LUND UNIVERSITY

Department of
AUTOMATIC CONTROL

Multivariable Control Exam

2018-04-06, 14.00–19.00

Points and grades

All answers must include a clear motivation and a well-formulated answer. Answers may be given in English or Swedish. The total number of points is 25. The maximum number of points is specified for each subproblem.

Accepted aid

The textbook *Glad & Ljung*, standard mathematical tables like TEFYMA, an authorized “Formelsamling i Reglerteknik”/”Collection of Formulas” and a pocket calculator. Handouts of lecture notes and lecture slides are also allowed.

Results

The result of the exam will be entered into LADOK. The result as well as solutions will be available on the course home page:
`http://www.control.lth.se/Education/EngineeringProgram/FRTN10.html`

1. Consider a linear system described by the following transfer function matrix:

$$G(s) = \begin{bmatrix} \frac{1}{s+1} & \frac{-1}{s+1} \\ \frac{2s}{s+1} & \frac{2s}{s+2} \end{bmatrix}$$

a. Calculate the (multivariable) poles and zeros of the system, including their multiplicity. (2 p)

b. Write down a minimal state-space realization of the system. (2 p)

c. Show that it is not possible to calculate the RGA of the system in stationarity. What is the fundamental problem? (1 p)

d. Show that the input transformation

$$\tilde{u}_1 = u_1 - u_2$$
$$\tilde{u}_2 = u_2$$

completely eliminates the cross-coupling in the first output. (1 p)

e. It can be shown that $G(s)$ has maximum gain when the input frequency approaches infinity. What is the maximum gain, and for what input direction is it attained? (1 p)

Hint: If you like, you may use the singular value decomposition

$$\begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2\sqrt{2} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ \sqrt{2}/2 & -\sqrt{2}/2 \end{bmatrix}$$

2. Derive a controller using the IMC method for the following system:

$$P(s) = \frac{4-2s}{s^2+7s+10}$$

Any free closed-loop poles should be given the same speed as the slowest open-loop pole. Check that the resulting controller is stable and proper. (3 p)

3. The linear system

$$G(s) = \begin{bmatrix} \frac{2s}{s+1} & \frac{4}{s+3} \end{bmatrix}$$

has been connected in a feedback loop with an unknown linear system $F(s)$ as illustrated in Figure **??**.

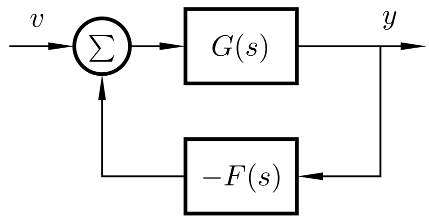

Figure 1 Setup for the system in problem **??**.

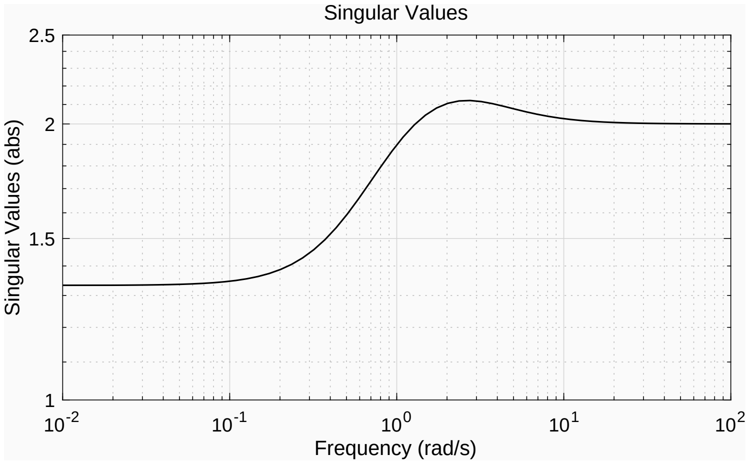

Figure 2 Singular values plot for $G(s)$ in problem **??**.

a. What are the dimensions of the signals v and y and of the system $F(s)$? (1 p)

b. A singular values plot for $G(s)$ is given in Figure **??**. State a set of (non-trivial) sufficient conditions on $F(s)$ under which the closed-loop system is guaranteed to be stable. (1 p)

c. Give an example of a system $F(s)$ that makes the closed-loop system unstable. Hint: It is enough to consider a static system. (1 p)

4. Consider an inverted pendulum that is mounted on a cart. If there is no friction, the transfer function from the force F on the cart, to the cart position p is given by

$$P_{pF}(s) = \frac{s^2 - 16}{s^2(s^2 - 9)}.$$

Show that it is impossible to find a linear controller that robustly stabilizes $P_{pF}(s)$.
(This problem illustrates that it is hard to stabilize the pendulum in the upright position by only measuring the cart position.) (2 p)

5. Consider the scalar linear system

$$G(s) = \frac{1}{s-1}$$

represented in state-space form with noise as

$$\begin{aligned} \dot{x} &= x + u + v_1 \\ z &= x \\ y &= x + v_2 \end{aligned}$$

The noises v_i are independent and white with intensities $R_i > 0$. We use the design criterion

$$J = \int_0^\infty (Q_1 x^2 + Q_2 u^2) dt$$

with $Q_i > 0$ and want to find the LQG controller.

a. Show that the controller is a function of $r = R_1/R_2$ and $q = Q_1/Q_2$ only, using the algebraic expressions for the relevant algebraic Riccati equations. (3 p)

b. Compute the poles of the closed-loop system as a function of r and q. (1 p)

c. Based on the results from subproblem **b**, explain what happens if we put more trust on the measurements than the process and vice versa. In which case does the state estimates converge faster? Motivate! (1 p)

d. Based on the results from subproblem **b**, explain what happens if we put larger penalty on the state variable than the control signal ($Q_1 \gg Q_2$) and vice versa. In which case does the process states converge faster to the origin? Motivate! (1 p)

6. The third order system

$$\dot{x} = \begin{bmatrix} -0.5 & 1 & 1 \\ 0 & -1 & 1.5 \\ 0 & 0 & -2 \end{bmatrix} x + \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 3 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} x$$

has the balanced realization

$$\dot{\xi} = \begin{bmatrix} -0.366 & 0.289 & -0.159 \\ -0.292 & -0.696 & -0.346 \\ 0.236 & 1.54 & -2.44 \end{bmatrix} \xi + \begin{bmatrix} -0.883 & -2.18 \\ -0.186 & -0.864 \\ -0.508 & 1.01 \end{bmatrix} u$$
$$y = \begin{bmatrix} -0.497 & -0.0194 & -1.10 \\ -2.30 & 0.883 & -0.258 \end{bmatrix} \xi$$

with the observability Gramian

$$O_\xi = \begin{bmatrix} 7.58 & 0 & 0 \\ 0 & 0.560 & 0 \\ 0 & 0 & 0.260 \end{bmatrix}.$$

a. What is the controllability Gramian S_ξ for the balanced realization? (0.5 p)

b. Reduce the system to a first order system without changing the behavior in stationarity. To aid the calculations, you are given the matrix inverse

$$\begin{bmatrix} -0.696 & -0.346 \\ 1.54 & -2.44 \end{bmatrix}^{-1} = \begin{bmatrix} -1.09 & 0.155 \\ -0.691 & -0.312 \end{bmatrix}$$

(3 p)

c. Do you think it would have been a big difference in the performance if you had only eliminated one of the states instead of two? Motivate. (0.5 p)