A New Framework for Stability Analysis of Quantized Feedback Systems

Dissertation

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Abstract

A quantized feedback system is a control system in which finite-level quantization of signal values is involved in the feedback loop. Quantized feedback is found in many engineering systems including mechanical systems and networked systems because multilevel-valued devices such as A/D (Analog-to-Digital) converters, on/off switching actuators, and digital communication networks are widely used in these systems. The purpose of this thesis is to establish a unified approach for the stability analysis of quantized feedback systems.

In this thesis, we first provide the motivation for developing a new framework for the stability analysis of quantized feedback systems. The stability analysis of quantized feedback systems is difficult in many cases because the traditional small gain theorem cannot be successfully applied owing to the nonlinearities caused by the finite-level quantization. This thesis shows the difficulties involved by using two examples: an uncertain networked control system with a rate-limited communication channel and a feedback system involving a uniform quantizer. Through the stability analysis of these examples, we discuss the need for introducing a new notion of stability and a new framework for analysis.

A new framework for the stability analysis of quantized feedback systems is then developed. In particular, we introduce a new notion of *small ℓ^p signal ℓ^p stability* in this thesis. This is a practical and reasonable notion for the stability of quantized feedback systems; in contrast, classical stability notions such as finite gain ℓ^p stability and asymptotic stability are occasionally too strong and not achievable in the presence of finite-level quantization of signals. *The small level theorem* is derived to give a sufficient condition for a feedback system to be small ℓ^p signal ℓ^p stable. A new class of uncertainty, *level bounded uncertainty*, is also introduced in this thesis. This is useful in approximating some classes of nonlinearities that include quantization errors. Using all these new notions and theorems, we provide a mathematical framework for the stability analysis of nonlinear systems that is applicable to a wide class of quantized feedback systems.

Finally, the use of the proposed framework is demonstrated by addressing two important issues related to quantized feedback systems: robust stabilization of an uncertain networked control systems over a rate-limited communication channel and stability analysis of a networked control system that is affected by finite-level quantization and packet dropouts. In the first example, quantitative analysis is provided for the combined effect of quantization and the model uncertainty in the system dynamics on the stability of the entire networked control system. In the second example, we elucidate the effect of quantization and packet dropouts on the stability of the overall networked control system. While the quantitative analyses of the stabilities of these two networked control systems have remained unresolved problems, the proposed framework provides us with a systematic approach for solving both of these important issues.

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Contents

Notations and Definitions

$\in$	belong to
$\subset$	subset (strict or not)
$\cup$	union
$\emptyset$	empty set
$\mathbb{R}$	set of real numbers
$\mathbb{R}_+$	set of nonnegative real numbers
$\mathbb{R}^n$	set of real n-dimensional vectors
$\mathbb{R}^{m\times n}$	set of $m \times n$ real matrices
$\mathbb{N}$	set of natural numbers
$\mathbb{Z}_+$	set of nonnegative integers
a_{ij}	the (i, j)-th component of a matrix A
$\lVert x\rVert_p$ $(1 \le p < \infty)$	p-norm of a vector $x \in \mathbb{R}^n$, i.e., $\lVert x\rVert_p := \left(\sum_{i=1}^n \lvert x_i\rvert^p\right)^{1/p}$
$\lVert x\rVert_\infty$	∞-norm of a vector $x \in \mathbb{R}^n$, i.e., $\lVert x\rVert_\infty := \max_{1\le i\le n} \lvert x_i\rvert$
$\lVert A\rVert_1$	1-norm of a constant matrix $A \in \mathbb{R}^{n\times m}$: $\lVert A\rVert_1 := \max_{1\le i\le n} \sum_{j=1}^m \lvert a_{ij}\rvert$
$f\vert_{[a,b]}$	restriction of a function f to an interval $[a, b]$
P_τ	truncation operator at time τ: $P_\tau f(t) := \begin{cases} f(t) & (0 \le t \le \tau), \\ 0 & (\tau < t). \end{cases}$
ℓ^p $(1 \le p < \infty)$	space of functions $f : \mathbb{Z}_+ \to \mathbb{R}^n$ s.t. $\sum_{t=0}^\infty \lVert f(t)\rVert_p^p < \infty$
ℓ^∞	space of functions $f : \mathbb{Z}_+ \to \mathbb{R}^n$ s.t. $\sup_{t\ge 0} \lVert f(t)\rVert_\infty < \infty$
ℓ_e	extended space of functions: $\ell_e := \{f : \mathbb{Z}_+ \to \mathbb{R}^n \mid P_\tau f \in \ell_p,\ \forall\tau \in \mathbb{Z}_+\}$
$\lVert f\rVert_{\ell^p}$ $(1 \le p \le \infty)$	ℓ^p norm of a signal $f \in \ell^p$, that is, $\lVert f\rVert_{\ell^p} = \begin{cases} \sum_{t=0}^\infty \lVert f(t)\rVert_p^p & (1 \le p < \infty), \\ \sup_{t\ge 0} \lVert f(t)\rVert_\infty & (p = \infty). \end{cases}$
$\lVert F\rVert_{\ell^p\text{-ind}}$ $(1 \le p \le \infty)$	ℓ^p induced norm of a map F

The following notations are also used.

- The upper and lower linear fractional transformations (LFTs) are denoted by

$$\mathcal{F}_{\mathrm{u}}\left(\left[\begin{smallmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{smallmatrix}\right], \Delta\right) = M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12},$$
$$\mathcal{F}_{\mathrm{l}}\left(\left[\begin{smallmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{smallmatrix}\right], \Delta\right) = M_{11} + M_{12}\Delta(I - M_{22}\Delta)^{-1}M_{21}.$$

- A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is said to belong to class $\mathcal{K}$ ($\alpha \in \mathcal{K}$) if it is continuous, zero at zero, and strictly increasing.

- A map $H : \ell_e \to \ell_e$ is said to be causal if

$$P_\tau H u = P_\tau H P_\tau u$$

holds for $\forall \tau \in \mathbb{Z}_+$ and $\forall u \in \ell_e$. H is said to be strictly causal if

$$P_\tau H u = P_\tau H P_{\tau-1} u$$

holds for $\forall \tau \in \mathbb{Z}_+$ and $\forall u \in \ell_e$.

Chapter 1

Introduction

1.1 Quantized feedback systems

A quantized feedback system is a control system in which the feedback loop involves finite-level quantization of signal values. Feedback control with quantized signals is used in many engineering systems such as mechanical systems and networked systems. Various digital devices with finite-level-valued signals are often involved in mechanical systems, e.g., A/D (Analog-to-Digital) converters, multilevel-valued sensors, on/off switching actuators, and so on. Communication channels in networked control systems can transmit only a finite amount of information per unit of time. Therefore, in such networked systems, we have to deal with signals that are quantized with finitely many levels. Because of the increasing and widespread use of communication networks and digital devices in recent times, quantized feedback systems have become prevalent in our society.

Quantized feedback systems give rise to challenging problems in the analysis and synthesis of control systems. Owing to the nonlinearities caused by the finite-level quantization of signals, the analysis and design of quantized feedback systems are complicated. In particular, it is often difficult to employ the classical small gain theorem and some traditional analysis tools to establish the stability of feedback systems. Many studies have focused on overcoming this difficulty and providing a systematic method for the analysis and synthesis of this important class of control systems (see, e.g., [23] and the references therein).

One of the most widely investigated quantized feedback systems is the single-loop feedback system shown in Figure 1.1. In this feedback system, the signal s is assumed to take its value in a given finite set. The development of this quantized feedback system was motivated by some applications

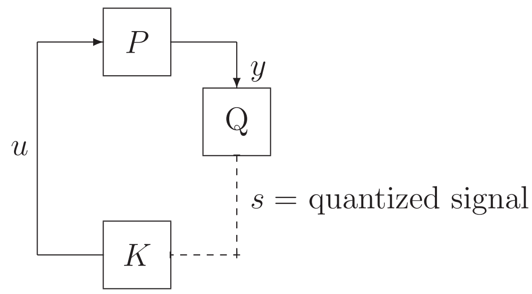

Figure 1.1: A quantized feedback system

of networked control systems and mechanical systems. In networked control systems, s represents the channel symbol that is transmitted over, for example, an intervening communication channel. The system P represents a plant to be controlled, and the devices Q and K are an encoder and a decoder-controller, respectively. Then, this feedback system can be viewed as a simplified model for a networked control system including a rate-limited communication channel. This quantized feedback system can also be considered as a model of a mechanical system with a multilevel-valued sensor/actuator. The device Q is, for example, a multilevel-valued sensor, and K is a controller that produces the control input u based on the quantized signals. In the last decade, the stability analysis of this quantized feedback systems has been actively studied with the increasing applications of mechanical systems and networked control systems involving quantized signals.

Noteworthy results on the stability analysis of this quantized feedback system have been reported in [37, 32, 15, 20, 21]. It should be noted that in these works, various notions of stability have been proposed because strong stabilities such as asymptotic stability and finite gain ℓ^p stability are occasionally not achievable when finite-level quantization of a signal is involved in the feedback loop. A weaker stability notion, containability, is introduced in [37]. The boundedness of the Euclidean norm of the plant state at each time instant is studied in [32] under the assumption that a bound on the Euclidean norm of the external disturbance at each time instant is given. The authors of [15] have discussed the problem of achieving the input-to-state stability to external disturbances for the quantized feedback systems. In [20], the tight condition for exponential stability has been derived for a feedback system that is not driven by external disturbances. With regard to works on stochastic setups, [21] is noteworthy in that it derives the tight condition on quantization for the mean square stabilizability of the quantized feedback systems.

There still remain many unresolved problems in the analysis and synthesis of quantized feedback systems. In fact, for the quantized feedback system shown in Figure 1.1, stability analysis is difficult when the controlled plant P includes uncertainty in its system dynamics. Because model uncertainty is indeed inevitable in practical control systems, robust stability analysis and robust stabilization of uncertain networked control systems are considered as important issues. Although [28] recently showed that the input-to-state stability of a networked control system involving finite-level quantization is robust with respect to some perturbation in the system dynamics, the quantitative analysis has yet been carried out. Another important issue is the stabilization of quantized feedback systems with "simple" quantizers. While most of the stabilizing quantizers employed in the previous works mentioned above are complicated and time-variant devices, the stabilization of systems with time-invariant quantizers, such as uniform quantizers and on/off switching actuators, is practically significant. The stabilization of the single-loop system shown in Figure 1.1 with a memoryless (static) quantizer has been recently discussed in [5, 9, 29]. The analysis and synthesis of more practical and complicated quantized feedback systems, e.g., networked control systems subject to combined effects of several channel properties of quantization, packet dropouts, and delays, have not yet been carried out.

Because of the increasing applications of such quantized feedback systems, nowadays many studies have focused on developing a systematic approach to resolve these unaddressed problems.

1.2 Robust control approach

This thesis develops a new framework for the stability analysis of quantized feedback systems, inspired by the basic concepts of Robust Control Theory. Using these concepts has been one of the most successful ways to analyze the stability of a wide class of nonlinear systems. Because the set of quantized feedback systems is a class of nonlinear systems, it is rather natural to expect some of the essential concepts pertaining to robust control frameworks to be helpful in the establishment of a framework for the stability analysis of quantized feedback systems.

First, it should be recalled that while robust control frameworks have been well known to enable us to deal with model uncertainties (see, e.g., textbooks [41, 8]), they also provide useful ways to analyze the stability of some classes of nonlinear systems. Figure 1.2 shows the concept. In this figure, a given nonlinear system is expressed as a feedback interconnection of a nominal system G and an error system Δ by appropriate system transformations. The nominal system is typically chosen from a set of tractable systems

(a) Original nonlinear system (b) Model for stability analysis

Figure 1.2: Robust control approach

such as the set of linear time-invariant (LTI) systems. The error system Δ is generally nonlinear and represents the gap between the nominal system and the original nonlinear system. Robust control frameworks provide theoretical and computational tools to establish the stability of the feedback system shown in Figure 1.2 (b) in the case in which G and Δ belong to particular sets of systems. For example, when the subsystems G and Δ both have bounded ℓ^p gains, the well-known small gain theorem [13] can be applied to the stability analysis of the nonlinear feedback system. S-procedure [38] is employed for stability analysis when Δ satisfies integral quadratic constraints [19]. In this manner, classical robust control frameworks provide us with methods to analyze the stability of these particular classes of nonlinear feedback systems.

Most recently, [35, 22] have attempted to extend these concepts pertaining to classical robust control frameworks to the stability analysis of quantized feedback systems. These studies addressed the stability analysis of quantized feedback systems by decomposing the original system into a feedback interconnection of some tractable subsystems, and they provided a theorem to establish the stability of the feedback system. Because the classical frameworks are not directly applicable to quantized feedback systems, new frameworks have been developed. Specifically for feedback systems with measurements and inputs with finitely many quantization levels, Tarraf, Megretski, and Dahleh [35] proposed a framework for robust stability analysis based on deterministic finite state machine models and several new stability notions. Liberzon and Nešić suggested the decomposition of a class of quantized feedback systems into the feedback interconnection of its discrete and continuous subsystems and the use of the input-to-state small-gain theorem [22] for stability analysis.

