm control:
Minimize sum of all time variances with *fixed total resources*.

Towards a Scalable Control Theory



- Linear quadratic control uses $O(n^3)$ flops, $O(n^2)$ memory
- Model Predictive Control requires even more
- **Today:** Exploiting monotone/positive systems

Towards a Scalable Control Theory



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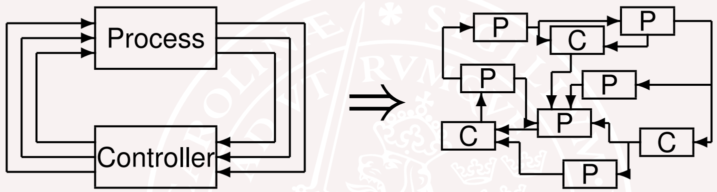

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- **Today:** Exploiting monotone/positive systems

Outline

- **Positive and Monotone Systems**
- Scalable Stability Analysis
- Input-Output Performance
- Trajectory Optimization
- Combination Therapy for HIV and Cancer

Positive systems

A linear system is called *positive* if the state and output remain nonnegative as long as the initial state and the inputs are nonnegative:

$$\frac{dx}{dt} = Ax + Bu \qquad y = Cx$$

Equivalently, A, B and C have nonnegative coefficients except for the diagonal of A.

Examples:

- Probabilistic models.
- Economic systems.
- Chemical reactions.
- Ecological systems.

Positive Systems and Nonnegative Matrices

Classics:

Mathematics: Perron (1907) and Frobenius (1912)
Economics: Leontief (1936)

Books:

Nonnegative matrices: Berman and Plemmons (1979)
Dynamical Systems: Luenberger (1979)

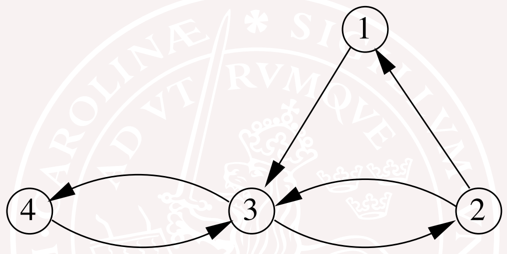
Recent control related work:

Biology inspired theory: Angeli and Sontag (2003)
Synthesis by linear programming: Rami and Tadeo (2007)
Switched systems: Liu (2009), Fornasini and Valcher (2010)
Distributed control: Tanaka and Langbort (2010)
Robust control: Briat (2013)

Example 1: Transportation Networks

- Cloud computing / server farms
- Heating and ventilation in buildings
- Traffic flow dynamics
- Production planning and logistics

A Transportation Network is a Positive System

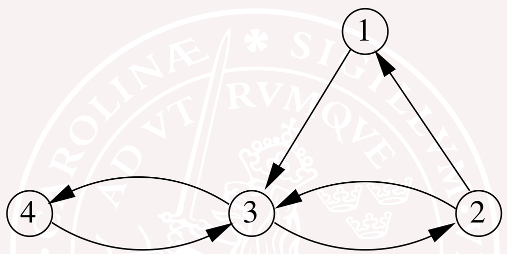


$$\begin{bmatrix}\dot x_1\\ \dot x_2\\ \dot x_3\\ \dot x_4\end{bmatrix} = \begin{bmatrix} -1-\ell_{31} & \ell_{12} & 0 & 0\\ 0 & -\ell_{12}-\ell_{32} & \ell_{23} & 0\\ \ell_{31} & \ell_{32} & -\ell_{23}-\ell_{43} & \ell_{34}\\ 0 & 0 & \ell_{43} & -4-\ell_{34}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} + \begin{bmatrix}w_1\\ w_2\\ w_3\\ w_4\end{bmatrix}$$

How do we select ℓ_{ij} to minimize the gain from w to x?

Example 2: A vehicle formation

Example 2: Vehicle Formations

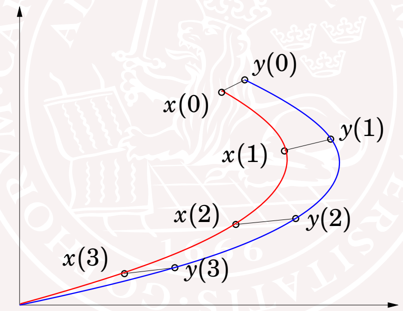


$$\begin{cases} \dot x_1 = -x_1 + \ell_{13}(x_3 - x_1) + w_1\\ \dot x_2 = \ell_{21}(x_1 - x_2) + \ell_{23}(x_3 - x_2) + w_2\\ \dot x_3 = \ell_{32}(x_2 - x_3) + \ell_{34}(x_4 - x_3) + w_3\\ \dot x_4 = -4x_4 + \ell_{43}(x_3 - x_4) + w_4 \end{cases}$$

How do we select ℓ_{ij} to minimize the gain from w to x?

