Distributed Control of Positive Systems

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Abstract

For positive systems, and more generally positively dominated systems, it is shown that distributed H_∞-optimal controllers can be computed using linear programming, with a complexity that scales linearly with the number of states and interconnections. Hence two fundamental advantages are achieved compared to classical methods for multivariable control: Distributed implementations and scalable computations. The results are illustrated by examples from control of mechanical structures, transportation networks and electrical power transmission.

I. Introduction

Classical methods for multi-variable control, such as LQG and H_∞, suffer from a lack of scalability that make them hard to use for large-scale systems. The difficulties come from both computational complexity and from the absence of distributed structure in the resulting controllers. The complexity can be traced back to the fact that even stability verification of a linear system with n states generally requires a Lyapunov function involving n^2 quadratic terms. This is true even if the system matrices are sparse. However, the situation improves drastically if we restrict our attention to system matrices with nonnegative off-diagonal entries. Then stability and performance can be verified using a Lyapunov function with only n linear terms. Sparsity can be exploited in performance verification and even synthesis of distributed controllers can be done with a complexity that grows linearly with the number of nonzero entries in the system matrices.

Given the striking difference between the two types of stability criteria, it is natural to ask how restrictive the second category is when it comes to applications. After all, linear state space models with negative off-diagonal elements are very common. However, there are strong indications that the basic ideas have far-reaching implications:

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1) The essential monotonicity property extends beyond system matrices with nonnegative off-diagonal entries. A sufficient assumption is that the transfer functions involved are "positively dominated".
2) The desired structure appears naturally in many important application areas, such as mechanical systems, economics, transportation networks, power systems and biology.
3) In control applications, the condition on positive dominance need not apply to the open loop process. Instead, a large-scale control system can often be structured into local control loops that give positive dominance, thereby enabling scalable methods for optimization of the global performance.

The paper is structured as follows: Sections II-III introduce background literature and notation. Stability criteria for positive systems are cited in section IV. These results are not new, but stated on a form convenient for later use and explained with emphasis on scalability. Section V shows how the stability criteria can be exploited in synthesis of stabilizing controllers using distributed linear programming. The techniques are then refined in section VI to optimize H_∞ and L_1 performance. Section VII extends the techniques to positively dominated transfer functions. An alternative approach to performance evaluation is given in section VIII. This approach relies on semi-definite programming, but enables optimization of multiple objectives simultaneously. An application to electrical power transmission is given.

II. Background

The study of matrices with nonnegative coefficients has a long history dating back to the Perron-Frobenius Theorem in 1912. A classic book on the topic is [2]. The theory is used in Leontief economics [14], where the states denote nonnegative quantities of commodities. Systems defined by nonnegative matrices (so called positive systems) appear in the study of Markov chains [21], where the states denote nonnegative probabilities and in compartment models [9], where the states could denote quantities of chemical species in an organism. A nice introduction to the subject is given in [15].

A fundamental property of linear maps described by a positive matrix is that they are contractive in Hilbert's projective metric [3], [12]. This metric is closely related to the Lyapunov function $\max\{x_1, \dots, x_n\} - \min\{x_1, \dots, x_n\}$ that is used for analysis of consensus algorithms [21], [24]. For more recent contributions, see [18], [22].

A nonlinear counterpart to positive systems is monotone systems, characterized by the property that a partial ordering of initial states is preserved. Such dynamical systems were studied in a series of papers by Hirsch [6], [7], showing that monotonicity under some additional assumptions implies convergence almost everywhere. Positive systems have also gained attention in the control literature and increasingly so during the last decade. See for example [25], [5], [10]. Feedback stabilization of positive linear systems was studied in [13], [19] and basic control theory for nonlinear monotone systems was developed in [1]. A recent result by Tanaka and Langbort [23] shows that decentralized controllers can be optimized for positive systems using semi-definite programming. The criterion is the closed loop H_∞ norm and the authors show that diagonal quadratic storage functions can be used without conservatism. Several of the main results in this paper can be viewed as extensions of this work.

III. Notation

Let $\mathbb{R}_+$ denote the set of nonnegative real numbers. The inequality $X > 0$ ($X \geq 0$) means that all elements of the matrix (or vector) X are positive (nonnegative). For a symmetric matrix X, the inequality $X \succ 0$ means that the matrix is positive definite. The matrix $A \in \mathbb{R}^{n\times n}$ is said to be *Hurwitz* if all eigenvalues have positive real part. It is *Schur* if all eigenvalues are strictly inside the unit circle. Finally, the matrix is said to be *Metzler* if all off-diagonal elements are nonnegative. The notation $\mathbb{CH}_\infty^{n\times m}$ represents the set of $n \times m$ matrices whose entries are analytic in the right half plane and continuous on the imaginary axis (including infinity) .

IV. Distributed Stability Verification

Proposition 1: Let $A \in \mathbb{R}^{n\times n}$ be Metzler. Then the following are equivalent:

(1.1) The matrix A is Hurwitz.

(1.2) There exists a $\xi \in \mathbb{R}^n$ such that $\xi > 0$ and $A\xi < 0$.

(1.3) There exists a $z \in \mathbb{R}^n$ such that $z > 0$ and $z^T A < 0$.

(1.4) There exists a *diagonal* matrix $P \succ 0$ such that $A^T P + PA \prec 0$.

(1.5) The matrix $-A^{-1}$ exists and has nonnegative entries.

Moreover, if $\xi = (\xi_1, \ldots, \xi_n)$ and $z = (z_1, \ldots, z_n)$ satisfy the conditions of (1.2) and (1.3) respectively, then $P = \operatorname{diag}(z_1/\xi_1, \ldots, z_n/\xi_n)$ satisfies the conditions of (1.4).