The research presented in this thesis is along similar lines and is motivated by a desire to establish a unified and systematic approach to the stability analysis of a wide class of quantized feedback systems. In particular, we introduce a new and practical notion of stability for quantized feedback systems and provide a set of mathematical tools for the stability analysis of a feedback system based on this stability notion. The proposed framework is applicable to a wide class of quantized feedback systems and enables us to tackle several important unresolved problems.

1.3 Contributions and organization of the thesis

A new framework for the stability analysis of quantized feedback systems is developed in this thesis. We introduce a new stability notion and provide a set of theorems to establish the stability of a feedback system based on the new stability notion. A new set of nonlinear systems is introduced to approximate some classes of nonlinearities including quantization errors. With all these new notions and mathematical tools, we provide a systematic approach to the stability analysis of a wide class of quantized feedback systems. The usefulness of the proposed framework is demonstrated by solving two unresolved problems: robust stabilization of an uncertain networked control system over a rate-limited communication channel and stability analysis of a networked control system that is affected by both finite-level quantization and packet dropouts. A simple uniform quantizer is employed in the latter problem.

The contributions of this thesis are as follows:

- The motivation for developing a new framework for the stability analysis of quantized feedback systems is provided. It is difficult or occasionally impossible to apply the classical small gain theorem to establish the stability of quantized feedback systems. By demonstrating these difficulties with two examples, we provide the motivation for introducing a reasonable notion of weak stability and a new framework for stability analysis. (Chapter 2)

- A new notion of *small ℓ^p signal ℓ^p stability* is introduced. This is a practical and reasonable stability notion for quantized feedback systems. (Chapter 3)

- *Small level theorem* is derived as a key theorem for the stability analysis of quantized feedback systems. This theorem gives a sufficient condition for a feedback system to be small ℓ^p signal ℓ^p stable. (Chapter 3)

- A novel class of uncertainty, *level bounded uncertainty*, is introduced. This new class of nonlinear maps enables us to effectively approximate some classes of model uncertainties and nonlinearities including quantization errors. (Chapter 3)

- A new framework for stability analysis and robust stability analysis is developed based on the ℓ^p signal ℓ^p stability. With regard to robust stability analysis, we derive conditions for the robust stability of feedback systems against two classes of uncertainties: the traditional gain bounded uncertainty and level bounded uncertainty. (Chapter 3)

- Robust stabilization of an uncertain networked control system over a rate-limited communication channel is studied based on the proposed framework. We derive a sufficient condition on the data rate of the communication channel for the existence of an encoder-controller pair that robustly stabilizes the networked control system. The combined effect of quantization and model uncertainty on the stabilizability of the entire system is evaluated by the derived condition. The robust stabilization of an uncertain networked control system demonstrates the usefulness of the proposed framework for the stability analysis of quantized feedback systems. (Chapter 4)

- Stability of a networked control system subject to finite-level quantization and packet dropouts is investigated. A sufficient condition on quantization parameters for the stability of the system is derived for a given triple of plant, controller, and packet dropout parameter. The effect of quantization and packet dropouts on the stability of the networked system is evaluated. This is another example that shows the usefulness of the proposed framework. (Chapter 5)

This thesis is organized as follows.

Chapter 2 provides the motivation of the thesis by discussing the difficulties involved in the stability analysis of two important examples of quantized feedback systems. In particular, we show that the classical small gain theorem is not applicable to these two examples.

In Chapter 3, a new framework for stability and robust stability analyses is developed. We first introduce a new notion of small ℓ^p signal ℓ^p stability and derive a condition for a feedback system to be small ℓ^p signal ℓ^p stable. A new class of uncertainty, level bounded uncertainty, is introduced in this chapter. We then examine the robust stability of a feedback system against two classes of uncertainties: level bounded uncertainty and the traditional gain bounded uncertainty.

Chapter 4 is devoted to the robust stabilization of an uncertain system over a rate-limited communication channel. By utilizing the framework proposed in Chapter 3, we evaluate the trade-off between the quantization and the model uncertainty in the plant dynamics for the robust stabilizability of the networked control system.

Chapter 5 studies the stability of a networked control system affected by finite-level quantization and packet dropouts. We elucidate the combined effect of the quantizer and packet dropouts on the stability of the networked control system.

Chapter 6 concludes the thesis.

Chapter 2

Difficulties Involved in Stability Analysis of Quantized Feedback Systems

This chapter discusses the motivation of this thesis for developing a new framework for the stability analysis of quantized feedback systems. Toward this end, we discuss the difficulties involved in applying the small gain theorem to the stability analysis of two examples of quantized feedback systems: an uncertain networked control system with a rate-limited communication channel and a feedback system involving a uniform quantizer. Through the stability analyses of these quantized feedback systems, we show the motivation for introducing a new and reasonable stability notion and a framework for stability analysis based on the stability.

This chapter is organized as follows. Section 2.1 serves as a preliminary and discusses the definitions of several stability notions. Section 2.2 introduces the small gain theorem. Section 2.3 discusses the stability analysis of two examples of quantized feedback systems. Section 2.4 summarizes this chapter.

2.1 Classical stability notions

This section discusses the definitions of several existing stability notions related to this thesis. These notions are repeatedly referred to throughout this thesis.

Definition 1. (ℓ^p *stability*, see, e.g., [13])
A map H: $\ell_e \to \ell_e$ is ℓ^p *stable* if there exist a class $\mathcal{K}$ function α and a

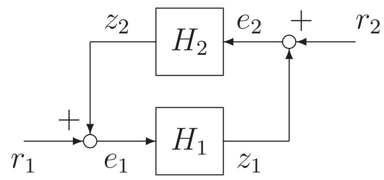

Figure 2.1: Feedback system for stability analysis

nonnegative constant β such that

$$\|H(u)|_{[0,\tau]}\|_{\ell^p} \leq \alpha(\|u|_{[0,\tau]}\|_{\ell^p}) + \beta \quad \forall u \in \ell_e,\ \forall \tau \in \mathbb{Z}_+. \tag{2.1}$$

In addition, H is called *unbiased* if we can take $\beta = 0$.

The feedback system shown in Figure 2.1 is called ℓ^p *stable* if there exist a class $\mathcal{K}$ function α and a nonnegative constant β such that

$$\|z|_{[0,\tau]}\|_{\ell^p} \leq \alpha(\|r|_{[0,\tau]}\|_{\ell^p}) + \beta \quad \forall r \in \ell_e,\ \forall \tau \in \mathbb{Z}_+, \tag{2.2}$$

where $z := (z_1, z_2)$ and $r := (r_1, r_2)$.

Definition 2. (*finite gain ℓ^p stability*, see, e.g., [13])
A map $H\colon\ \ell_e \to \ell_e$ is *finite gain ℓ^p stable* if there exist nonnegative constants γ and β such that

$$\|H(u)|_{[0,\tau]}\|_{\ell^p} \leq \gamma\|u|_{[0,\tau]}\|_{\ell^p} + \beta \quad \forall u \in \ell_e,\ \forall \tau \in \mathbb{Z}_+. \tag{2.3}$$

The infimum of γ satisfying the above inequality is called the ℓ^p *gain* of H.

In addition, H is called *unbiased* if we can take $\beta = 0$.

Definition 3. (*small signal ℓ^p stability*, [36]) A map $H\colon\ \ell_e \to \ell_e$ is *small signal ℓ^p stable* if there exist nonnegative constants γ and c such that

$$\llbracket\ u \in \ell_e \cap \{u \mid \|u\|_{\ell^\infty} \leq c\} \Rightarrow \|H(u)|_{[0,\tau]}\|_{\ell^p} \leq \gamma\|u|_{[0,\tau]}\|_{\ell^p}\ \rrbracket \quad \forall u \in \ell_e,\ \forall \tau \in \mathbb{Z}_+. \tag{2.4}$$

Definition 4. (*local ℓ^p stability*, [2])
A map $H\colon\ \ell_e \to \ell_e$ is *local ℓ^p stable* if there exist positive constant ϵ and nonnegative constant γ such that

$$\llbracket\ \|u|_{[0,\tau]}\|_{\ell^p} \leq \epsilon \Rightarrow \|H(u)|_{[0,\tau]}\|_{\ell^p} \leq \gamma\|u|_{[0,\tau]}\|_{\ell^p}\ \rrbracket \quad \forall u \in \ell_e,\ \forall \tau \in \mathbb{Z}_+. \tag{2.5}$$

2.2 Small gain theorem

The small gain theorem provides a condition for the feedback system shown in Figure 2.1 to be ℓ^p stable. It plays a key role in the classical robust control framework. See, e.g., the standard textbook [13] for the proof of this theorem.

Theorem 1. *(Small gain theorem) Consider the feedback system shown in Figure 2.1, and assume that the following three conditions hold.*

(i) For the subsystem $H_1 : e_1 \mapsto z_1$, there exist nonnegative constants γ_1 and β_1 such that

$$\|z_1|_{[0,\tau]}\|_{\ell^p} \leq \gamma_1 \|e_1|_{[0,\tau]}\|_{\ell^p} + \beta_1 \quad \forall e_1 \in \ell_e,\ \forall \tau \in \mathbb{Z}_+. \tag{2.6}$$

(ii) For the subsystem $H_2 : e_2 \mapsto z_2$, there exist nonnegative constants γ_2 and β_2 such that

$$\|z_2|_{[0,\tau]}\|_{\ell^p} \leq \gamma_2 \|e_2|_{[0,\tau]}\|_{\ell^p} + \beta_2 \quad \forall e_2 \in \ell_e,\ \forall \tau \in \mathbb{Z}_+. \tag{2.7}$$

(iii) $\gamma_1\gamma_2 < 1$.

Then, the feedback system is ℓ^p stable. In particular,

$$\|z_1|_{[0,\tau]}\|_{\ell^p} \leq \frac{\gamma_1\|r_1|_{[0,\tau]}\|_{\ell^p} + \gamma_1\gamma_2\|r_2|_{[0,\tau]}\|_{\ell^p} + \beta_1 + \gamma_1\beta_2}{1-\gamma_1\gamma_2} \tag{2.8}$$

$$\|z_2|_{[0,\tau]}\|_{\ell^p} \leq \frac{\gamma_1\gamma_2\|r_1|_{[0,\tau]}\|_{\ell^p} + \gamma_2\|r_2|_{[0,\tau]}\|_{\ell^p} + \beta_2 + \gamma_2\beta_1}{1-\gamma_1\gamma_2} \tag{2.9}$$

hold true for all $r_1 \in \ell^p_e$, $r_2 \in \ell^p_e$, and $\tau \in [0, \infty)$.

2.3 Motivating examples

While Theorem 1 has been successfully applied to the stability analysis of many applications of nonlinear systems, it is difficult and occasionally impossible to establish the stability of quantized feedback systems based on this theorem. In this section, we examine the stabilities of two examples of quantized feedback systems and discuss the difficulties involved in applying the small gain theorem to them.

2.3.1 Robust stabilization of an uncertain networked control system over a rate-limited channel

In this subsection, we investigate the robust stabilization of a networked control system in which an uncertain plant is controlled over a rate-limited

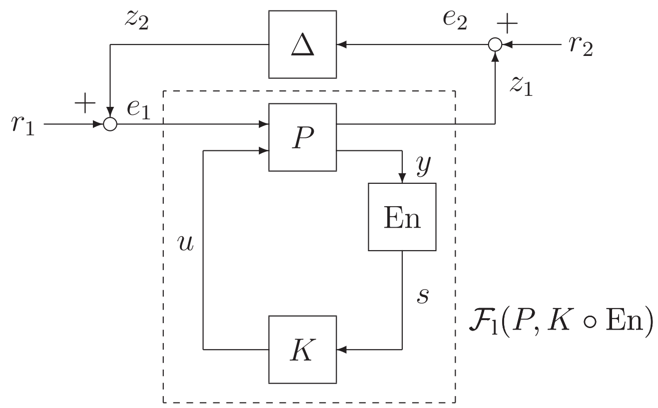

Figure 2.2: Uncertain networked control system

communication channel. As can easily be expected, the stability of such a networked control system is affected by quantization, or *a rate constraint*, at a channel and model uncertainty in the plant dynamics. We wish to quantitatively evaluate the combined effect of quantization and model uncertainty on the stability of the entire networked control system.

System description

Consider the uncertain networked control system shown in Figure 2.2. The plant is modeled as the feedback interconnection of an LTI nominal plant P and an ℓ^p stable uncertainty Δ. This uncertain plant is controlled over a rate-limited communication channel.

Each system is described in detail below.

Nominal plant $\boldsymbol{P}$: A nominal plant is defined as a causal LTI map P: $\ell_e \to \ell_e$ that maps (e_1, u) to (z_1, y):

$$\begin{bmatrix} z_1 \\ y \end{bmatrix} = P \begin{bmatrix} e_1 \\ u \end{bmatrix}, \quad P := \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}, \tag{2.10}$$

where u and y are the control input and the measured output, respectively. The nominal plant is connected with the uncertainty through (e_1, z_1).