Nonlinear Monotone Systems

For the nonlinear system $\dot x = f(x)$, let $x(t) = \phi(x_0, t)$ be the solution starting from x_0. The system is called *monotone* if $x_0 \le y_0$ implies $\phi(x_0, t) \le \phi(y_0, t)$ for all $t \ge 0$.



Macroscopic Models of Traffic Flow

1. Partial differential equation by Lighthill/Whitham (1955), Richards (1956) based on mass-conservation:

$$0 = \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x} f(\rho)$$

where $\rho(x,t)$ is traffic density in position x at time t and $f(\rho)$ expresses flow as function of density.

2. Spatial discretization by Daganzo (1994).

Both models are monotone systems!

Exploited for lines: [Gomes/Horowitz/Kurzhanskiy/Varaiya/Kwon, 2008].
Exploited for networks: [Lovisari/Como/Rantzer/Savla, MTNS-14].

Outline

- Positive and Monotone Systems
- **Scalable Stability Analysis**
- Input-Output Performance
- Trajectory Optimization
- Combination Therapy for HIV and Cancer

Stability of Positive systems

Suppose the matrix A has nonnegative off-diagonal elements. Then the following conditions are equivalent:

(i) The system $\frac{dx}{dt} = Ax$ is exponentially stable.

(ii) There is a *diagonal* matrix $P \succ 0$ such that $A^T P + PA \prec 0$

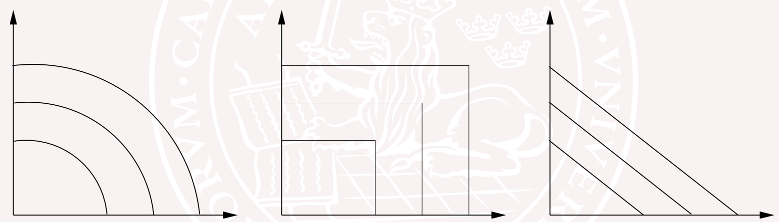
(iii) There exists a vector $\xi > 0$ such that $A\xi < 0$. (The vector inequalities are elementwise.)

(iv) There exits a vector $z > 0$ such that $A^T z < 0$.

Lyapunov Functions of Positive systems

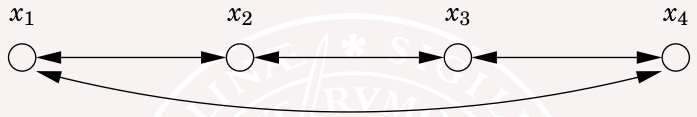
Solving the three alternative inequalities gives three different Lyapunov functions:

$A^T P + PA \prec 0$ | $A\xi < 0$ | $A^T z < 0$



$V(x) = x^T P x$ | $V(x) = \max_k (x_k/\xi_k)$ | $V(x) = z^T x$

A Scalable Stability Test



Stability of $\dot{x} = Ax$ follows from existence of $\xi_k > 0$ such that

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & 0 & a_{14} \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{32} \\ a_{41} & 0 & a_{43} & a_{44} \end{bmatrix}}_{A} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

The first node verifies the inequality of the first row.

The second node verifies the inequality of the second row.

...

Verification is scalable!

A Distributed Search for Stabilizing Gains

Suppose $\begin{bmatrix} a_{11} - \ell_1 & a_{12} & 0 & a_{14} \\ a_{21} + \ell_1 & a_{22} - \ell_2 & a_{23} & 0 \\ 0 & a_{32} + \ell_2 & a_{33} & a_{32} \\ a_{41} & 0 & a_{43} & a_{44} \end{bmatrix} \geq 0$ for $\ell_1, \ell_2 \in [0, 1]$.

For stabilizing gains ℓ_1, ℓ_2, find $0 < \mu_k < \xi_k$ such that

$$\begin{bmatrix} a_{11} & a_{12} & 0 & a_{14} \\ a_{21} & a_{22} & a_{23} & 0 \\ 0 & a_{32} & a_{33} & a_{32} \\ a_{41} & 0 & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and set $\ell_1 = \mu_1/\xi_1$ and $\ell_2 = \mu_2/\xi_2$. Every row gives a local test.

Distributed synthesis by linear programming (gradient search).