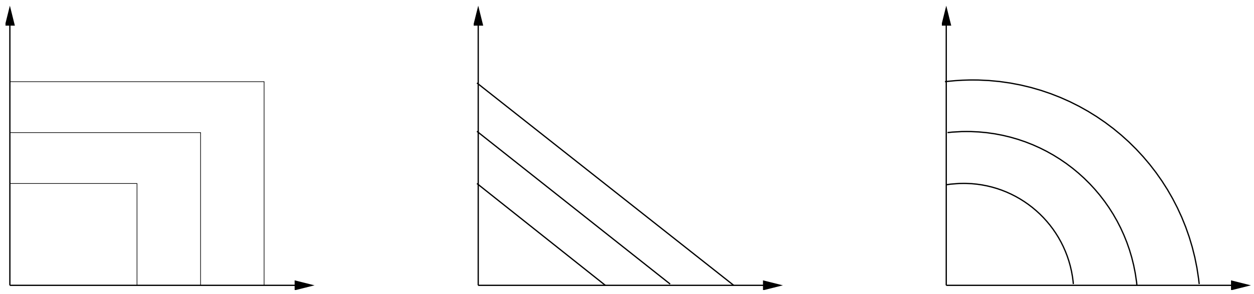

Fig. 1. Level curves of Lyapunov functions corresponding to the conditions (1.2), (1.3) and (1.4) in Proposition 1: If $A\xi < 0$, then $V(x) = \max_i(x_i/\xi_i)$ is a Lyapunov function with rectangular level curves. If $z^T A < 0$, then $V(x) = z^T x$ is a linear Lyapunov function. Finally if $A^T P + PA \prec 0$ and $P \succ 0$, then $V(x) = x^T P x$ is a quadratic Lyapunov function for the system $\dot{x} = Ax$.

Remark 1. Each of the conditions (1.2), (1.3) and (1.4) corresponds to a Lyapunov function of a specific form. See Figure 1.

Remark 2. One of the main observations of this paper is that verification and synthesis of positive control systems can be done with methods that scale linearly with the number of interconnections. For stability, this claim follows directly from Proposition 1: Given ξ, verification of the inequality $A\xi < 0$ requires a number of scalar additions and multiplications that is directly proportional to the number of nonzero elements in the matrix A. In fact, the search for a feasible ξ also scales linearly, since integration of the differential equation $\dot{\xi} = A\xi$ with $\xi(0) = \xi_0$ for an arbitrary $\xi_0 > 0$ generates a feasible $\xi(t)$ in finite time provided that A is Metzler and Hurwitz.

Proof of Proposition 1. The equivalence between (1.1), (1.2), (1.4) and (1.5) is the equivalence between the statements G_{20}, I_{27}, H_{24} and N_{38} in [2, Theorem 6.2.3]. The equivalence between (1.1) and (1.3) is obtained by applying the equivalence between (1.1) and (1.2) to the transpose of A. Moreover, if $\xi = (\xi_1, \dots, \xi_n)$ and $z = (z_1, \dots, z_n)$ satisfy the conditions of (1.2) and (1.3) respectively, then $P = \mathrm{diag}(z_1/\xi_1, \dots, z_n/\xi_n)$ gives $(A^T P + PA)\xi = A^T z + PA\xi < 0$ so the symmetric matrix $A^T P + PA$ is Hurwitz and (1.4) follows. □

Example 1. Transportation network. Consider a dynamical system interconnected according

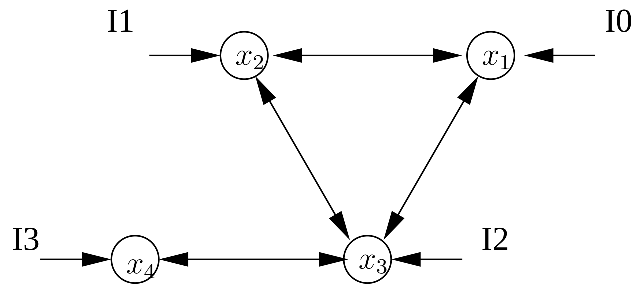

Fig. 2. A graph of interconnected systems. In Example 1 the interpretation is a transportation network and each arrow indicates a transportation link. In Example 2 the interpretation is instead a vehicle formation and each arrow indicates the use of a distance measurement.

to the graph illustrated in Figure 2:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -1-\ell_{31} & \ell_{12} & 0 & 0 \\ 0 & 2-\ell_{12}-\ell_{32} & \ell_{23} & 0 \\ \ell_{31} & \ell_{32} & 3-\ell_{23}-\ell_{43} & \ell_{34} \\ 0 & 0 & \ell_{43} & -4-\ell_{34} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \tag{1}$$

The model could for example be used to describe an transportation network connecting four buffers. The states x_1, x_2, x_3, x_4 represent the contents of the buffers and the parameter ℓ_{ij} determines the rate of transfer from buffer j to buffer i. Without such transfer the content of the second and third buffer would grow exponentially due to the unstable internal dynamics of those buffers.

Notice that the dynamics can be written as $\dot{x} = Ax$ where A is a Metzler matrix provided that every ℓ_{ij} is nonnegative. Hence, by Proposition 1, stability is equivalent to existence of numbers $\xi_1, \dots, \xi_4 > 0$ such that

$$\begin{bmatrix} -1-\ell_{31} & \ell_{12} & 0 & 0 \\ 0 & 2-\ell_{12}-\ell_{32} & \ell_{23} & 0 \\ \ell_{31} & \ell_{32} & 3-\ell_{23}-\ell_{43} & \ell_{34} \\ 0 & 0 & \ell_{43} & -4-\ell_{34} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Given these numbers, stability can be verified by a distributed test where the first buffer verifies the first inequality, the second buffer verifies the second and so on. $\square$

Example 2. Vehicle formation or Distributed Kalman Filter. Another dynamical system, which can be viewed as a dual of the previous one, is the following:

$$\begin{cases} \dot{x}_1 = -x_1 + \ell_{13}(x_3 - x_1) \\ \dot{x}_2 = 2x_2 + \ell_{21}(x_1 - x_2) + \ell_{23}(x_3 - x_2) \\ \dot{x}_3 = 3x_3 + \ell_{32}(x_2 - x_3) + \ell_{34}(x_4 - x_3) \\ \dot{x}_4 = -4x_4 + \ell_{43}(x_3 - x_4) \end{cases} \tag{2}$$

The model could for example be used to describe a platoon of four vehicles. The parameters ℓ_{ij} represent position adjustments based on distance measurements between the vehicles. Without these adjustments only the first and last vehicle maintain a stable position, while the position error in the second and third vehicle grow exponentially. Again, stability can be verified by a distributed test where the first vehicle verifies the first inequality, the second vehicle verifies the second inequality and so on. □

A discrete time counterpart to Proposition 1 can be stated as follows:

Proposition 2: Let $B \in \mathbb{R}_+^{n\times n}$. Then the following statements are equivalent:

(2.1) The matrix B is Schur stable.