Uncertainty $\boldsymbol{\Delta}$: The plant uncertainty Δ is modeled as a possibly nonlinear and time-varying unbiased map that has a finite ℓ^p gain less than a given level $1/\gamma > 0$. In other words, Δ belongs to the set $\mathbf{B}_\Delta^\gamma$ defined below.

$$\mathbf{B}_\Delta^\gamma := \left\{ \Delta : \ell_e \to \ell_e \mid \|\Delta(e_2)|_{[0,\tau]}\|_{\ell^p} \le \frac{1}{\gamma} \|e_2|_{[0,\tau]}\|_{\ell^p}, \quad \forall \tau \in \mathbb{Z}_+, \ \forall e_2 \in \ell_e \right\}. \tag{2.11}$$

Channel: The communication channel is modeled as a noiseless digital channel that transmits one of 2^R symbols at each time instant. That is,

$$s(t) \in \mathcal{A} := \{0, 1, \cdots 2^R - 1\} \tag{2.12}$$

where $\mathcal{A}$ is the alphabet of the channel symbols and $s(t)$ represents the channel symbol at time t. We call R *the data rate*, or simply *rate*, of this channel.
Encoder En: The encoder En is a causal map from the observed output to the channel symbol, such that

$$\mathrm{En} : (y(0), \cdots, y(t)) \mapsto s(t). \tag{2.13}$$

That is, En quantizes the measured outputs and sends the quantized symbol $s(t)$ to the communication channel. Note that because R is finite, the information about the measured output transmitted to the controller is limited. This is called *the rate constraint*.
Controller $\boldsymbol{K}$: The controller K is a causal map that generates the control input from the received channel symbols:

$$K : (s(0), \cdots, s(t)) \mapsto u(t). \tag{2.14}$$

In this example, we assume that the encoder and the controller know the exact dynamics of each other. For the encoder, this is equivalent to saying that the past inputs $u(0), u(1), \cdots, u(t-1)$ as well as $y(0), y(1), \cdots, y(t)$ are available, namely,

$$\mathrm{En} : (y(0), \cdots, y(t)) \times (u(0), \cdots, u(t-1)) \mapsto s(t). \tag{2.15}$$

In turn, this assumption is necessary for the controller to estimate the plant state from the received channel symbols.

Robust stabilization

Within the setup described above, we study the robust stabilization of the uncertain feedback system. In particular, when an uncertain plant is given, we wish to derive a condition on R for the existence of (En, K) that robustly stabilize the networked control system. A stabilizing encoder-controller pair should be given if R satisfies the derived condition.

The small gain theorem is not applicable to the robust stability analysis of the networked control system. Because the uncertainty Δ has a bounded ℓ^p gain less than or equal to $1/\gamma$, the small gain theorem tells us that the uncertain feedback system is ℓ^p stable for all $\Delta \in \mathbf{B}_\Delta^\gamma$ if the ℓ^p gain of the

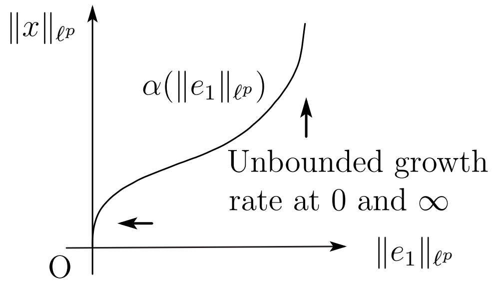

Figure 2.3: Achievable input-output relation

nominal part $\mathcal{F}_l(P, K \circ \mathrm{En})$ shown in Figure 2.2 is less than γ. However, it can be shown that under a finite rate constraint, there are no causal encoder-controller pairs such that $\mathcal{F}_l(P, K \circ \mathrm{En})$ has finite ℓ^p gain even in a simple case.

To be more specific, consider the case in which P is an unstable LTI system and $z_1(t) = y(t) = x(t)$, the plant state. Suppose that there exists an encoder-controller pair such that the closed-loop system $\mathcal{F}_l(P, K \circ \mathrm{En})$ is unbiased ℓ^p stable (see Definition 1 in Section 2.1 for the unbiased ℓ^p stability), namely, there exists a class $\mathcal{K}$ function α satisfying

$$\|x\|_{\ell^p} \leq \alpha(\|e_1\|_{\ell^p}). \tag{2.16}$$

It is shown by Martins [16] that for any encoder-controller pair and any α satisfying the above, there holds

$$\sup_{\|e_1\|_{\ell^p} \in (0, 2^{\delta_1})} \frac{\alpha(\|e_1\|_{\ell^p})}{\|e_1\|_{\ell^p}} = \infty \qquad \forall \delta_1 > 0, \tag{2.17}$$

$$\sup_{\|e_1\|_{\ell^p} > 2^{-\delta_2}} \frac{\alpha(\|e_1\|_{\ell^p})}{\|e_1\|_{\ell^p}} = \infty \qquad \forall \delta_2 > 0. \tag{2.18}$$

Inequalities (2.17) and (2.18) imply that α has an unbounded growth rate at $\|e_1\|_{\ell^p} = 0$ and $\|e_1\|_{\ell^p} = \infty$ (Figure 2.3). Thus, we cannot make $\mathcal{F}_l(P, K \circ \mathrm{En})$ finite gain ℓ^p stable, and we cannot use the traditional small gain theorem to establish the robust stability of the feedback system shown in Figure 2.2.

This example implies that in such an uncertain networked control system involving finite-level quantization, the traditional ℓ^p stability is too restrictive and the small gain theorem is not successfully applicable. We thus need a new reasonable stability notion and a framework for stability analysis based on it for quantized feedback systems.

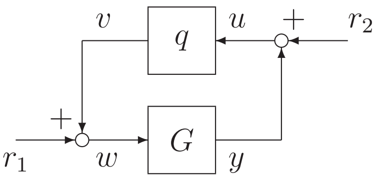

Figure 2.4: System with a uniform quantizer

2.3.2 Stability analysis of systems involving a uniform quantizer

We consider a feedback system including a uniform quantizer in this subsection. This quantized feedback system is also an important example because devices such as on/off switching actuators and memoryless multilevel-valued sensors are widely used in mechanical systems today. While such quantizers are practically important, their use gives rise to difficulties in deriving a non-conservative condition for the stability of the quantized feedback system.

System description

Consider the quantized feedback system shown in Figure 2.4. The system G is a single-input, single-output LTI system. The function $q\colon \mathbb{R} \to \mathbb{V} := \{0, \pm d, \pm 2d, \cdots, \pm md\}$ denotes a uniform static quantizer, where $d \in \mathbb{R}_+$ and $m \in \mathbb{N}$ are positive constants. As shown in Figure 2.5, this quantizer produces a quantized symbol by rounding its input to the nearest discrete value in $\mathbb{V}$:

$$q(u) = \begin{cases} md, & \text{if } \left(m - \frac{1}{2}\right) d \le u, \\ (m-1)d, & \text{if } \left(m - \frac{3}{2}\right) d \le u < \left(m - \frac{1}{2}\right) d, \\ \vdots & \vdots \\ 0, & \text{if } -\frac{1}{2}d \le u < \frac{1}{2}d, \\ \vdots & \vdots \\ -md, & \text{if } u < -\left(m - \frac{1}{2}\right) d. \end{cases} \tag{2.19}$$

The constant d is the step size, whereas

$$M := 2m + 1 \tag{2.20}$$

is the number of quantization levels.

Stability analysis

For the quantized feedback system described above, we wish to elucidate the effect of quantization on the stability of the entire system. In particular, we

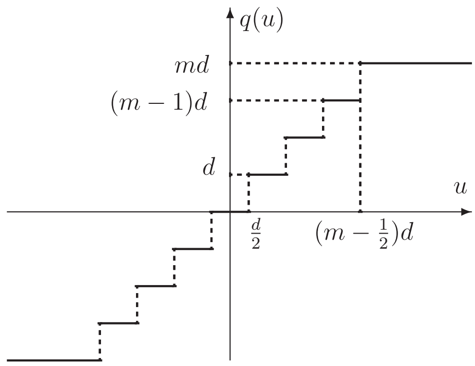

Figure 2.5: Uniform quantizer q

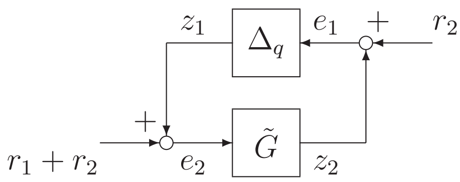

Figure 2.6: Equivalent system to Figure 2.4

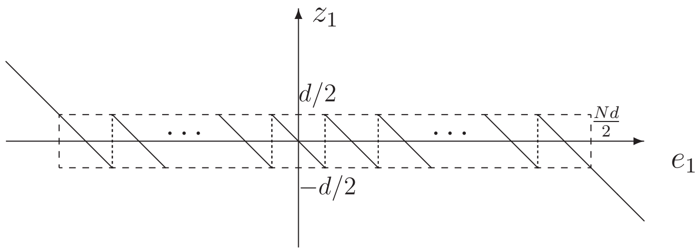

Figure 2.7: Quantization error Δ_q

want to obtain a stability condition in terms of the quantization parameters M and d.

The feedback system shown in Figure 2.4 is equivalently transformed to the system shown in Figure 2.6, where

$$e_1 = u, \quad z_1 = v - u, \quad e_2 = w - u, \quad z_2 = y. \tag{2.21}$$

The system $\Delta_q : e_1 \mapsto z_1$ represents the quantization error at the static quantizer. Its input-output relation is shown in Figure 2.7. The system $\tilde{G}$ is the corresponding linear system.

A simple application of the small gain theorem to this feedback system leads to a conservative stability condition. In fact, for any value of M and

d, the nonlinear map Δ_q satisfies

$$\|z_1|_{[0,\tau]}\|_{\ell^\infty} \leq \|e_1|_{[0,\tau]}\|_{\ell^\infty} \quad \forall e_1 \in \ell_e,\ \forall \tau \in \mathbb{Z}_+. \tag{2.22}$$

This implies that Δ_q belong to the set $\mathbf{B}^1_\Delta$, which is defined in (2.11). By using the small gain theorem, it can be shown that the feedback system is ℓ^∞ stable if the linear system $\tilde{G}$ satisfies

$$\|\tilde{G}\|_{\ell^\infty\text{-ind}} < 1. \tag{2.23}$$

However, the stability analysis does not give any information on the relationship between the stability and the step size d or the number M of quantization levels of the quantizer. This indicates that the traditional ℓ^∞ gain bounded uncertainty does not effectively approximate the quantization error and leads to a conservative stability condition.

This observation motivates us to introduce a new class of uncertainty that is suitable for approximating the quantization error.

2.4 Summary

In this chapter, we have provided a motivation for introducing a new notion of stability and a new framework for the stability analysis of quantized feedback systems. Specifically, we have discussed the difficulties involved in applying the small gain theorem to two examples of quantized feedback systems. In the first example, it was shown that the small gain theorem is not applicable to the robust stability analysis of a networked control system. In the second example, a simple application of the small gain theorem led to a conservative condition for the stability of a feedback system with a uniform quantizer. On account of these examples, we were motivated to introduce a reasonable stability notion for quantized feedback systems and a framework for stability analysis based on this notion.

Chapter 3

A New Framework for Stability Analysis of Quantized Feedback Systems

A new framework for the stability analysis of quantized feedback systems is developed in this chapter. In the previous chapter, we have shown the difficulties involved in successfully applying the small gain theorem to the stability analysis of quantized feedback systems. Motivated by our observation, we develop a new framework that is applicable to the stability analysis of quantized feedback systems. In particular, we introduce a new notion of *small ℓ^p signal ℓ^p stability*. This is a reasonable and practical notion of stability for quantized feedback systems. We then derive a sufficient condition for the stability of a feedback system in the sense of the new stability notion. A new class of uncertainty, *level bounded uncertainty*, is also introduced. This set of systems is useful for approximating some classes of nonlinearities that include quantization errors.

This chapter is organized as follows. Section 3.1 introduces a new notion of *small ℓ^p signal ℓ^p stability* and discusses the relationship with some other stability notions. Section 3.2 presents the derivation of *the small level theorem* and provides a condition for a feedback system to be small ℓ^p signal ℓ^p stable. Section 3.3 discusses robust stability analysis against two classes of uncertainties. Section 3.4 demonstrates the usefulness of a new class of uncertainty, *level bounded uncertainty*, with an example. Section 3.5 summarizes this chapter. All the proofs of the theorems used in this chapter are presented in Appendices.

3.1 Small ℓ^p signal ℓ^p stability

The new notion of *small ℓ^p signal ℓ^p stability* is defined as follows.

Definition 5. *(small ℓ^p signal ℓ^p stability)* A map $H\colon \ell_e \to \ell_e$ is said to be *small ℓ^p signal ℓ^p stable* with attenuation level γ and input bound ϵ if

$$\llbracket\ \|u|_{[0,\tau]}\|_{\ell^p} \le \epsilon \Rightarrow \|H(u)|_{[0,\tau]}\|_{\ell^p} \le \gamma\epsilon\ \rrbracket \quad \forall u \in \ell_e,\ \forall \tau \in \mathbb{Z}_+ \tag{3.1}$$

holds for given positive constants ϵ and γ. The map H is simply said to be *small ℓ^p signal ℓ^p stable* if there exist some positive constants ϵ and γ satisfying (3.1).