Max-separable Lyapunov Functions

Max-separable: $V(x) = \max\{V_1(x_1), \dots, V_n(x_n)\}$

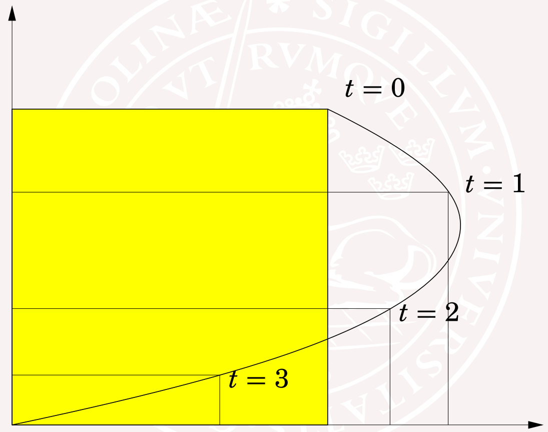
Theorem. Let $\dot{x} = f(x)$ be a monotone system such that the origin globally asymptotically stable and the compact set $X \subset \mathbb{R}^n_+$ is invariant. Then there exist strictly increasing functions $V_k : \mathbb{R}_+ \to \mathbb{R}_+$ for $k = 1, \dots, n$, such that $V(x) = \max\{V_1(x_1), \dots, V_n(x_n)\}$ satisfies

$$\frac{d}{dt} V(x(t)) = -V(x(t))$$

along all trajectories in X.

[Rantzer, Rüffer, Dirr, CDC-13]

Proof idea



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Performance of Positive systems

Suppose that $\mathbf{G}(s) = C(sI - A)^{-1}B + D$ where $A \in \mathbb{R}^{n\times n}$ is Metzler, while $B \in \mathbb{R}_+^{n\times 1}$, $C \in \mathbb{R}_+^{1\times n}$ and $D \in \mathbb{R}_+$. Define $\|\mathbf{G}\|_\infty = \sup_\omega |G(i\omega)|$. Then the following are equivalent:

(i) The matrix A is Hurwitz and $\|\mathbf{G}\|_\infty < \gamma$.

(ii) The matrix $\begin{bmatrix} A & B \\ C & D-\gamma \end{bmatrix}$ is Hurwitz.

Optimizing H_∞/L_1 Performance

Let $\mathcal{D}$ be the set of diagonal matrices with entries in $[0,1]$. Suppose $B, C, D \geq 0$ and $A + ELF$ is Metzler for all $L \in \mathcal{D}$.

If $F \geq 0$, then the following are equivalent:

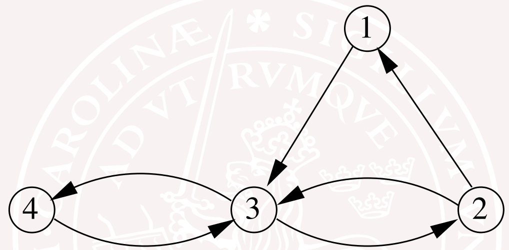
(i) There exists $L \in \mathcal{D}$ such that $A + ELF$ is Hurwitz and $\|C[sI - (A + ELF)]^{-1}B + D\|_\infty < \gamma$.

(ii) There exist $\xi \in \mathbb{R}_+^n$, $\mu \in \mathbb{R}_+^m$ with

$$A\xi + E\mu + B < 0 \qquad C\xi + D < \gamma \qquad \mu \leq F\xi$$

If ξ, μ satisfy (ii) , then (i) holds for every L such that $\mu = LF\xi$.

Example 1: Transportation Networks



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -1-\ell_{31} & \ell_{12} & 0 & 0 \\ 0 & -\ell_{12}-\ell_{32} & \ell_{23} & 0 \\ \ell_{31} & \ell_{32} & -\ell_{23}-\ell_{43} & \ell_{34} \\ 0 & 0 & \ell_{43} & -4-\ell_{34} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} w \\ w \\ w \\ w \end{bmatrix}$$

How do we select $\ell_{ij} \in [0,1]$ to minimize the gain from w to $\sum_i x_i$?

Example 1: Transportation Networks

$A = \text{diag}\{-1, 0, 0, -4\}$ $\quad B = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}^T$

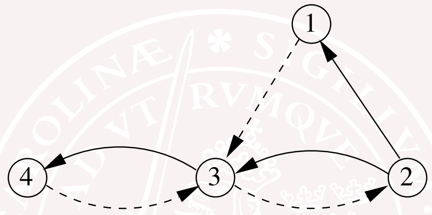
$C = \begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix}$ $\quad K = 0$

$L = \text{diag}\{\ell_{31}, \ell_{12}, \ell_{32}, \ell_{23}, \ell_{43}, \ell_{34}\}$

$$E = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \quad F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The closed loop matrix is $A + ELF$.