(2.2) There is a $\xi \in \mathbb{R}^n$ such that $\xi > 0$ and $B\xi < \xi$.

(2.3) There exists a $z \in \mathbb{R}^n$ such that $z > 0$ and $B^T z < z$.

(2.4) There is a *diagonal* matrix $P \succ 0$ such that $B^T PB \prec P$.

(2.5) The matrix $(I - B)^{-1}$ exists and has nonnegative entries.

Moreover, if $\xi = (\xi_1, \dots, \xi_n)$ and $z = (z_1, \dots, z_n)$ satisfy the conditions of (2.2) and (2.3) respectively, then $P = \text{diag}(z_1/\xi_1, \dots, z_n/\xi_n)$ satisfies the conditions of (2.4).

Proof. The equivalence between (2.1) and (2.5) is proved by [2, Lemma 6.2.1]. Setting $A = B - I$ gives the equivalence between (2.2), (2.3) and (2.5) from the equivalence between (1.2), (1.3) and (1.5).

Suppose $\xi = (\xi_1, \dots, \xi_n)$ and $z = (z_1, \dots, z_n)$ satisfy the conditions of (2.2) and (2.3) respectively. Set $P = \text{diag}(z_1/\xi_1, \dots, z_n/\xi_n)$ and $y_k = \sqrt{\xi_k z_k}$ for $k = 1, \dots, n$. Then

$$P^{-1/2}B^T PBP^{-1/2}y = P^{-1/2}B^T PB\xi < P^{-1/2}B^T P\xi = P^{-1/2}B^T z < P^{-1/2}z = y$$

so $P^{-1/2}B^T PBP^{-1/2}$ is Schur. Hence $P^{-1/2}B^T PBP^{-1/2} \prec I$ and (2.4) follows. Finally, the implication from (2.4) to (2.1) is standard. □

V. Distributed Stabilization by Linear Programming

The next step is to search for stabilizing feedback laws by distributed optimization. This can be done using the following theorem:

Theorem 3: Let the matrices $A \in \mathbb{R}^{n\times n}, E \in \mathbb{R}^{n\times m}, F \in \mathbb{R}^{m\times n}, K \in \mathbb{R}^{m\times m}$ be given and let $\mathcal{D}$ be the set of $m \times m$ diagonal matrices with entries in $[0, 1]$. Suppose that $(I - LK)^{-1}$ exists and $A + E(I - LK)^{-1}LF$ is Metzler for all $L \in \mathcal{D}$. If F and K have nonnegative coefficients, then the following two conditions are equivalent:

(3.1) There exists $L \in \mathcal{D}$ such that $A + E(I - LK)^{-1}LF$ is Hurwitz.

(3.2) There exist $\xi \in \mathbb{R}^n_+$, $\mu \in \mathbb{R}^m_+$ with $\mu \leq F\xi + K\mu$ and $A\xi + E\mu < 0$.

Alternatively, if E and K have nonnegative coefficients, then (3.1) is equivalent to

(3.3) There exist $p \in \mathbb{R}^n_+$, $q \in \mathbb{R}^m_+$ with $q \leq E^T p + K^T q$ and $A^T p + F^T q < 0$.

Remark 3. It is natural to compare the expression $A+E(I-LK)^{-1}LF$ with the "state feedback" expression $A+BL$ of standard linear quadratic optimal control. A major difference is the presence of F and K which make the optimization into a problem of "static output feedback" rather than state feedback. Another difference is the diagonally structured L instead of a full matrix. The diagonal structure gives a much higher degree of flexibility, particularly in the specification of distributed controllers.

Remark 4. If the diagonal elements of $\mathcal{D}$ are restricted to $\mathbb{R}_+$ instead of $[0, 1]$, then the condition $\mu \leq F\xi + K\mu$ is replaced by $0 < F\xi + K\mu$.

Remark 5. Each row of the vector inequalities can be verified separately to get a distributed test.

Proof of Theorem 3. Suppose (3.1) holds. Let $A + E(I - LK)^{-1}LF$ be Hurwitz and define $\xi \in \mathbb{R}^n_+$ with $[A + E(I - LK)^{-1}LF]\xi < 0$. Let $\mu = (I - LK)^{-1}LF\xi$. Then $\mu = L(F\xi + K\mu)$ and $A\xi + E\mu = (A + E(I - LK)^{-1}LF)\xi < 0$.

Conversely, suppose that (3.2) holds. Choose $L \in \mathcal{D}$ to get $\mu = (I - LK)^{-1}LF\xi$. Then

$$[A + E(I - LK)^{-1}LF]\xi = A\xi + E\mu < 0$$

so $A+E(I-LK)^{-1}LF$ is Hurwitz. The equivalence between (3.1) and (3.3) follows immediately by replacing $A + E(I - LK)^{-1}LF$ with its transpose. □

Example 3 Consider the system (1) and the problem to find feedback gains $\ell_{ij} \in [0, \overline{\ell}\,]$ that stabilize the transportation network. The problem can be solved by applying linear programming to condition (3.2) with

$$A = \text{diag}\{-1, 2, 3, -4\} \qquad K = 0 \qquad L = \text{diag}\{\ell_{31}, \ell_{12}, \ell_{32}, \ell_{23}, \ell_{43}, \ell_{34}\}/\overline{\ell}$$

$$E = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 \\ 1 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \qquad F = \begin{pmatrix} \overline{\ell} & 0 & 0 & 0 \\ 0 & \overline{\ell} & 0 & 0 \\ 0 & \overline{\ell} & 0 & 0 \\ 0 & 0 & \overline{\ell} & 0 \\ 0 & 0 & \overline{\ell} & 0 \\ 0 & 0 & 0 & \overline{\ell} \end{pmatrix}$$

It turns out that the linear program is feasible if and only if $\overline{\ell} > 2$, in which case $L = \text{diag}\{0, 1, 0, 1, 1, 0\}$ is stabilizing.