The feedback system shown in Figure 2.1 is called small ℓ^p signal ℓ^p stable if there exist positive constants ϵ and γ such that

$$\llbracket\ \|r|_{[0,\tau]}\|_{\ell^p} \le \epsilon \Rightarrow \|z|_{[0,\tau]}\|_{\ell^p} \le \gamma\epsilon\ \rrbracket \quad \forall r \in \ell_e,\ \forall \tau \in \mathbb{Z}_+. \tag{3.2}$$

Small ℓ^p signal ℓ^p stability is a weaker stability notion than the traditional ℓ^p stability (see Definition 1 in Section 2.1 for ℓ^p stability). In fact, small ℓ^p signal ℓ^p stability implies ℓ^p boundedness of the output only for the inputs with small ℓ^p norm whereas ℓ^p stability guarantees the ℓ^p boundedness of the output for all ℓ^p inputs. Moreover, small ℓ^p signal ℓ^p stability differs from finite gain ℓ^p stability [13], small signal ℓ^p stability [36], and local ℓ^p stability [2] (see Definitions 2, 3, and 4 for the definitions of these notions) in that it does not use the notion of ℓ^p gains. Instead, the attenuation level of a map is expressed in terms of the ratio of the local upper bounds of the ℓ^p norms of input-output signals.

While the small ℓ^p signal ℓ^p stability is generally a weaker than ℓ^p stability, as described above, in the special case of linear maps, these stability notions are equivalent, as shown in the following theorem.

Theorem 2. *Suppose that $H\colon \ell_e \to \ell_e$ is a linear map. Then, H is small ℓ^p signal ℓ^p stable if and only if it is ℓ^p stable.*

Proof. See Appendix 3.A. □

3.2 Small level theorem

We derive a sufficient condition for the small ℓ^p signal ℓ^p stability of the feedback system shown in Figure 2.1.

Theorem 3. *(Small level theorem) For the feedback system shown in Figure 2.1, assume that the following four conditions hold true.*

(i) For the subsystem $H_1 : e_1 \mapsto z_1$*, there exist positive constants* ϵ_1 *and* γ_1 *such that*

$$[\![\ \|e_1|_{[0,\tau-1]}\|_{\ell^p} \leq \epsilon_1 \Rightarrow \|z_1|_{[0,\tau]}\|_{\ell^p} \leq \gamma_1\epsilon_1 \ ]\!] \quad \forall e_1 \in \ell_e, \ \forall \tau \in \mathbb{Z}_+. \tag{3.3}$$

(ii) For the subsystem $H_2 : e_2 \mapsto z_2$*, there exist positive constants* ϵ_2 *and* γ_2 *such that*

$$[\![\ \|e_2|_{[0,\tau]}\|_{\ell^p} \leq \epsilon_2 \Rightarrow \|z_2|_{[0,\tau]}\|_{\ell^p} \leq \gamma_2\epsilon_2 \ ]\!] \quad \forall e_2 \in \ell_e, \ \forall \tau \in \mathbb{Z}_+. \tag{3.4}$$

(iii) $\epsilon_1 > \gamma_2\epsilon_2$

(iv) $\epsilon_2 > \gamma_1\epsilon_1$

Then, the feedback system is small ℓ^p *signal* ℓ^p *stable. In particular,*

$$[\![\ \|r|_{[0,\tau]}\|_{\ell^p} \leq \epsilon \Rightarrow \left(\|z_1|_{[0,\tau]}\|_{\ell^p} \leq \delta_1 \text{ and } \|z_2|_{[0,\tau]}\|_{\ell^p} \leq \delta_2\right) \ ]\!] \quad \forall r \in \ell_e, \ \forall \tau \in \mathbb{Z}_+ \tag{3.5}$$

holds for

$$\epsilon := \min\{\epsilon_2 - \gamma_1\epsilon_1, \epsilon_1 - \gamma_2\epsilon_2\}, \quad \delta_1 := \gamma_1\epsilon_1, \quad \delta_2 := \gamma_2\epsilon_2. \tag{3.6}$$

Proof. See Appendix 3.B. □

This theorem is not simply a local version of the small gain theorem. In fact, we do not assume that each of the subsystems has a finite ℓ^p gain. Instead, we assume small ℓ^p signal ℓ^p stability for each subsystem and derive a condition for the small ℓ^p signal ℓ^p stability of the feedback system. Note that conditions (iii) and (iv) in the above theorem imply $\gamma_1\gamma_2 < 1$, which is analogous to the small gain condition in the traditional small gain theorem [13]. In this thesis, we call this theorem *the small level theorem*.

It should also be noted that the bounds on ℓ^p norms of signals are characterized in (3.5) and (3.6) in terms of the attenuation levels and input bounds of both subsystems. Because small ℓ^p signal ℓ^p stability is a weak stability notion if there is no specification on its attenuation level or input bound, it should be occasionally important to achieve the small ℓ^p signal ℓ^p stability of an entire control system for some fixed values of ϵ and γ. Theorem 3 provides a sufficient condition on the attenuation levels and input bounds of the subsystems for achieving a specified small ℓ^p signal ℓ^p stability in the feedback system.

Two special cases of the above theorem can be easily verified.

First, in the case in which each of the subsystems H_1 or H_2 is unbiased linear or finite gain ℓ^p stable with a gain less than or equal to γ_1 or γ_2, respectively, Theorem 3 can be reduced to a special case of Theorem 1, the traditional small gain theorem with zero biases.

Second, in the case in which one of the subsystems, H_2, is linear or finite gain ℓ^p stable with gain γ_2, we have the following theorem.

Theorem 4. *For the feedback system shown in Figure 2.1, assume that the following three conditions hold true.*

(i) For the subsystem $H_1 : e_1 \mapsto z_1$, there exist positive constants ϵ_1 and γ_1 such that

$$[\![\; \|e_1|_{[0,\tau-1]}\|_{\ell^p} \le \epsilon_1 \Rightarrow \|z_1|_{[0,\tau]}\|_{\ell^p} \le \gamma_1\epsilon_1 \;]\!] \quad \forall e_1 \in \ell_e, \; \forall \tau \in \mathbb{Z}_+. \tag{3.7}$$

(ii) The subsystem $H_2 : e_2 \mapsto z_2$ is finite gain ℓ^p stable with gain γ_2 and bias β_2, namely,

$$\|z_2|_{[0,\tau]}\|_{\ell^p} \le \gamma_2\|e_2|_{[0,\tau]}\|_{\ell^p} + \beta_2 \quad \forall e_2 \in \ell_e, \; \forall \tau \in \mathbb{Z}_+. \tag{3.8}$$

(iii) The positive constants ϵ_1, γ_1, γ_2, and β_2 satisfy

$$(1 - \gamma_1\gamma_2)\epsilon_1 > \beta_2. \tag{3.9}$$

Then, the feedback system is small ℓ^p signal ℓ^p stable. In particular,

$$[\![\; \|r|_{[0,\tau]}\|_{\ell^p} \le \epsilon \Rightarrow \big(\|z_1|_{[0,\tau]}\|_{\ell^p} \le \delta_1 \text{ and } \|z_2|_{[0,\tau]}\|_{\ell^p} \le \delta_2\big) \;]\!] \\ \forall r \in \ell_e, \; \forall \tau \in \mathbb{Z}_+ \tag{3.10}$$

holds for

$$\epsilon = \frac{(1 - \gamma_1\gamma_2)\epsilon_1 - \beta_2}{1 + \gamma_2}, \quad \delta_1 = \gamma_1\epsilon_1, \quad \delta_2 = \frac{(1 + \gamma_1)\gamma_2\epsilon_1 + \beta_2}{1 + \gamma_2}. \tag{3.11}$$

Proof. See Appendix 3.C. □

When we can take the bias $\beta_2 = 0$ in the above theorem, the condition (3.9) is replaced by an inequality $\gamma_1\gamma_2 < 1$.

Example 1. *Recall the feedback system with a uniform quantizer described in Subsection 2.3.2. Assume, to be more specific, a state space representation of linear system $\tilde{G}$ in Figure 2.6 is given by*

$$\tilde{G} : \left\{ \begin{array}{rcl} x(t+1) & = & Ax(t) + Be_2(t), \quad x(0) = x_0, \\ z_2(t) & = & Cx(t), \end{array} \right. \tag{3.12}$$

where $A \in \mathbb{R}^{n\times n}$ is Schur stable, and $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ are matrices.

Let $g(t)$ denote the impulse response of the LTI system, namely,

$$g(t) = \begin{cases} 0 & \text{if } t = 0, \\ CA^{t-1}B & \text{if } t \geq 1. \end{cases} \tag{3.13}$$

It is then well-known [6],[40] that

$$\begin{aligned} \gamma_2 &:= \max_{i=1,\cdots,q} \sum_{\tau=0}^{+\infty} \sum_{j=1}^{m} |g_{i,j}(\tau)| = \max_{i=1,\cdots,q} \sum_{j=1}^{m} \sum_{\tau=0}^{+\infty} |g_{i,j}(\tau)| < +\infty, \\ \kappa_2 &:= \max_{0\leq t<\infty} \|CA^t\|_1 < +\infty. \end{aligned} \tag{3.14}$$

and the input-output relation

$$\|z_2|_{[0,\tau]}\|_{\ell^\infty} \leq \gamma_2 \|e_2|_{[0,\tau]}\|_{\ell^\infty} + \kappa_2 \|x_0\|_\infty \tag{3.15}$$

holds true for the LTI subsystem.

It can also be shown that the nonlinear system Δ_q satisfies

$$\left[\!\!\left[\|e_1|_{[0,\tau-1]}\|_{\ell^\infty} \leq \frac{Nd}{2} \Rightarrow \|z_1|_{[0,\tau]}\|_{\ell^\infty} \leq \frac{d}{2} \right]\!\!\right] \quad \forall e_1 \in \ell_e, \ \forall \tau \in \mathbb{Z}_+.$$

Then, it is implicit from Theorem 4 that if $\gamma_2 < N$, then we can define a non-empty set of initial states

$$\mathcal{X}_0 := \left\{ x \,\middle|\, \|x\|_\infty \leq (1 - \frac{\gamma_2}{N}) \frac{Nd}{2\kappa_2} \right\} \tag{3.16}$$

and the feedback system shown in Figure 2.6 is small ℓ^∞ signal ℓ^∞ stable for all initial states $x_0 \in \mathcal{X}_0$.

This example shows that conditions on admissible initial states, in addition to gains, of subsystems are required for the small level theorem. In fact, the bias term β_2 and the attenuation levels γ_1 and γ_2 in Theorems 3 and 4 are dependent on initial conditions as is shown in the above example. When we consider the attenuation level of small ℓ^∞ signal ℓ^∞ stability, instead of the gain, of H_2, the effect of its initial condition is "absorbed" in the attenuation level constant. Specifically in the above example, the subsystem H_2 defined by (3.12) satisfies

$$[\![\ \|e_2|_{[0,\tau]}\|_{\ell^\infty} \leq \epsilon_2 \Rightarrow \|z_2|_{[0,\tau]}\|_{\ell^\infty} \leq \gamma_2' \epsilon_2 \ ]\!] \quad \forall e_2 \in \ell_e, \ \forall \tau \in [0, \infty) \tag{3.17}$$

for any fixed $\epsilon_2 > 0$ where

$$\gamma_2' := \gamma_2 + \frac{\beta_2}{\epsilon_2}. \tag{3.18}$$

It can be seen that the effect of β_2 is included in the attenuation level γ_2'.

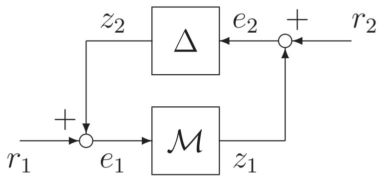

Figure 3.1: Feedback system for robust stability analysis

The following proposition can be directly shown from the definition of strict causality.

Theorem 5. *Assume that the subsystem $H_1 : e_1 \mapsto z_1$ is strictly causal. Namely,*

$$P_\tau H_1 e_1 = P_\tau H_1 P_{\tau-1} e_1 \tag{3.19}$$

holds for $\forall \tau \in \mathbb{Z}_+$ and $\forall e_1 \in \ell_e$. Then, condition (3.3), or equivalently (3.7), is satisfied if

$$[\![\, \|e_1\|_{\ell^p} \le \epsilon_1 \Rightarrow \|z_1\|_{\ell^p} \le \gamma_1 \epsilon_1 \,]\!] \quad \forall e_1 \in \ell^p \tag{3.20}$$

holds for the positive constants ϵ_1 and γ_1.

3.3 Robust stability analysis

Based on the results obtained in the previous section, we examine the robust stability of the feedback system shown in Figure 3.1, where $\mathcal{M}$ is the nominal system and Δ is the uncertainty. In this section, we consider two important classes of uncertainties: the traditional gain bounded uncertainty and level bounded uncertainty. The latter is a novel class of uncertainty introduced in this chapter. It enables us to effectively approximate some classes of nonlinearities that include quantization errors. Against each class of uncertainties, in this section, we derive a sufficient condition for the robust small ℓ^p signal ℓ^p stability of the feedback system.