Example 1: Transportation Networks



Minimize $\sum_i \xi_i$ subject to

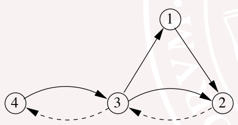
$$\begin{aligned} 0 &\geq -\xi_1 - \mu_{31} + \mu_{12} + 1 \\ 0 &\geq -\mu_{12} - \mu_{32} + \mu_{23} + 1 \\ 0 &\geq \mu_{31} + \mu_{32} - \mu_{23} - \mu_{43} + \mu_{34} + 1 \\ 0 &\geq -4\xi_4 + \mu_{43} - \mu_{34} + 1 \end{aligned}$$

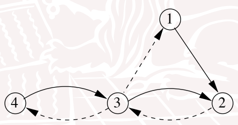
and $0 \leq \mu_{ij} \leq \xi_j$. Then define $\ell_{ij} = \mu_{ij}/\xi_j$.

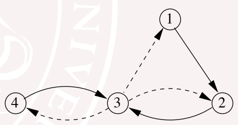
Optimal solution $\ell_{12} = \ell_{32} = \ell_{43} = 1$ and $\ell_{31} = \ell_{23} = \ell_{34} = 0$.

Example 2: Vehicle Formations

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -1-\ell_{13} & 0 & \ell_{13} & 0 \\ \ell_{21} & -\ell_{21}-\ell_{23} & \ell_{23} & 0 \\ 0 & \ell_{32} & -\ell_{32}-\ell_{34} & \ell_{34} \\ 0 & 0 & \ell_{43} & -4-\ell_{43} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + Bw$$







$B = \begin{bmatrix} 10 \\ 10 \\ 1 \\ 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 1 \\ 1 \\ 10 \\ 10 \end{bmatrix}$

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Monotone Input-output Systems

The system

$$\dot{x}(t) = f(x(t), u(t)), \qquad x(0) = a$$

is called *monotone* if

$$\begin{cases} a_0 \le a_1 \\ u_0(\tau) \le u_1(\tau), \quad \tau \in [0,t] \end{cases} \implies \phi_t(a_0, u_0) \le \phi_t(a_1, u_1)$$

[Angeli and Sontag, 2003]

Trajectory Optimization

The monotone system $\dot{x} = f(x,u)$ is a *convex monotone system* if every row of f is also convex.

Theorem:

For a convex monotone system $\dot{x} = f(x,u)$, each component of the trajectory $\phi_t(a,u)$ is a convex function of (a,u).

[Rantzer and Bernhardsson, 2014]

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Evolution to resistance in mice with chronic HIV infection

Antibody trimix

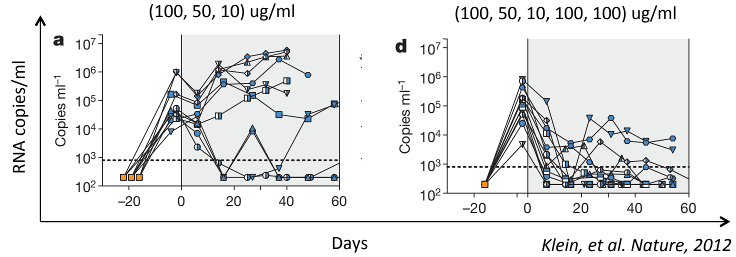
(3BC176, PG16, 45-46G54W)

(100, 50, 10) ug/ml

Antibody pentamix

(3BC176, PG16, 45-46G54W, PGT128, 10-1074)

(100, 50, 10, 100, 100) ug/ml



Klein, et al. Nature, 2012

Challenge: Design combination of drugs that gives fastest possible decay rate of virus population in spite of mutations.

Combination Therapy is a Control Problem

Evolutionary dynamics:

$$\dot{x} = \left(A - \sum_i u_i D^i\right) x$$

Each state x_k is the concentration of a mutant. (There can be hundreds!) Each input u_i is a drug dosage.

A describes the mutation dynamics without drugs, while $D^1, \dots, D^m$ are diagonal matrices modeling drug effects.

Determine $u_1, \dots, u_m \geq 0$ with $u_1 + \cdots + u_m \leq 1$ such that x decays as fast as possible!