If we instead consider the vehicle formation, E and F are replaced by F^T and E^T respectively, so we need to use condition (3.3) instead. □

A discrete time counterpart to Theorem 3 is given without proof:

Theorem 4: Let the matrices $A \in \mathbb{R}^{n\times n}, E \in \mathbb{R}^{n\times m}, F \in \mathbb{R}^{m\times n}, K \in \mathbb{R}^{m\times m}$ be given and let $\mathcal{D}$ be the set of $m \times m$ diagonal matrices with entries in $[0, 1]$. Suppose that $(I - LK)^{-1}$ exists and $A + E(I - LK)^{-1}LF$ is non-negative for all $L \in \mathcal{D}$. If F and K have nonnegative coefficients, then the following are equivalent:

(4.1) There is $L \in \mathcal{D}$ such that $A + E(I - LK)^{-1}LF$ is Schur.

(4.2) There exist $\xi \in \mathbb{R}^n_+$, $\mu \in \mathbb{R}^m_+$ with $\mu \leq F\xi + K\mu$ and $A\xi + E\mu < \xi$.

Alternatively, if E and K have nonnegative coefficients, then (4.1) is equivalent to

(4.3) There exist $p \in \mathbb{R}^n_+$, $q \in \mathbb{R}^m_+$ with $q \leq E^T p + K^T q$ and $A^T p + F^T q < p$.

VI. Distributed Optimization of Input-Output Performance

It is also natural to move beyond stability and optimize input-output performance. The connection between stability and performance is established by the following theorem.

Theorem 5: Suppose that $\mathbf{G}(s) = C(sI - A)^{-1}B + D$ where $A \in \mathbb{R}^{n\times n}$ is Metzler, while $B \in \mathbb{R}^{n\times 1}_+$, $C \in \mathbb{R}^{1\times n}_+$ and $D \in \mathbb{R}_+$. Then the following two conditions are equivalent:

(5.1) The matrix A is Hurwitz and $\|\mathbf{G}\|_\infty < \gamma$.

(5.2) The matrix $\begin{bmatrix} A & B \\ C & D-\gamma \end{bmatrix}$ is Hurwitz.

Moreover, if A is Hurwitz, then $\|\mathbf{G}\|_\infty = \mathbf{G}(0)$.

Proof. First note that the maximum $\max_\omega |\mathbf{G}(i\omega)|$ must be attained at $\omega = 0$ since

$$|\mathbf{G}(i\omega)| = \left|D + \int_0^\infty Ce^{At}B\right| dt \leq |D| + \int_t^\infty |Ce^{At}B| dt = D - CA^{-1}B = \mathbf{G}(0)$$

Hence $\|\mathbf{G}\|_\infty < \gamma$ may equivalently be written

$$D - CA^{-1}B < \gamma$$

Suppose that (5.1) holds. By Proposition 1 there exists $\xi > 0$ such that $A\xi < 0$. Define $x = \xi - A^{-1}B$. Then $x > 0$ since $-A^{-1} \geq 0$ and

$$Ax + B = A\xi < 0$$

If ξ is sufficiently small, we also get $Cx + D < \gamma$ so

$$\begin{bmatrix} A & B \\ C & D-\gamma \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3}$$

and (5.2) holds. Conversely, (5.2) implies that (3) holds for some x, so $Ax < 0$ and

$$-A^{-1}B < x \qquad\qquad D - CA^{-1}B < Cx + D < \gamma$$

so (5.1) follows. □

A discrete time version can be stated as follows.

Theorem 6: Let $\mathbf{G}(z) = C(zI - A)^{-1}B + D$ where $A \in \mathbb{R}_+^{n\times n}$, $B \in \mathbb{R}_+^{n\times 1}$, $C \in \mathbb{R}_+^{1\times n}$ and $D \in \mathbb{R}_+$. Then the following two conditions are equivalent:

(6.1) The matrix A is Schur and $\|\mathbf{G}\|_\infty < \gamma$.

(6.2) The matrix $\begin{bmatrix} A & B \\ \gamma^{-1}C & \gamma^{-1}D \end{bmatrix}$ is Schur.

Moreover, if A is Schur, then $\|\mathbf{G}\|_\infty = \mathbf{G}(1)$.

Combining Theorem 5 with Theorem 3 gives a linear programming formulation of the problem to minimize input-output gain:

Corollary 7: Let $\mathcal{D}$ be the set of $m \times m$ diagonal matrices with entries in $[0, 1]$. Suppose that D is scalar and that $A + ELF$ is Metzler for all $L \in \mathcal{D}$.

If the matrices B, C, D and F have nonnegative coefficients, then the following two conditions are equivalent:

(7.1) There exists $L \in \mathcal{D}$ such that $A + ELF$ is Hurwitz and

$$\|C[sI - (A + ELF)]^{-1}B + D\|_\infty < \gamma. \tag{4}$$

(7.2) There exist $\xi \in \mathbb{R}^n_+$, $\mu \in \mathbb{R}^m_+$ with

$$A\xi + E\mu + B < 0 \qquad C\xi + D < \gamma \qquad \mu \leq F\xi$$

If ξ, μ satisfy (7.2), then (7.1) holds for every L such that $\mu = LF\xi$.

Alternatively, if B, C, D and E are nonnegative, then (7.1) is equivalent to

(7.3) There exist $p \in \mathbb{R}^n_+$, $q \in \mathbb{R}^m_+$ with

$$A^T p + F^T q + C^T < 0 \qquad B^T p + D < \gamma \qquad q \leq E^T p$$

If p, q satisfy (7.3), then (7.1) holds for every L such that $q = LE^T p$.