The class of ℓ^p gain bounded uncertainty is defined by a set of possibly nonlinear and time-varying maps with gains less than or equal to a given level $1/\gamma > 0$:

$$\mathbf{B}_\Delta^\gamma := \{\Delta : \ell_e \to \ell_e \mid \|\Delta(e_2)|_{[0,\tau]}\|_{\ell^p} \le \frac{1}{\gamma}\|e_2|_{[0,\tau]}\|_{\ell^p} \quad \forall e_2 \in \ell_e, \ \forall \tau \in \mathbb{Z}_+\}. \tag{3.21}$$

The class of ℓ^p level bounded uncertainty is defined by a set of small ℓ^p signal ℓ^p stable maps with given input bounds $\epsilon > 0$ and attenuation levels less than or equal to $1/\gamma > 0$.

$$\mathbf{SB}_\Delta^{\epsilon,\gamma} := \{\Delta : \ell_e \to \ell_e \mid \|e_2|_{[0,\tau]}\|_{\ell^p} \leq \epsilon \Rightarrow \|\Delta(e_2)|_{[0,\tau]}\|_{\ell^p} \leq \frac{\epsilon}{\gamma} \quad \forall e_2 \in \ell_e,\ \forall \tau \in \mathbb{Z}_+\}. \tag{3.22}$$

Note that the inclusion relation $\mathbf{B}_\Delta^\gamma \subset \mathbf{SB}_\Delta^{\epsilon,\gamma}$ holds for any fixed ϵ and γ.

We wish to answer the following two questions in this section:

[RS-$\mathbf{B}_\Delta^\gamma$] *Robust stability analysis against ℓ^p gain bounded uncertainty:* Find a condition on $\mathcal{M}$ shown in Figure 3.1 such that the feedback system is small ℓ^p signal ℓ^p stable for all $\Delta \in \mathbf{B}_\Delta^\gamma$.

[RS-$\mathbf{SB}_\Delta^{\epsilon,\gamma}$] *Robust stability analysis against ℓ^p level bounded uncertainty:* Find a condition on $\mathcal{M}$ shown in Figure 3.1 such that the feedback system is small ℓ^p signal ℓ^p stable for all $\Delta \in \mathbf{SB}_\Delta^{\epsilon,\gamma}$.

For simplicity, we use the following assumption hereafter.

Assumption 1. *$\mathcal{M}$ is strictly causal.*

Under Assumption 1, we shall derive a sufficient condition on $\mathcal{M}$ for the robust small ℓ^p signal ℓ^p stability of the feedback system shown in Figure 3.1.

It is implicit from Theorems 3 and 4 that the conditions for robust stability conditions on $\mathcal{M}$ shown in Figure 3.1 against $\mathbf{SB}_\Delta^{\epsilon,\gamma}$ and $\mathbf{B}_\Delta^\gamma$ are given in the following theorems.

Theorem 6. **[RS-$\mathbf{SB}_\Delta^{\epsilon,\gamma}$]** *Suppose that for the nominal system $\mathcal{M} : e_1 \mapsto z_1$, there exist positive constants ϵ_1 and $\gamma_1 < \gamma$ such that*

(i)
$$[\![\ \|e_1\|_{\ell^p} \leq \epsilon_1 \Rightarrow \|z_1\|_{\ell^p} \leq \gamma_1 \epsilon_1\ ]\!] \quad \forall e_1 \in \ell^p, \tag{3.23}$$

(ii)
$$\frac{\epsilon}{\gamma} < \epsilon_1 < \frac{\epsilon}{\gamma_1}. \tag{3.24}$$

Then, under Assumption 1, the feedback system shown in Figure 3.1 is small ℓ^p signal ℓ^p stable for all $\Delta \in \mathbf{SB}_\Delta^{\epsilon,\gamma}$. In particular, for all $\Delta \in \mathbf{SB}_\Delta^{\epsilon,\gamma}$

$$[\![\ \|r|_{[0,\tau]}\|_{\ell^p} \leq \epsilon' \Rightarrow \left(\|z_1|_{[0,\tau]}\|_{\ell^p} \leq \delta_1 \text{ and } \|z_2|_{[0,\tau]}\|_{\ell^p} \leq \delta_2\right)\ ]\!] \quad \forall r \in \ell_e,\ \forall \tau \in \mathbb{Z}_+ \tag{3.25}$$

holds for

$$\epsilon' := \min\left\{\epsilon - \gamma_1\epsilon_1, \epsilon_1 - \frac{\epsilon}{\gamma}\right\},\ \delta_1 := \gamma_1\epsilon_1,\ \delta_2 := \frac{\epsilon}{\gamma}. \tag{3.26}$$

Proof. See Appendix 3.D. □

Theorem 7. [RS-$\mathbf{B}_{\boldsymbol{\Delta}}^{\gamma}$] *Suppose that $\mathcal{M}$ is small ℓ^p signal ℓ^p stable with attenuation level less than γ, namely, there exist positive constants ϵ_1 and $\gamma_1 < \gamma$ such that*

$$\llbracket\ \|e_1\|_{\ell^p} \le \epsilon_1 \Rightarrow \|z_1\|_{\ell^p} \le \gamma_1\epsilon_1\ \rrbracket \quad \forall e_1 \in \ell^p. \tag{3.27}$$

Then, under Assumption 1, the feedback system shown in Figure 3.1 is small ℓ^p signal ℓ^p stable for all $\Delta \in \mathbf{B}_\Delta^\gamma$. In particular, for all $\Delta \in \mathbf{B}_\Delta^\gamma$,

$$\llbracket\ \|r|_{[0,\tau]}\|_{\ell^p} \le \epsilon' \Rightarrow \big(\|z_1|_{[0,\tau]}\|_{\ell^p} \le \delta_1 \text{ and } \|z_2|_{[0,\tau]}\|_{\ell^p} \le \delta_2\big)\ \rrbracket \\ \forall r \in \ell_e,\ \forall \tau \in \mathbb{Z}_+ \tag{3.28}$$

holds for

$$\epsilon' = \frac{\gamma - \gamma_1}{\gamma + 1}\epsilon_1, \quad \delta_1 = \gamma_1\epsilon_1, \quad \delta_2 = \frac{\gamma_1 + 1}{\gamma + 1}\epsilon_1. \tag{3.29}$$

Proof. See Appendix 3.E. □

Note also that we have condition (3.24) on the input bound of $\mathcal{M}$ for the robust stability against $\mathbf{SB}_\Delta^{\epsilon,\gamma}$, whereas there is no such condition for the robust stability against $\mathbf{B}_\Delta^\gamma$.

When $\mathcal{M}$ is an LTI system, we have the following theorems for the robust small ℓ^p signal ℓ^p stability in Figure 3.1 against each class of uncertainties. In particular, a necessary and sufficient condition for robust small ℓ^p signal ℓ^p stability is derived against $\mathbf{B}_\Delta^\gamma$.

Theorem 8. [RS-$\mathbf{SB}_{\boldsymbol{\Delta}}^{\epsilon,\gamma}$] *Suppose that $\mathcal{M} : e_1 \mapsto z_1$ is an LTI system. The feedback system shown in Figure 3.1 is small ℓ^p signal ℓ^p stable for all $\Delta \in \mathbf{SB}_\Delta^{\epsilon,\gamma}$ if $\mathcal{M}$ satisfies*

(i)

$$\llbracket\ \|z_1\|_{\ell^p} \le \gamma_1\|e_1\|_{\ell^p} + \beta_1\ \rrbracket\ \forall e_1 \in \ell^p, \tag{3.30}$$

(ii)

$$\left(1 - \frac{\gamma_1}{\gamma}\right)\epsilon > \beta_1. \tag{3.31}$$

In particular,

$$\llbracket\ \|r|_{[0,\tau]}\|_{\ell^p} \le \epsilon \Rightarrow \big(\|z_1|_{[0,\tau]}\|_{\ell^p} \le \delta_1 \text{ and } \|z_2|_{[0,\tau]}\|_{\ell^p} \le \delta_2\big)\ \rrbracket \\ \forall r \in \ell_e,\ \forall \tau \in \mathbb{Z}_+ \tag{3.32}$$

holds for

$$\epsilon = \frac{\left(1 - \frac{\gamma_1}{\gamma}\right)\epsilon - \beta_2}{1+\gamma_1}, \quad \delta_1 = \frac{\epsilon}{\gamma}, \quad \delta_2 = \frac{\left(1 + \frac{1}{\gamma}\right)\gamma_1\epsilon + \beta_1}{1+\gamma_1}. \tag{3.33}$$

Proof. Theorem 8 is immediate from Theorem 4. □

Theorem 9. [RS-$\mathbf{B}_{\Delta}^{\gamma}$] *Suppose $\mathcal{M} : e_1 \mapsto z_1$ is an LTI system. The feedback system shown in Figure 3.1 is small ℓ^p signal ℓ^p stable for all $\Delta \in \mathbf{B}_{\Delta}^{\gamma}$ if and only if*

$$\|\mathcal{M}\|_{\ell^p\text{-ind}} < \gamma. \tag{3.34}$$

In particular, if $\mathcal{M}$ satisfies

$$[\![\; \|z_1\|_{\ell^p} \le \gamma_1 \|e_1\|_{\ell^p} + \beta_1 \;]\!] \;\; \forall e_1 \in \ell^p, \tag{3.35}$$

then

$$\|z_1|_{[0,\tau]}\|_{\ell^p} \le \frac{\gamma\gamma_1 \|r_1|_{[0,\tau]}\|_{\ell^p} + \gamma_1 \|r_2|_{[0,\tau]}\|_{\ell^p} + \gamma\beta_1}{\gamma - \gamma_1} \tag{3.36}$$

$$\|z_2|_{[0,\tau]}\|_{\ell^p} \le \frac{\gamma_1 \|r_1|_{[0,\tau]}\|_{\ell^p} + \|r_2|_{[0,\tau]}\|_{\ell^p} + \beta_1}{\gamma - \gamma_1} \tag{3.37}$$

hold true for all $r_1 \in \ell_e^p$, $r_2 \in \ell_e^p$, and $\tau \in [0, \infty)$.

Proof. See Appendix 3.F. □

3.4 Contribution of level bounded uncertainty

The new class of uncertainty, level bounded uncertainty, is useful in approximating some classes of nonlinearity that include quantization errors. In this section, we demonstrate the advantage of the use of this new class of uncertainty by recalling the example described in Subsection 2.3.2.

Let us again consider the stability analysis of the feedback system with a uniform quantizer, which has been described in Subsection 2.3.2. In this example, the quantization error Δ_q (see Figure 2.7 for the input-output relation of Δ_q) has not been effectively approximated by an ℓ^∞ gain bounded uncertainty, and therefore, a conservative stability condition has been derived. Specifically, the ℓ^∞ gain of Δ_q is neither dependent on the quantization parameter d or M. The effect of quantization on the stability of the feedback system has therefore not been assessed.

The level bounded uncertainty, in contrast, effectively approximates the quantization error. It is easily shown that for the system $\Delta_q : e_1 \mapsto z_1$, there holds

$$\left[\!\left[\|e_1|_{[0,\tau]}\|_{\ell^\infty} \le \frac{Md}{2} \Rightarrow \|z_1|_{[0,\tau]}\|_{\ell^\infty} \le \frac{d}{2} \right]\!\right] \quad \forall e_1 \in \ell_e,\ \forall \tau \in \mathbb{Z}_+.$$

That is, Δ_q is small ℓ^∞ signal ℓ^∞ stable with attenuation level

$$\gamma_1 := \frac{\frac{d}{2}}{\frac{Md}{2}} = \frac{1}{M} \tag{3.38}$$

and input bound

$$\epsilon_1 := \frac{1}{2}Md. \tag{3.39}$$

In other words, Δ_q belongs to the set $\mathbf{SB}_{\Delta}^{\epsilon_1,\gamma_1}$. It should be emphasized that $\mathbf{SB}_{\Delta}^{\epsilon_1,\gamma_1}$ is parametrized by the step size and the number of levels of quantization.