[Hernandez-Vargas, Colaneri and Blanchini, JRNC 2011]
[Jonsson, Rantzer,Murray, ACC 2014]

Optimizing Decay Rate

Stability of the matrix $A - \sum_i u_i D^i + \gamma I$ is equivalent to existence of $\xi > 0$ with

$$(A - \sum_i u_i D^i + \gamma I)\xi < 0$$

For row k, this means

$$A_k \xi - \sum_i u_i D^i_k \xi_k + \gamma \xi_k < 0$$

or equivalently

$$\frac{A_k \xi}{\xi_k} - \sum_i u_i D^i_k + \gamma < 0$$

Maximizing γ is convex optimization in $(\log \xi_i, u_i, \gamma)$!

Using Measurements of Virus Concentrations

Evolutionary dynamics:

$$\dot{x}(t) = \left(A - \sum_i u_i(t) D^i\right) x(t)$$

Can we get faster decay using time-varying $u(t)$ based on measurements of $x(t)$?

Using Measurements of Virus Concentrations

The evolutionary dynamics can be written as a convex monotone system:

$$\frac{d}{dt} \log x_k(t) = \frac{A_k x(t)}{x_k(t)} - \sum_i u_i(t) D^i_k$$

Hence the decay of $\log x_k$ is a convex function of the input and optimal trajectories can be found even for large systems.

Example

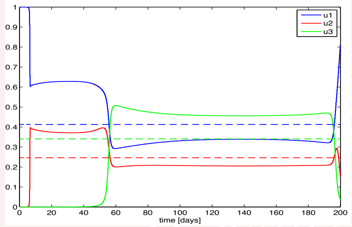
$$A = \begin{bmatrix} -\delta & \mu & \mu & 0 \\ \mu & -\delta & 0 & \mu \\ \mu & 0 & -\delta & \mu \\ 0 & \mu & \mu & -\delta \end{bmatrix}$$

clearance rate $\delta = 0.24\,\text{day}^{-1}$, mutation rate $\mu = 10^{-4}\,\text{day}^{-1}$ and replication rates for viral variants and therapies as follows

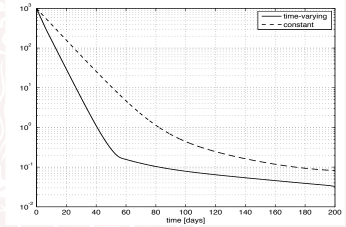
Variant	Therapy 1	Therapy 2	Therapy 3
Wild type (x_1)	$D_1^1 = 0.05$	$D_1^2 = 0.10$	$D_1^3 = 0.30$
Genotype 1 (x_2)	$D_2^1 = 0.25$	$D_2^2 = 0.05$	$D_2^3 = 0.30$
Genotype 2 (x_3)	$D_3^1 = 0.10$	$D_3^2 = 0.30$	$D_3^3 = 0.30$
HR type (x_4)	$D_4^1 = 0.30$	$D_4^2 = 0.30$	$D_4^3 = 0.15$

Example

Optimized drug doses:



Total virus population:



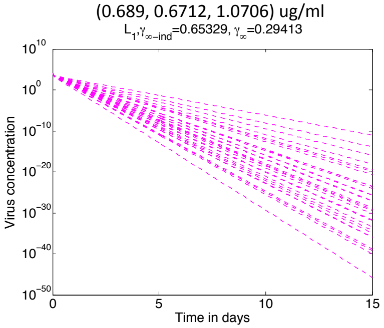
Comparison with the experimental trimix

SIMULATION

Robustly stabilizing controller, TRIMIX

(3BC176, 45-46G54W, PGT128)

(0.689, 0.6712, 1.0706) ug/ml

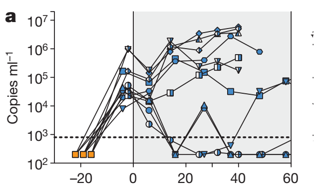


EXPERIMENT

TRIMIX

(3BC176, PG16, 45-46G54W)

0.5 mg weekly/ 2 per week injection



Klein, et al, Nature, 2012

(3BC176, PG16, 45-46G54W) at (1, 1, 1) ug/ml: Input-output gain 0.75

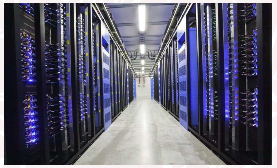
(3BC176, 45-46G54W, PGT128) at (0.689, 0.6712, 1.07) ug/ml: Input-output gain 0.65

For Scalable Control — Use Positive Systems!

- Verification and synthesis scale linearly
- Distributed controllers by linear programming
- No need for global information
- Optimal trajectories by convex optimization

Many Research Challenges Remain

- Optimal Dynamic Controllers in Positive Systems
- Analyze Trade-off Between Performance and Scalability
- Distributed Controllers for Nonlinear Monotone Systems


Thanks!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Martina Maggio


Alessandro Papadopoulos


Bo Bernhardsson


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