Proof. According to Theorem 5, condition (7.1) holds if and only if there exists $\xi \in \mathbb{R}^n_+$ with

$$\begin{bmatrix} A + ELF & B \\ C & D - \gamma \end{bmatrix} \begin{bmatrix} \xi \\ 1 \end{bmatrix} < 0 \tag{5}$$

Given (5), the inequalities of (7.2) hold with $\mu = LF\xi$. Conversely, given (7.2), the inequalities of (5) follow provided that $\mu = LF\xi$. This proves the desired equivalence between (7.1) and (7.2). The equivalence between (7.1) and (7.3) follows immediately by replacing $G(s)$ with its transpose. □

We conclude the section by pointing out that for scalar positive systems, all induced norms are equal:

Theorem 8: For a scalar impulse response $g(t)$ and $w \in \mathbf{L}_p[0, \infty)$, let $g * w$ denote the convolution of g and w. Suppose that $g(t) \geq 0$ and $\int_0^\infty g(t)dt < \infty$. Then the induced norm $\|g\|_{p-\text{ind}} = \sup_w \frac{\|g*w\|_p}{\|w\|_p}$ satisfies

$$\|g\|_{p-\text{ind}} = \int_0^\infty g(t)dt \qquad p \in [1, \infty]$$

Proof. It is well known that $\|g\|_{2-\text{ind}} = \max_\omega |\mathbf{G}(e^{i\omega})|$ where $\mathbf{G}(s) = \int_0^\infty g(t)e^{-st}dt$. When $g(t) \geq 0$, the maximum must be attained at $\omega = 0$ since

$$|\mathbf{G}(i\omega)| = \left| \int_0^\infty g(t)e^{i\omega t}dt \right| \leq \int_0^\infty g(t)dt = \mathbf{G}(0)$$

Moreover

$$\begin{aligned}\|y(t)\|_1 &= \int_0^\infty \left| \int_0^t g(t-\tau)w(\tau)d\tau \right| dt \leq \int_0^\infty \int_0^t g(t-\tau)|w(\tau)|d\tau dt \\ &= \int_0^\infty \left(\int_\tau^\infty g(t-\tau)dt \right) \big|w(\tau)\big| d\tau = \left(\int_0^\infty g(t)dt \right) \|w\|_1\end{aligned}$$

with equality when $w(t) \geq 0$ for all t. Similarly

$$|y(t)| = \left| \int_0^\infty g(\tau)w(t-\tau)d\tau \right| \leq \int_0^\infty g(\tau)\big|w(t-\tau)\big|d\tau \leq \left(\int_0^\infty g(\tau)d\tau \right) \|w\|_\infty$$

with equality if w is constant. Hence the desired equality

$$\|g\|_{p-\text{ind}} = \int_0^\infty g(t)dt \tag{6}$$

has been proved for $p = 1$, $p = 2$ and $p = \infty$. The Riesz-Thorin convexity theorem [8, Theorem 7.1.12] shows that $\|g\|_{p-\text{ind}}$ is a convex function of p for $1 \leq p \leq \infty$, so (6) must hold for all $p \in [1, \infty]$. □

VII. Positively Dominated Systems

As indicated before, a transfer matrix $\mathbf{G} \in \mathbb{CH}_\infty^{m\times n}$ is called *positively dominated* if every matrix entry satisfies $|\mathbf{G}_{jk}(i\omega)| \leq \mathbf{G}_{jk}(0)$ for $\omega \in \mathbb{R}$. The set of all such matrices is denoted $\mathbb{DH}_\infty^{m\times n}$. Some properties follow immediately:

Proposition 9: Let $\mathbf{G}, \mathbf{H} \in \mathbb{DH}_\infty^{n\times n}$. Then $\mathbf{GH} \in \mathbb{DH}_\infty^{n\times n}$ and $a\mathbf{G} + b\mathbf{H} \in \mathbb{DH}_\infty^{n\times n}$ when $a, b \in \mathbb{R}_+$. Moreover $\|\mathbf{G}\|_\infty = \|\mathbf{G}(0)\|$.

The following property is also fundamental:

Theorem 10: Let $\mathbf{G} \in \mathbb{DH}_\infty^{n\times n}$. Then $(I - \mathbf{G})^{-1} \in \mathbb{DH}_\infty^{n\times n}$ if and only if $\mathbf{G}(0)$ is Schur.

Proof. That $(I - \mathbf{G})^{-1}$ is stable and positively dominated implies that $[I - \mathbf{G}(0)]^{-1}$ exists and is nonnegative, so $\mathbf{G}(0)$ must be Schur according to Proposition 2. On the other hand, if $\mathbf{G}(0)$ is Schur we may choose $\xi \in \mathbb{R}_+$ and $\epsilon > 0$ with $\mathbf{G}(0)\xi < (1-\epsilon)\xi$. Then for every $z \in \mathbb{C}^n$ with $0 < |z| < \xi$ and $s \in \mathbb{C}$ with Re $s \geq 0$ we have

$$|\mathbf{G}(s)^t z| \leq \mathbf{G}(0)^t|z| < (1-\epsilon)^t|z| \qquad \text{for } t = 1, 2, 3, \ldots$$

Hence $\sum_{k=0}^{\infty} \mathbf{G}(s)^t z$ is convergent and bounded above by $\sum_{k=0}^{\infty} \mathbf{G}(0)^t |z| = [I - \mathbf{G}(0)]^{-1}|z|$. The sum of the series solves the equation $[I - \mathbf{G}(s)] \sum_{k=0}^{\infty} \mathbf{G}(s)^t z = z$, so therefore $\sum_{k=0}^{\infty} \mathbf{G}(s)^t z = [I - \mathbf{G}(s)]^{-1} z$. This proves $(I - \mathbf{G})^{-1}$ is stable and positively dominated and the proof is complete. □

Theorem 11: Let $\mathcal{D}$ be the set of $m \times m$ diagonal matrices with entries in $[0, 1]$. Suppose that $\mathbf{B} \in \mathbb{DH}_\infty^{n\times 1}$, $\mathbf{C} \in \mathbb{DH}_\infty^{1\times n}$, $\mathbf{D} \in \mathbb{DH}_\infty$ and $\mathbf{A} + \mathbf{E}L\mathbf{F} \in \mathbb{CH}_\infty^{n\times n}$ for all $L \in \mathcal{D}$. Assume that the *off-diagonal entries* of $\mathbf{A} + \mathbf{E}L\mathbf{F}$ are positively dominated for all $L \in \mathcal{D}$.