By using this result, some relationships between the quantization parameters and the small ℓ^∞ signal ℓ^∞ stability of the feedback system can be obtained. It is implicit from Theorem 8 that the feedback system shown in Figure 2.4 is small ℓ^∞ signal ℓ^∞ stable if $\tilde{G} : e_2 \mapsto z_2$ satisfies

(i)
$$[\![\ \|z_2\|_{\ell^p} \le \gamma_2 \|e_2\|_{\ell^p} + \beta_2\ ]\!]\ \ \forall u \in \ell^p, \tag{3.40}$$

(ii)
$$\frac{1}{2}\left(1 - M\gamma_1\right) Md > \beta_2. \tag{3.41}$$

In particular, when β_2 can be taken as zero, that is, when $\tilde{G}$ is unbiased finite gain stable, the stability condition is given by

$$\gamma_2 := \|\tilde{G}\|_{\ell^\infty\text{-ind}} < M.$$

From the above stability condition, it is clear that the number of quantization levels and the step size affect the small ℓ^∞ signal ℓ^∞ stability. Moreover, the step size and the number of quantization levels affect the bound on the ℓ^∞ norm of the signals of the feedback system. In fact,

$$\left[\!\left[\|r|_{[0,\tau]}\|_{\ell^\infty} \le \epsilon \Rightarrow \begin{array}{c} \|(v-y)|_{[0,\tau]}\|_{\ell^\infty} = \|z_1|_{[0,\tau]}\|_{\ell^\infty} \le \delta_1, \\ \text{and} \\ \|y|_{[0,\tau]}\|_{\ell^\infty} = \|z_2|_{[0,\tau]}\|_{\ell^\infty} \le \delta_2 \end{array} \right]\!\right]$$
$$\forall r \in \ell_e,\ \forall \tau \in \mathbb{Z}_+$$

holds for

$$\epsilon = \frac{1}{4}Md(1 - M\gamma_2) - \frac{\beta_2}{2}, \quad \delta_1 = \frac{d}{2}, \quad \delta_2 = \frac{(M+1)d\gamma_2 + 2\beta_2}{2(1+\gamma_2)}$$

for this quantized feedback system. The bound on the ℓ^∞ norm of the quantization error depends on the step size, and the bound on the ℓ^∞ norm of the admissible disturbance input and the bound on $\|y|_{[0,\tau]}\|_{\ell^\infty}$ depend on the step size and the number of quantization levels of the quantizer.

This example shows the advantage of employing the new class of uncertainty in approximating the nonlinearity caused by quantization for the stability analysis of the quantized feedback system.

3.5 Summary

In this chapter, we have developed a new framework for the stability analysis of quantized feedback systems based on the new notion of small ℓ^p signal ℓ^p stability. The small level theorem has been derived as a key theorem for stability analysis. In the section dealing with robust stability analysis, sufficient conditions for robust small ℓ^p signal ℓ^p stability have been derived against two classes of uncertainties: gain bounded uncertainty and level bounded uncertainty. We have also shown the usefulness of the class of level bounded uncertainty for a feedback system including a uniform quantizer.

Appendix 3.A Proof of Theorem 2

We first discuss the sufficiency. Assume that $H : u \mapsto z$ is ℓ^p stable, namely, there exist a class $\mathcal{K}$ function α and a nonnegative constant β such that

$$\|z|_{[0,\tau]}\|_{\ell^p} \leq \alpha(\|u|_{[0,\tau]}\|_{\ell^p}) + \beta \quad \forall u \in \ell_e, \ \forall \tau \in \mathbb{Z}_+. \tag{3.42}$$

Then, if we take $\gamma := (\alpha(\epsilon) + \beta)/\epsilon$ for an arbitrarily fixed positive constant ϵ, (3.1) holds from the monotonic increase in α.

With regard to the necessity, assume that H is small ℓ^p signal ℓ^p stable. From the definition of small ℓ^p signal ℓ^p stability, we obtain

$$[\![\ \|u|_{[0,\tau]}\|_{\ell^p} = \epsilon \Rightarrow \|z|_{[0,\tau]}\|_{\ell^p} \leq \gamma\epsilon = \gamma\|u|_{[0,\tau]}\|_{\ell^p} \ ]\!] \quad \forall \tau \in \mathbb{Z}_+.$$

From the above inequality and the linearity of H, there exist some nonnegative constants $0 < \gamma' < \gamma$ and β such that

$$\|z|_{[0,\tau]}\|_{\ell^p} \leq \gamma'\|u|_{[0,\tau]}\|_{\ell^p} + \beta \quad \forall u \in \ell_e, \ \forall \tau \in \mathbb{Z}_+,$$

which implies, in turn, that there holds (3.42) for a class $\mathcal{K}$ function

$$\alpha(x) := \gamma' x$$

and a nonnegative constant β. Therefore, a map H is finite gain ℓ^p stable and ℓ^p stable.

Appendix 3.B Proof of Theorem 3

Define

$$T_\epsilon := \left\{ \tau \geq 1 \,\middle|\, \exists r \in \ell_e, \|r|_{[0,\tau]}\|_{\ell^p} \leq \epsilon \text{ s.t. } \begin{array}{c} \|e_1|_{[0,\tau]}\|_{\ell^p} > \epsilon_1 \\ \text{or} \\ \|e_2|_{[0,\tau]}\|_{\ell^p} > \epsilon_2 \end{array} \right\} \tag{3.43}$$

for the positive constant ϵ defined in (3.6). Because $T_\epsilon = \emptyset$ implies

$$\begin{aligned} &[\![\, \|r|_{[0,\tau]}\|_{\ell^p} \leq \epsilon \Rightarrow \big(\|e_1|_{[0,\tau]}\|_{\ell^p} \leq \epsilon_1 \text{ and } \|e_2|_{[0,\tau]}\|_{\ell^p} \leq \epsilon_2\big) \,]\!] \\ &\qquad \forall r \in \ell_e, \ \forall \tau \in \mathbb{Z}_+, \end{aligned} \tag{3.44}$$

we first prove $T_\epsilon = \emptyset$ for the boundedness of e_1 and e_2. Assume on the contrary that T_ϵ is non-empty, and we define $k := \min T_\epsilon$. This implies that $\|r|_{[0,k-1]}\|_{\ell^p} \leq \epsilon$, $\|e_1|_{[0,k-1]}\|_{\ell^p} \leq \epsilon_1$ and $\|e_2|_{[0,k-1]}\|_{\ell^p} \leq \epsilon_2$. It is implicit from assumptions (i) and (ii) that $\|z_1|_{[0,k]}\|_{\ell^p} \leq \gamma_1\epsilon_1$ and $\|z_2|_{[0,k-1]}\|_{\ell^p} \leq \gamma_2\epsilon_2$, and thus,

$$\begin{aligned} \|e_2|_{[0,k]}\|_{\ell^p} &\leq \|z_1|_{[0,k]}\|_{\ell^p} + \|r_2|_{[0,k]}\|_{\ell^p} \leq \gamma_1\epsilon_1 + \|r_2|_{[0,k]}\|_{\ell^p} \\ &\leq \gamma_1\epsilon_1 + \|r|_{[0,k]}\|_{\ell^p}. \end{aligned} \tag{3.45}$$

Because $\epsilon \leq \epsilon_2 - \gamma_1\epsilon_1$, $\|e_2|_{[0,k]}\|_{\ell^p} \leq \epsilon_2$ holds for every $r \in \ell_e$ satisfying $\|r|_{[0,k]}\|_{\ell^p} \leq \epsilon$. It follows, in turn, from assumption (ii) that

$$\|e_1|_{[0,k]}\|_{\ell^p} \leq \|r_1|_{[0,k]}\|_{\ell^p} + \|z_2|_{[0,k]}\|_{\ell^p} \leq \|r|_{[0,k]}\|_{\ell^p} + \gamma_2\epsilon_2. \tag{3.46}$$

This implies that $\|e_1|_{[0,k]}\|_{\ell^p} \leq \epsilon_1$ and $\|e_2|_{[0,k]}\|_{\ell^p} \leq \epsilon_2$ for every $r \in \ell_e$ satisfying $\|r|_{[0,k]}\|_{\ell^p} \leq \epsilon$, which contradicts the definition of k. Thus, T_ϵ must be empty, and hence, (3.44) is satisfied. It then follows from (3.3), (3.4), and (3.44) that

$$\begin{aligned} &[\![\, \|r|_{[0,\tau]}\|_{\ell^p} \leq \epsilon \Rightarrow \|z_1|_{[0,\tau]}\|_{\ell^p} \leq \|z_1|_{[0,\tau+1]}\|_{\ell^p} \leq \gamma_1\epsilon_1 \,]\!] \\ &\qquad \forall r \in \ell_e, \ \forall \tau \in \mathbb{Z}_+ \end{aligned} \tag{3.47}$$

and

$$[\![\ \|r|_{[0,\tau]}\|_{\ell^p} \le \epsilon \Rightarrow \|z_2|_{[0,\tau]}\|_{\ell^p} \le \gamma_2\epsilon_2\ ]\!] \quad \forall r \in \ell_e,\ \forall \tau \in \mathbb{Z}_+. \tag{3.48}$$

By combining (3.47) and (3.48), we obtain

$$\|z|_{[0,\tau]}\|_{\ell^p} \le \|z_1|_{[0,\tau]}\|_{\ell^p} + \|z_2|_{[0,\tau]}\|_{\ell^p} \le \gamma_1\epsilon_1 + \gamma_2\epsilon_2 \tag{3.49}$$

This proves the small ℓ^p signal ℓ^p stability of the feedback system, namely,

$$[\![\ \|r|_{[0,\tau]}\|_{\ell^p} \le \epsilon \Rightarrow \|z|_{[0,\tau]}\|_{\ell^p} \le \delta_z\ ]\!] \quad \forall r \in \ell_e,\ \forall \tau \in \mathbb{Z}_+,$$

where

$$\delta_z := \gamma_1\epsilon_1 + \gamma_2\epsilon_2. \tag{3.50}$$

Furthermore, (3.47) and (3.48) imply the relation (3.5).

Appendix 3.C Proof of Theorem 4

Define

$$\begin{aligned} \epsilon &:= \frac{(1-\gamma_1\gamma_2)\epsilon_1 - \beta_2}{\gamma_2 + 1}, \\ T_\epsilon &:= \left\{\tau \ge 1 | \exists r \text{ s.t. } \|r|_{[0,\tau]}\|_{\ell^p} \le \epsilon \text{ and } \|e_1|_{[0,\tau]}\|_{\ell^p} > \epsilon_1\right\}. \end{aligned} \tag{3.51}$$

Because $T_\epsilon = \emptyset$ implies

$$[\![\ \|r|_{[0,\tau]}\|_{\ell^p} \le \epsilon \Rightarrow \|e_1|_{[0,\tau]}\|_{\ell^p} \le \epsilon_1\ ]\!] \quad \forall \tau \in \mathbb{Z}_+, \tag{3.52}$$

we first prove $T_\epsilon = \emptyset$ for the boundedness of e_1 and z_1. Assume on the contrary that T_ϵ is non-empty, and we define $k := \min T_\epsilon$. This implies that $\|r|_{[0,k-1]}\|_{\ell^p} \le \epsilon$ and $\|e_1|_{[0,k-1]}\|_{\ell^p} \le \epsilon_1$. It is implicit from assumption (i) that $\|z_1|_{[0,k]}\|_{\ell^p} \le \gamma_1\epsilon_1$. Then, it follows from assumption (ii) that

$$\begin{aligned} \|e_1|_{[0,k]}\|_{\ell^p} &\le \|r_1|_{[0,k]}\|_{\ell^p} + \|z_2|_{[0,k]}\|_{\ell^p} \\ &\le \|r_1|_{[0,k]}\|_{\ell^p} + \gamma_2\|e_2|_{[0,k]}\|_{\ell^p} + \beta_2 \\ &\le \|r_1|_{[0,k]}\|_{\ell^p} + \gamma_2(\|r_2|_{[0,k]}\|_{\ell^p} + \|z_1|_{[0,k]}\|_{\ell^p}) + \beta_2 \\ &\le \|r_1|_{[0,k]}\|_{\ell^p} + \gamma_2\|r_2|_{[0,k]}\|_{\ell^p} + \gamma_1\gamma_2\epsilon_1 + \beta_2 \\ &\le (\gamma_2 + 1)\|r|_{[0,k]}\|_{\ell^p} + \gamma_1\gamma_2\epsilon_1 + \beta_2. \end{aligned} \tag{3.53}$$

This implies that $\|e_1|_{[0,k]}\|_{\ell^p} \leq \epsilon_1$ holds for every $r \in \ell_e$ satisfying $\|r|_{[0,k]}\|_{\ell^p} \leq \epsilon$, which contradicts the definition of k. Thus, T_ϵ must be empty, and hence, (3.52) is satisfied. It then follows from (3.53) and (3.52) that

$$[\![\ \|r|_{[0,\tau]}\|_{\ell^p} \leq \epsilon \Rightarrow \|z_1|_{[0,\tau]}\|_{\ell^p} \leq \|z_1|_{[0,\tau+1]}\|_{\ell^p} \leq \gamma_1\epsilon_1 \ ]\!] \ \ \forall \tau \in \mathbb{Z}_+. \tag{3.54}$$

We next evaluate the upper bound on $\|z_2|_{[0,\tau]}\|_{\ell^p}$. We see from (3.8), (3.51), and (3.54) that if $\|r|_{[0,\tau]}\|_{\ell^p} \leq \epsilon$, then there holds

$$\begin{aligned}\|z_2|_{[0,\tau]}\|_{\ell^p} &\leq \gamma_2\|e_2|_{[0,\tau]}\|_{\ell^p} + \beta_2\\ &\leq \gamma_2(\|r_2|_{[0,\tau]}\|_{\ell^p} + \gamma_1\epsilon_1) + \beta_2\\ &\leq \gamma_2\frac{(1-\gamma_1\gamma_2)\epsilon_1 - \beta_2}{\gamma_2+1} + \gamma_1\gamma_2\epsilon_1 + \beta_2\\ &= \frac{(\gamma_1+1)\gamma_2\epsilon_1 + \beta_2}{\gamma_2+1}.\end{aligned} \tag{3.55}$$

By combining (3.54) and (3.55), we obtain

$$\begin{aligned}\|z|_{[0,\tau]}\|_{\ell^p} &\leq \|z_1|_{[0,\tau]}\|_{\ell^p} + \|z_2|_{[0,\tau]}\|_{\ell^p}\\ &\leq \frac{(\gamma_1+\gamma_2+2\gamma_1\gamma_2)\epsilon_1 + \beta_2}{\gamma_2+1}.\end{aligned} \tag{3.56}$$

This proves the small ℓ^p signal ℓ^p stability of the feedback system, namely,

$$[\![\ \|r|_{[0,\tau]}\|_{\ell^p} \leq \epsilon \Rightarrow \|z|_{[0,\tau]}\|_{\ell^p} \leq \delta_z \ ]\!] \ \ \forall r \in \ell_e, \ \forall \tau \in \mathbb{Z}_+, \tag{3.57}$$

where

$$\delta_z := \frac{(\gamma_1+\gamma_2+2\gamma_1\gamma_2)\epsilon_1 + \beta_2}{1+\gamma_2}. \tag{3.58}$$

Furthermore, (3.54) and (3.55) imply relation (3.11).