If $\mathbf{F} \in \mathbb{DH}_\infty^{m\times n}$, then the following two conditions are equivalent:

(11.1) There exists $L \in \mathcal{D}$ such that $(I - \mathbf{A} - \mathbf{E}L\mathbf{F})^{-1} \in \mathbb{CH}_\infty^{n\times n}$ is positively dominated and $\|\mathbf{C}(I - \mathbf{A} - \mathbf{E}L\mathbf{F})^{-1}\mathbf{B} + \mathbf{D}\|_\infty < \gamma$.

(11.2) There exist $\xi \in \mathbb{R}_+^n$, $\mu \in \mathbb{R}_+^m$ with

$$\mathbf{A}(0)\xi + \mathbf{E}(0)\mu + \mathbf{B}(0) < \xi \qquad \mathbf{C}(0)\xi + \mathbf{D}(0) < \gamma \qquad \mu \leq \mathbf{F}(0)\xi$$

If ξ, μ satisfy (11.2), then (11.1) holds for every L such that $\mu = L\mathbf{F}(0)\xi$.

Alternatively, $\mathbf{E} \in \mathbb{DH}_\infty^{n\times m}$, then (11.1) is equivalent to

(11.3) There exist $p \in \mathbb{R}_+^n$, $q \in \mathbb{R}_+^m$ with

$$\mathbf{A}(0)^T p + \mathbf{F}(0)^T q + \mathbf{C}(0)^T < p \qquad \mathbf{B}(0)^T p + \mathbf{D}(0) < \gamma \qquad q \leq \mathbf{E}(0)^T p$$

If p, q satisfy (11.3), then (11.1) holds for every L such that $q = L\mathbf{E}(0)^T p$.

Proof. Theorem 10 shows that (11.1) holds if and only if $\mathbf{A}(0) - \mathbf{E}(0)L\mathbf{F}(0)$ is Schur and $\mathbf{C}[I - \mathbf{A}(0) - \mathbf{E}(0)L\mathbf{F}(0)]^{-1}\mathbf{B}(0) + \mathbf{D}(0) < \gamma$. According to Theorem 6, this is true if and only if

$$\begin{bmatrix} \mathbf{A}(0) + \mathbf{E}(0)L\mathbf{F}(0) & \mathbf{B}(0) \\ \gamma^{-1}\mathbf{C}(0) & \gamma^{-1}\mathbf{D}(0) \end{bmatrix} \tag{7}$$

is Schur. By Proposition 2 this is equivalent to existence of $\xi \in \mathbb{R}_+^n$ such that

$$\begin{bmatrix} \mathbf{A}(0) + \mathbf{E}(0)L\mathbf{F}(0) & \mathbf{B}(0) \\ \gamma^{-1}\mathbf{C}(0) & \gamma^{-1}\mathbf{D}(0) \end{bmatrix} \begin{bmatrix} \xi \\ 1 \end{bmatrix} < \begin{bmatrix} \xi \\ 1 \end{bmatrix}$$

This is equivalent to (11.2) if we set $\mu = L\mathbf{F}(0)\xi$, so the desired equivalence between (11.1) and (11.2) in Theorem 11 follows. The equivalence between (11.1) and (11.3) is obtained by replacing $G(s)$ with its transpose. □

Example 4 Consider a mechanical structure consisting of N point-masses connected by springs. The dynamics is described by the equations

$$\ddot{x}_i = \sum_j \ell_{ij}(x_j - x_i) + u_i + w_i \qquad\qquad i = 1, \ldots, N$$

where u_i is an external control force, w_i is a disturbance and ℓ_{ij} is the spring constant between the point masses i and j. Suppose that local control laws $u_i = -k_i x_i - d_i \dot{x}$ are given with $d_i \geq k_i$ and consider the problem to find spring constants ℓ_{ij} that minimize the gain from w_1 to x_1.

The closed loop system has the following frequency domain description

$$X_i(s) = \frac{1}{s^2 + d_i s + k_i}\left[\sum_j \ell_{ij}(X_j(s) - X_i(s)) + W_i(s)\right] \qquad\qquad i = 1, \ldots, N$$

Similarly to Example 3, we write this on matrix form as

$$X = \mathbf{G}(ELF + W)$$

where $L = \text{diag}\{\ell_{12}, \ell_{13}, \ell_{23}, \ldots\}$, $\mathbf{G} = \text{diag}\{\mathbf{G}_1, \ldots, \mathbf{G}_n\}$, $\mathbf{G}_i(s) = (s^2 + d_i s + k_i)^{-1}$ and the matrix E is nonnegative. Theorem 11 can then be applied with

$$\begin{aligned} &\mathbf{A} = 0 && \mathbf{D} = 0 \\ &\mathbf{B} = \begin{bmatrix} \mathbf{G}_1 & 0 & \ldots & 0 \end{bmatrix}^T && \mathbf{E} = \mathbf{G}E \\ &\mathbf{C} = \begin{bmatrix} 1 & 0 & \ldots & 0 \end{bmatrix} && \mathbf{F} = F \end{aligned}$$

to find the optimal spring constants. However, notice that ℓ_{ij} and ℓ_{ji} must be optimized separately, even though by symmetry they must be equal at optimum. □

VIII. THE KYP LEMMA FOR POSITIVE SYSTEMS

For multi-variable input-output gains we may follow the path suggested by [23] and replace Theorem 5 by a positive systems counterpart of the Kalman-Yakubovich-Popov lemma:

Theorem 12: Let $A \in \mathbb{R}^{n\times n}$ be Metzler and Hurwitz, while $B \in \mathbb{R}_+^{n\times m}$. Suppose that all entries of $Q \in \mathbb{R}^{(n+m)\times(n+m)}$ are nonnegative, except for the last m diagonal elements. Then the following statements are equivalent:

(12.1) $$\begin{bmatrix} (i\omega I - A)^{-1}B \\ I \end{bmatrix}^* Q \begin{bmatrix} (i\omega I - A)^{-1}B \\ I \end{bmatrix} \preceq 0 \text{ for } \omega \in [0, \infty).$$