Appendix 3.D Proof of Theorem 6

Suppose that (3.23) and (3.24) hold. From Assumption 1 and Theorem 5, for the subsystem $\mathcal{M}$, we obtain

$$[\![\ \|e_1|_{[0,\tau-1]}\|_{\ell^p} \leq \epsilon_1 \Rightarrow \|z_1|_{[0,\tau]}\|_{\ell^p} \leq \gamma_1\epsilon_1 \ ]\!] \quad \forall e_1 \in \ell_e, \ \forall \tau \in \mathbb{Z}_+.$$

The definition of $\mathbf{SB}_\Delta^{\epsilon,\gamma}$ and (3.24) implies $\epsilon_1 > \gamma_2\epsilon_2$, $\epsilon_2 > \gamma_1\epsilon_1$ and

$$[\![\ \|e_2|_{[0,\tau]}\|_{\ell^p} \leq \epsilon_2 \Rightarrow \|z_2|_{[0,\tau]}\|_{\ell^p} \leq \gamma_2\epsilon_2 \ ]\!] \quad \forall e_1 \in \ell_e, \ \forall \tau \in \mathbb{Z}_+$$

where $\gamma_2 := 1/\gamma$ and $\epsilon_2 := \epsilon$.

It is then directly verified from Theorem 3 that the feedback system is small ℓ^p signal ℓ^p stable for all $\Delta \in \mathbf{SB}_\Delta^{\epsilon,\gamma}$. Moreover, (3.25) and (3.26) are obtained from relationships (3.10) and (3.11).

Appendix 3.E Proof of Theorem 7

Suppose that (3.27) holds. From Assumption 1 and Theorem 5,

$$[\![\|e_1|_{[0,\tau-1]}\|_{\ell^p} \leq \epsilon_1 \Rightarrow \|z_1|_{[0,\tau]}\|_{\ell^p} \leq \gamma_1\epsilon_1]\!] \quad \forall e_1 \in \ell_e,\ \forall \tau \in \mathbb{Z}_+$$

holds for the subsystem $\mathcal{M}$. We have

$$\|z_2|_{[0,\tau]}\|_{\ell^p} \leq \frac{1}{\gamma}\|e_2|_{[0,\tau]}\|_{\ell^p} \quad \forall e_2 \in \ell_e,\ \forall \tau \in \mathbb{Z}_+ \tag{3.59}$$

by the definition of $\mathbf{B}_\Delta^\gamma$. Because $\gamma_1 < \gamma$, it is implicit from Theorem 4 that the feedback system is small ℓ^p signal ℓ^p stable for all $\Delta \in \mathbf{B}_\Delta^\gamma$. Relationships (3.28) and (3.29) are obtained from (3.10) and (3.11).

Appendix 3.F Proof of Theorem 9

The sufficiency is immediate from Theorem 7.

We next prove the necessity part. Assume that $\|\mathcal{M}\|_{\ell^p\text{-ind}} \geq 1$. From Theorem 2 in [39] ($1 \leq p < \infty$) and Theorem 1 in [14] ($p = \infty$), there exists a linear system $\Delta_0 \in \mathbf{B}_\Delta$ such that the feedback system shown in Figure 3.1 is not ℓ^p stable. Because the feedback interconnection of Δ_0 and $\mathcal{M}$ is a linear system, it is shown from Theorem 3 that for $\Delta_0 \in \mathbf{B}_\Delta$, the feedback system is not small ℓ^p signal ℓ^p stable.

Inequalities (3.36) and (3.37) are implicit from Theorem 1.

Chapter 4

Robust Stabilization over a Rate-limited Communication Channel

This chapter shows the usefulness of the framework that has been proposed in Chapter 3. Toward this end, we demonstrate the use of the framework with one unresolved problem: the robust stabilization of an uncertain networked control system. By addressing this problem, we show the advantages of the proposed framework in the stability analysis of quantized feedback systems.

The robust stabilization of a networked control system in which an uncertain plant is controlled over a rate-limited communication channel has been one of the important issues in the analysis and synthesis of quantized feedback systems. The stability of such an uncertain networked control system is affected by both quantization at the channel and model uncertainty in the plant dynamics. A low data rate at the communication channel, equivalent to a coarse quantization, generally leads to a degradation of the control performance of the system and can lead to instability in some cases. The uncertainty in the plant dynamics should also affect the stability of the networked control system. Although it has expectedly been assumed that there should exist a trade-off between the data rate and the degree of uncertainty for the robust stabilizability of the networked control system, the quantitative analysis has not yet been carried out.

In this section, we use the framework proposed in Chapter 3 to derive a condition on the data rate for the robust small ℓ^∞ signal ℓ^∞ stabilizability of a networked control system. When the data rate satisfies the derived condition, a stabilizing encoder and controller pair will be given. Through robust stability analysis and robust stabilization, this chapter shows the advantage of the use of the proposed framework in the stability analysis of quantized feedback systems.

The remainder of this chapter is organized as follows. Section 4.1 discusses the problem formulation. Section 4.2 discusses the robust stabilization of the networked control system. Section 4.3 presents numerical examples and simulation results. Section 4.4 summarizes this chapter.

4.1 Problem formulation

Consider the networked control system shown in Figure 2.2. A detailed description of the same has already been given in Subsection 2.3.1.

In this chapter, we make the following assumption.

Assumption 2. *A state-space realization of the nominal plant P is given by*

$$x(t+1) = Ax(t) + Bu(t) + Ee_1(t),\ x(0) = 0, \tag{4.1a}$$

$$z_1(t) = Cx(t), \tag{4.1b}$$

$$y(t) = x(t), \tag{4.1c}$$

where $x(t) \in \mathbb{R}^n$ represents the plant state. (A, B) is stabilizable and the encoder directly observes the plant state $x(t)$. The initial state $x(0)$ is assumed to be zero.

This chapter derives a condition on the data rate for the robust small ℓ^∞ signal ℓ^∞ stabilizability ($p = \infty$) of the feedback system shown in Figure 2.2. Namely, when an uncertain plant is given, we wish to derive under Assumption 2 a condition on R for the existence of (En, K) such that

$$[\![\ \|r|_{[0,\tau]}\|_{\ell^\infty} \le \epsilon \Rightarrow \|z|_{[0,\tau]}\|_{\ell^\infty} \le \gamma\epsilon\ ]\!] \quad \forall r \in \ell_e,\ \forall \tau \in \mathbb{Z}_+ \tag{4.2}$$

holds for some positive constants ϵ and γ, where $r := (r_1, r_2)$ and $z := (z_1, z_2)$.

4.2 Robust stabilization under rate constraint

Noting that $\mathcal{F}_l(P,\ K \circ \mathrm{En})$: $e_1 \mapsto z_1$ is strictly causal for any causal encoder-controller pairs under Assumption 2, it is implicit from Theorem 7 that the robust small ℓ^∞ signal ℓ^∞ stability of the feedback system for the fixed (En, K) can be established in the following proposition.

Proposition 1. *Fix the encoder-controller pair (K, En). Assume that there exist positive constants ϵ_1 and $\gamma_1 < \gamma$ such that*

$$[\![\ \|e_1\|_{\ell^\infty} \le \epsilon_1 \Rightarrow \|z_1\|_{\ell^\infty} \le \gamma_1\epsilon_1\ ]\!] \quad \forall e_1 \in \ell^\infty \tag{4.3}$$

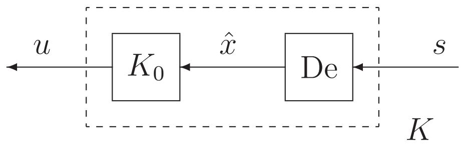

Figure 4.1: Structure of K

holds for the nominal system $\mathcal{F}_l(P, K \circ \text{En})$. Then, the feedback system shown in Figure 2.2 is small ℓ^∞ signal ℓ^∞ stable for all $\Delta \in \mathbf{B}_\Delta^\gamma$. In particular,

$$\left[\!\left[\ \|r|_{[0,\tau]}\|_{\ell^\infty} \le \epsilon \Rightarrow \left(\ \|z_1|_{[0,\tau]}\|_{\ell^\infty} \le \delta_1 \text{ and } \|z_2|_{[0,\tau]}\|_{\ell^\infty} \le \delta_2\ \right)\ \right]\!\right] \quad \forall r \in \ell_e,\ \forall \tau \in \mathbb{Z}_+ \tag{4.4}$$

holds for

$$\epsilon = \frac{\gamma - \gamma_1}{1+\gamma}\epsilon_1, \quad \delta_1 = \gamma_1 \epsilon_1, \quad \delta_2 = \frac{1+\gamma_1}{1+\gamma}\epsilon. \tag{4.5}$$

We then investigate a necessary or sufficient condition on R for the existence of (En, K) such that the nominal system $\mathcal{F}_l(P, K \circ \text{En})$ satisfies (4.3). A sufficient condition will be obtained when the nominal plant P is a multi-order system. If P is a scalar nominal plant, a necessary and sufficient condition on the data rate for the existence of (En, K) satisfying (4.3) is derived.

In order to prove the sufficiency part, we employ a constructive approach. Toward this end, we hereafter consider the special class of K that has the structure of the cascade connection of the following two subsystems (Figure 4.1):

- **Decoder De:** The decoder De is a causal map which produces a state estimate $\hat{x}$ from the received channel symbols $(s(0), \cdots, s(t))$.

- **LTI Controller K_0:** K_0 is an LTI system which produces the control input $u(t)$ from the state estimate $(\hat{x}(0), \cdots, \hat{x}(t))$.

It is easily seen from (2.10) and $u = K_0\hat{x} = K_0 x + K_0(\hat{x} - x)$ that

$$z_1 = \{P_{11} + P_{12}(I - K_0 P_{22})^{-1} K_0 P_{22}\} e_1 + P_{12}(I - K_0 P_{22})^{-1} K_0 (\hat{x} - x). \tag{4.6}$$

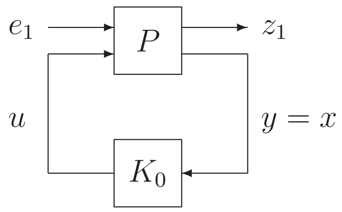

Figure 4.2: Nominal control system without a digital channel

We now define

$$\begin{aligned}\mathbf{K}_0^\gamma = \{&K_0 : \ell_e \to \ell_e \mid \\ &K_0 \text{ stabilizes the feedback system shown in Figure 4.2} \\ &\text{in the sense of } \ell^\infty \text{ stability } ^1, \\ &\text{and } \|P_{11} + P_{12}(I - K_0P_{22})^{-1}K_0P_{21}\|_{\ell^\infty\text{-ind}} < \gamma\}\end{aligned}$$

Assumption 3. *The set $\mathbf{K}_0^\gamma$ is non-empty, and we are given a controller $K_0 \in \mathbf{K}_0^\gamma$.*

If the set $\mathbf{K}_0^\gamma$ is non-empty, a controller $K_0 \in \mathbf{K}_0^\gamma$ can be obtained by standard ℓ^1 control techniques (see, e.g., [4]).

By the small gain theorem and Theorem 9, this assumption implies that the uncertain plant $P_\Delta = \mathcal{F}_u(P, \Delta)$ is robustly stabilizable in the sense of ℓ^∞ stability and small ℓ^∞ signal ℓ^∞ stability by the LTI controller K_0 in the absence of the data rate constraint, namely, in the situation in which the output $x(t)$, instead of $\hat{x}(t)$, is directly available to the controller (see Figure 4.2). Therefore, $\mathbf{K}_0^\gamma \neq \emptyset$ is a necessary condition for the robust stabilizability in our setting.