(12.2) $\begin{bmatrix} -A^{-1}B \\ I \end{bmatrix}^* Q \begin{bmatrix} -A^{-1}B \\ I \end{bmatrix} \preceq 0.$

(12.3) There exists a diagonal $P \succeq 0$ such that $Q + \begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} \preceq 0.$

Our proof of Theorem 12 will rely on the following result, which can be found in [11, Theorem 3.1]:

Proposition 13 (Positive Quadratic Programming): Suppose $M_0, \ldots, M_K \in \mathbb{R}^{n \times n}$ are Metzler matrices and $b_1, \ldots, b_K \in \mathbb{R}$. Then

$$\begin{array}{lllll} \max & x^T M_0 x & = & \max & \mathrm{trace}(M_0 X) \\ \text{s.t.} & x \in \mathbb{R}^n_+ & & \text{s.t.} & X \succeq 0 \\ & x^T M_k x \ \geq \ b_k & & & \mathrm{trace}(M_k X) \geq b_k \\ & k = 1, \ldots, K & & & k = 1, \ldots, K \end{array} \tag{8}$$

Moreover, the maximum of (8) is finite if and only if there exist $\tau_1, \ldots, \tau_K \geq 0$ such that $M_0 + \sum_{k=1}^K \tau_k M_k$ is negative semi-definite.

Remark 6. The problem on the right is always convex and readily solvable by semidefinite programming. The problem on the left is generally not a convex program, since the matrices M_k may be indefinite. However, the maximization on the left is concave in $(x_1^2, \ldots, x_n^2)$ [16]. This is because every product $x_i x_j$ is the geometric mean of two such variables, hence concave [4, p. 74].

Proof. Every x satisfying the constraints on the left hand side of (8) corresponds to a matrix $X = xx^T$ satisfying the constraints on the right hand side. This shows that the right hand side of (8) is at least as big as the left.

On the other hand, let $X = (x_{ij})$ be any positive definite matrix. In particular, the diagonal elements $x_{11}, \ldots, x_{nn}$ are non-negative and $x_{ij} \leq \sqrt{x_{ii} x_{jj}}$. Let $x = (\sqrt{x_{11}}, \ldots, \sqrt{x_{nn}})$. Then the matrix xx^T has the same diagonal elements as X, but has off-diagonal elements $\sqrt{x_{ii} x_{jj}}$ instead of x_{ij}. The fact that xx^T has off-diagonal elements at least as big as those of X, together with the assumption that the matrices M_k are Metzler, gives $x^T M_k x \geq \mathrm{trace}(M_k X)$ for $k = 1, \ldots, K$. This shows that the left hand side of (8) is at least as big as the right.

For the last statement, note that the conditions $\mathrm{trace}(M_k X) \geq b_k$ are linear in X, so strong duality holds [20, Theorem 28.2] and the right hand side of (8) has a finite maximum if and only if $M_0 + \sum_{k=1}^K \tau_k M_k \preceq 0$ for some $\tau_1, \ldots, \tau_K \geq 0$. □

Proof of Theorem 12. Putting $\omega = 0$ gives (12.2) from (12.1). The matrix $-A^{-1}$ is nonnegative, so (12.2) gives $\begin{bmatrix} x \\ w \end{bmatrix}^T Q \begin{bmatrix} x \\ w \end{bmatrix} \leq 0$ for all $x \in \mathbb{R}^n_+$, $w \in \mathbb{R}^m_+$ with

$$x \leq -A^{-1}Bw \tag{9}$$

The inequality (9) follows from (but is not equivalent to) the constraint $0 \leq Ax + Bw$, which can also be written $0 \leq A_i x + B_i w$ for $i = 1, \ldots, n$, where A_i and B_i denote the i:th rows of A and B respectively. For non-negative x and w, this is equivalent to

$$0 \leq x_i(A_i x + B_i w) \qquad\qquad i = 1, \ldots, n \tag{10}$$

Hence (12.2) implies that $\begin{bmatrix} x \\ w \end{bmatrix}^T Q \begin{bmatrix} x \\ w \end{bmatrix} \leq 0$ for $x \in \mathbb{R}^n_+$, $w \in \mathbb{R}^m_+$ satisfying (10). By Proposition 13, the same bound must hold for $x \in \mathbb{R}^n$, $w \in \mathbb{R}^m$ and there exist $\tau_1, \ldots, \tau_n \geq 0$ such that the quadratic form

$$\sigma(x, w) = \begin{bmatrix} x \\ w \end{bmatrix}^T Q \begin{bmatrix} x \\ w \end{bmatrix} + \sum_i \tau_i x_i (A_i x + B_i w)$$

is negative semi-definite. Define $P = \mathrm{diag}(\tau_1, \ldots, \tau_n) \succeq 0$. Integrating over t gives

$$0 \geq \int_0^\infty \left(\begin{bmatrix} x \\ w \end{bmatrix}^T Q \begin{bmatrix} x \\ w \end{bmatrix} + x^T P(Ax + Bw) \right) dt$$

For square integrable solutions to $\dot{x} = Ax + Bw$, $x(0) = 0$ we get

$$0 \geq \int_0^\infty \left(\begin{bmatrix} x \\ w \end{bmatrix}^T Q \begin{bmatrix} x \\ w \end{bmatrix} + \frac{d}{dt}(x^T P x) \right) dt = \int_0^\infty \begin{bmatrix} x(t) \\ w(t) \end{bmatrix}^T Q \begin{bmatrix} x(t) \\ w(t) \end{bmatrix} dt$$

which in frequency domain implies (12.1). Hence (12.1)⇔(12.2)⇔(12.3). □

For interconnected systems it is common to have constraints on several subsystems. For such situations, the following theorem is useful:

Theorem 14: Let $A \in \mathbb{R}^{n\times n}$ be Metzler and Hurwitz, while $B \in \mathbb{R}_+^{n\times m}$. Suppose that all entries of $Q_1, \dots, Q_K \in \mathbb{R}^{(n+m)\times(n+m)}$ are nonnegative, except for the last m diagonal elements. Let $q_1, \dots, q_K \in \mathbb{R}$. Then the following statements are equivalent:

(14.1) The equation $\dot{x} = Ax + Bw$, $x(0) = 0$ has a square integrable solution with

$$\int_0^\infty \begin{bmatrix} x \\ w \end{bmatrix}^T Q_k \begin{bmatrix} x \\ w \end{bmatrix} dt > q_k \qquad\qquad k = 1, \dots, K$$

(14.2) There exists no $(\tau_1, \dots, \tau_K) \neq 0$ with τ_k being nonnegative numbers and $P \succeq 0$ a diagonal matrix such that

$$\sum_{k=0}^K \tau_k Q_k + \begin{bmatrix} A^T P + PA & PB \\ B^T P & 0 \end{bmatrix} \preceq 0 \qquad\qquad \sum_{k=0}^K \tau_k q_k \geq 0$$

If (14.1) is true, then there exist $\widehat{w} \in \mathbb{R}_+^m$ and $T_\epsilon > 0$ such that the conditions of (14.1) hold when $w(t) = \widehat{w}$ for $t \in [0, T_\epsilon]$ and $w(t) = 0$ for $t > T_\epsilon$.

Proof. For $k = 1, \dots, K$ and $w \in \mathbf{L}_2^m[0, \infty)$, define

$$\sigma_k(w) = \int_0^\infty \begin{bmatrix} x \\ w \end{bmatrix}^T Q_k \begin{bmatrix} x \\ w \end{bmatrix} dt - q_k$$

where $\dot{x} = Ax + Bw$, $x(0) = 0$. It was shown in [17, Theorem 4.1] that the set

$$\Sigma = \Big\{ (\sigma_1(w), \dots, \sigma_K(w)) \, : \, w \in \mathbf{L}_2^m[0, \infty) \Big\}$$

has convex closure $\overline{\Sigma}$ in $\mathbb{R}^K$. Failure of the statement (14.1) means that $\overline{\Sigma}$ does not intersect the open positive orthant. This implies existence of separating hyperplane, i.e. a vector $(\tau_1, \dots, \tau_K) \neq 0$ with $\tau_k \geq 0$ and

$$\sum_k \tau_k \sigma_k(w) \leq 0 \qquad \text{for all } w \in \mathbf{L}_2^m[0, \infty) \tag{11}$$

The condition (11) holds if and only if $\sum_{k=0}^K \tau_k q_k \geq 0$ and

$$\begin{bmatrix} (i\omega I - A)^{-1} B \\ I \end{bmatrix}^* \left(\sum_{k=0}^m \tau_k Q_k \right) \begin{bmatrix} (i\omega I - A)^{-1} B \\ I \end{bmatrix} \preceq 0 \qquad\qquad \text{for } \omega \in [0, \infty) \tag{12}$$

so (14.2) fails according to Theorem 12.

Conversely, if (14.1) holds then

$$\int_0^\infty \begin{bmatrix} x \\ w \end{bmatrix}^T \left(\sum_{k=0}^m \tau_k Q_k \right) \begin{bmatrix} x \\ w \end{bmatrix} dt > \sum_{k=0}^K \tau_k q_k$$

so either $\sum_{k=0}^K \tau_k q_k < 0$, or the left hand side is strictly positive, in which case (12) fails. Hence (14.2) holds by Theorem 12. □

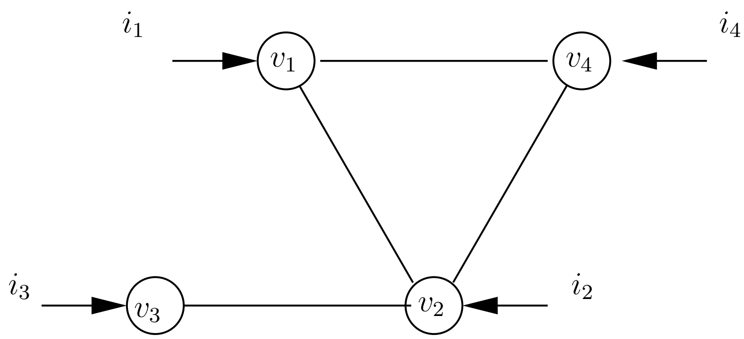

Fig. 3. Illustration of the power transmission network described by equation (13).

Example 5. Electrical Power Transmission. Consider a power transmission network illustrated in Figure 3 and described by the equations

$$\begin{cases} C_1\dot{v}_1 = Y_{12}(v_2 - v_1) + Y_{14}(v_4 - v_1) + i_1 \\ C_2\dot{v}_2 = Y_{12}(v_1 - v_2) + Y_{23}(v_3 - v_2) + Y_{24}(v_4 - v_2) + i_2 \\ C_3\dot{v}_3 = Y_{23}(v_2 - v_3) + i_3 \\ C_4\dot{v}_4 = Y_{14}(v_1 - v_4) + Y_{24}(v_2 - v_4) + i_4 \end{cases} \tag{13}$$

The power generated in node k at time t is given by $v_k(t)i_k(t)$. Consider the problem to minimize losses in the network subject to constraints on power demands in the nodes:

Minimize $\sum_{k=1}^{4}\int_0^\infty v_k(t)i_k(t)dt$

subject to (13) with $\int_0^\infty v_k i_k dt \leq P_k$ and $\int_0^\infty v_k^2 dt \leq V_k^{\max}$ for $k = 1, \dots, 4$

The state dynamics as well as the quadratic forms have the Metzler structure required by Theorem 14, so the minimal losses can be found by semi-definite programming over τ_k using condition (14.2). Moreover, the theorem states that minimal losses can be achieved by keeping the voltages constant for a long period of time. □

IX. Conclusions

The results above indicate that the monotonicity properties of positive systems and positively dominated systems bring remarkable benefits to control theory. Most important is the opportunity for scalable verification and synthesis of H_∞ optimal performance. In particular, the optimal solution comes with a certificate (the numbers ξ_k, μ_k) that makes it possible to verify optimality locally, without access to a global model.

Many important problems remain open for future research. Here are two examples:

- How can the scalable methods for verification be extended to monotone nonlinear systems in a nonconservative way?
- How can local controllers be designed to get positively dominated interactions with optimal properties? (This would be in contrast with the mass-spring example where the local control parameters d_i and k_i were fixed.)

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