We also define

$$\begin{aligned}\gamma_0 &:= \|P_{11} + P_{12}(I - K_0P_{22})^{-1}K_0P_{21}\|_{\ell^\infty\text{-ind}} < \gamma, \\ \kappa_0 &:= \|P_{12}(I - K_0P_{22})^{-1}K_0\|_{\ell^\infty\text{-ind}}.\end{aligned}$$

In the remainder of this section, we employ the *primitive quantizer* proposed by Tatikonda and Mitter [32] for the encoding-decoding algorithm with a slight modification to satisfy the ℓ^∞ setting. See Appendix 4.A for the detailed description of the primitive quantizer. It should be noted that the primitive quantizer is implemented under the following assumption.

[1] Note that the feedback system shown in Figure 4.2 is an LTI one.

Assumption 4. *The upper bound ϵ_1 on $\|e_1|_{[0,\tau]}\|_{\ell^\infty}$, $\tau \geq 0$ is available as prior information for both of the encoder and the decoder.*

We denote the encoder and decoder with the primitive quantizer by $\mathrm{E}^{\epsilon_1}_{\mathrm{pq}}$ and $\mathrm{D}^{\epsilon_1}_{\mathrm{pq}}$, respectively.

We can now state the main result of this chapter. The following theorem gives a sufficient condition for the robust stabilizability under the data rate constraint in the sense of small ℓ^∞ signal ℓ^∞ stability.

Theorem 10. *Let ϵ_1 be an arbitrary positive constant. Under Assumptions 2, 3, and 4, assume that there exist nonnegative integers $R_1, \cdots, R_n$ satisfying*

$$R_1 + R_2 + \cdots + R_n \leq R, \tag{4.7}$$

$$\left\|F_{\mathcal{R}}\bar{J}\right\|_1 < 1, \tag{4.8}$$

$$\gamma_1 := \gamma_0 + \frac{\hbar^2}{1 - \left\|F_{\mathcal{R}}\bar{J}\right\|_1}\|T^{-1}\|_1\|TE\|_1\|F_{\mathcal{R}}\|_1\kappa_0 < \gamma, \tag{4.9}$$

where $F_{\mathcal{R}}$, $\bar{J}$, T, and $\hbar$ are defined in Appendix 4.A.

(i) With the encoder-controller pair $(K, \mathrm{En}) = (K_0 \circ \mathrm{D}^{\epsilon_1}_{\mathrm{pq}}, \mathrm{E}^{\epsilon_1}_{\mathrm{pq}})$, $\mathcal{F}_1(P, K \circ \mathrm{En})\colon e_1 \mapsto z_1$ satisfies

$$[\![\ \|e_1\|_{\ell^\infty} \leq \epsilon_1 \Rightarrow \|z_1\|_{\ell^\infty} \leq \gamma_1\epsilon_1 \ ]\!] \quad \forall e_1 \in \ell^\infty, \tag{4.10}$$

where $\gamma_1 \in (0, \gamma)$ is given in (4.9).

(ii) With the above encoder-controller pair, the feedback system shown in Figure 2.2 is small ℓ^∞ signal ℓ^∞ stable for all $\Delta \in \mathbf{B}^\gamma_\Delta$. In particular, for all $\Delta \in \mathbf{B}^\gamma_\Delta$,

$$[\![\ \|r|_{[0,\tau]}\|_{\ell^\infty} \leq \epsilon \Rightarrow \left(\ \|z_1|_{[0,\tau]}\|_{\ell^\infty} \leq \delta_1 \text{ and } \|z_2|_{[0,\tau]}\|_{\ell^\infty} \leq \delta_2 \ \right) \ ]\!] \quad \forall r \in \ell_e, \ \forall \tau \in \mathbb{Z}_+ \tag{4.11}$$

holds for

$$\epsilon = \frac{\gamma - \gamma_1}{1 + \gamma}\epsilon_1, \quad \delta_1 = \gamma_1\epsilon_1, \quad \delta_2 = \frac{1 + \gamma_1}{1 + \gamma}\epsilon \tag{4.12}$$

where $\gamma_1 \in (0, \gamma)$ is defined in (4.9).

Proof. See Appendix 4.B. □

The conditions (4.7) and (4.8) are a sufficient condition for the existence of an encoder-controller pair that stabilizes the nominal plant P. In fact, it is well known that there exists an encoder-controller pair stabilizing the nominal plant if $F_{\mathcal{R}}\bar{J}$ is Schur stable [32].

It can easily be verified that the second term on the left-hand side of (4.9) decays to zero when $\min_i R_i$ goes to infinity. It is natural that the admissible level of small ℓ^∞ signal ℓ^∞ stability of the nominal part is bounded by γ when we consider $R_1, \cdots, R_n$ to be sufficiently large, because γ is the bound on the level for the robust small ℓ^∞ signal ℓ^∞ stabilizability of the networked control system in the absence of the data rate constraint.

Equation (4.9) gives the trade-off between the stability margin and the data rate. That is, for a large uncertainty, i.e., for a small γ, we need a large R in order to guarantee the robust small ℓ^∞ signal ℓ^∞ stabilizability of the feedback system.

In view of Proposition 1, the feedback system shown in Figure 2.2 is small ℓ^∞ signal ℓ^∞ stable for all $\Delta \in \mathbf{B}_\Delta^\gamma$ if $\mathcal{F}_l(P, K \circ \mathrm{En})$ satisfies (4.10) for some $\epsilon_1 > 0$ and $\gamma_1 \in (0, \gamma)$. In contrast, Theorem 10 gives a stronger result because we can choose ϵ_1 *arbitrarily* when applying the encoder-controller pair $(K_0 \circ \mathrm{D}_{\mathrm{pq}}^{\epsilon_1}, \mathrm{E}_{\mathrm{pq}}^{\epsilon_1})$. Moreover, it is seen from (4.12) that for fixed ϵ_1 and γ_1, the admissible ℓ^∞ norm bound on the exogenous signal r is given by

$$\epsilon = \frac{\gamma - \gamma_1}{1+\gamma}\epsilon_1,$$

and conversely, if ϵ and γ_1 are given, then ϵ_1 is calculated as

$$\epsilon_1 = \frac{1+\gamma}{\gamma - \gamma_1}\epsilon. \tag{4.13}$$

Therefore, if ϵ is given as prior information rather than ϵ_1, we can implement $\mathrm{E}_{\mathrm{pq}}^{\epsilon_1}$ and $\mathrm{D}_{\mathrm{pq}}^{\epsilon_1}$ for any $\epsilon_1 > 0$ by using (4.13) (γ_1 can be computed by (4.9) independently of ϵ and ϵ_1). This implies that the boundedness of $z := (z_1, z_2)$ is robustly guaranteed by $(K_0 \circ \mathrm{D}_{\mathrm{pq}}^{\epsilon_1}, \mathrm{E}_{\mathrm{pq}}^{\epsilon_1})$ for an arbitrarily large exogenous signal r under the assumption that the upper bound ϵ on $\|r\|_{\ell^\infty}$ is known *a priori*.

If the nominal plant is a scalar system, i.e.,

$$\begin{cases} x(t+1) &= ax(t) + u(t) + e_1(t), \\ z_1(t) &= cx(t), \\ y(t) &= x(t), \end{cases} \tag{4.14}$$

it is possible to derive the strict bound on R for the existence of an encoder-controller pair that achieves (4.10) without restricting $(\mathrm{En}, \mathrm{De})$ to $(\mathrm{E}_{\mathrm{pq}}^{\epsilon_1}, \mathrm{D}_{\mathrm{pq}}^{\epsilon_1})$.

Corollary 1. *Assume that there exists an encoder-controller pair for which* $\mathcal{F}_l(P, K \circ \mathrm{En})$ *satisfies (4.10) for some* $\epsilon_1 > 0$ *and* $\gamma_1 \in (0, \gamma)$. *Then, there* *hold*

$$|a| < 2^R, \tag{4.15}$$

$$\frac{|c|}{1 - \frac{|a|}{2^R}} \leq \gamma. \tag{4.16}$$

Conversely, if R satisfies

$$|a| < 2^R, \tag{4.17}$$

$$\frac{|c|}{1 - \frac{|a|}{2^R}} < \gamma, \tag{4.18}$$

then (4.10) is satisfied for any fixed $\epsilon_1 > 0$ and for

$$\gamma_1 := \frac{|c|}{1 - \frac{|a|}{2^R}} \in (0, \gamma) \tag{4.19}$$

by $(K, \mathrm{En}) = (K_0 \circ \mathrm{D}_{\mathrm{pq}}^{\epsilon_1}, \mathrm{E}_{\mathrm{pq}}^{\epsilon_1})$, $K_0 = -a \in \mathbf{K}_0^{\gamma}$ (constant feedback gain).

Proof. See Appendix 4.C. □

The only difference between the necessary condition and the sufficient condition in the above corollary is that the inequality (4.16) in the sufficient condition is strict, whereas (4.18) in the necessary condition is not a strict inequality. Thus, the obtained bound on the data rate is tight for the existence of an encoder-controller pair satisfying (4.10).

Conditions (4.17) and (4.18) with $|c| = 1$ correspond to the robust stabilizability conditions in Theorem 3.2 of [17] in the case in which there is no parametric uncertainty in a.

Remark 1. *It should be noted that the contribution of this chapter lies in the robust stabilizability conditions in Theorem 10 and Corollary 1 and not in the use of the primitive quantizer. In fact, we have employed it only to derive a tight bound of R for the existence of an encoder-controller pair such that $\mathcal{F}_l(P, K \circ \mathrm{En})$ satisfies (4.10) for some $\gamma_1 \in (0, \gamma)$. (Recall that the conditions in Theorem 10 are also necessary in the case in which the nominal plant is a scalar system.) This enables us to derive the conditions (4.7) - (4.9), which represent the trade-off between the stability margin of the closed-loop system and the data rate constraint.*

4.3 Numerical example

This section is devoted to numerical examples and simulation results.

Consider the third-order linear plant described by the following state-space realization.

$$\begin{aligned} x(t+1) &= \begin{bmatrix} 2 & 0 & 0 \\ 1 & -2 & \sqrt{3} \\ 0 & \sqrt{3} & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e_1(t),\ x(0) = 0 \\ z_1(t) &= \begin{bmatrix} 1 & \frac{17}{6} & \frac{19}{18}\sqrt{3} \end{bmatrix} x(t), \\ y(t) &= x(t). \end{aligned}$$

The transfer function matrix of this plant is given by

$$P(z) = \frac{1}{(z-1)(z-2)(z+3)} \left[\begin{array}{c|c} \frac{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}{(z-1)(z+3)} & \frac{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{3}\right)}{(z-1)(z+3)} \\ \hline z & z \\ \sqrt{3} & \sqrt{3} \end{array} \right].$$

From Theorem 3.1 of [7], we recall that because $P_{12}(z)$ has only minimum phase zeros,

$$\min_{K_0:\ \text{stabilizing}} \|\mathcal{F}_l(P, K_0)\|_{\ell^\infty\text{-ind}}$$

is attained by the linear static feedback controller that assigns the closed-loop poles at the above minimum phase zeros and at the origin. In this example, this optimal static feedback controller is given by

$$K_0 : u(t) = \begin{bmatrix} \frac{5}{6} & -\frac{53}{6} & \frac{17}{6}\sqrt{3} \end{bmatrix} x(t)$$

with the optimal cost $\min_{K_0} \|\mathcal{F}_l(P, K_0)\|_{\ell^\infty\text{-ind}} = 1$. For this controller K_0 and the plant, we have

$$\begin{aligned} \gamma_0 &= \|\mathcal{F}_l(P, K_0)\|_{\ell^\infty\text{-ind}} = 1, \\ \kappa_0 &= \|P_{12}(I - K_0 P_{22})^{-1} K_0\|_{\ell^\infty\text{-ind}} = 14.5741. \end{aligned}$$

Then, conditions (4.7) - (4.9) in Theorem 10 are reduced to

$$R_1 + R_2 + R_3 \le R, \tag{4.20}$$

$$\max\left\{\frac{3}{2^{R_1}}, \frac{1}{2^{R_2}}, \frac{2}{2^{R_3}}\right\} < 1, \tag{4.21}$$

$$1 + \frac{14.5741 \max\left\{\frac{1}{2^{R_1}}, \frac{1}{2^{R_2}}, \frac{1}{2^{R_3}}\right\}}{1 - \max\left\{\frac{3}{2^{R_1}}, \frac{1}{2^{R_2}}, \frac{2}{2^{R_3}}\right\}} < \gamma. \tag{4.22}$$

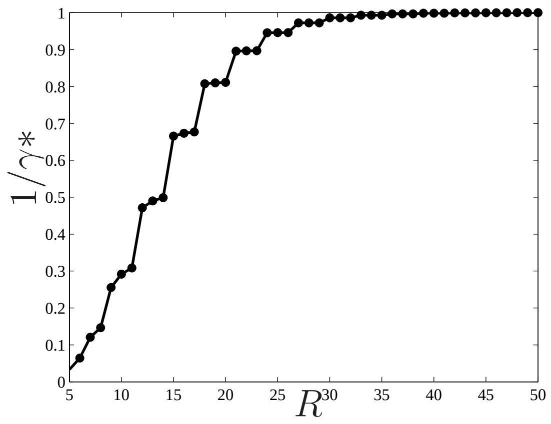

Figure 4.3: Trade-off between the stability margin and the data rate constraint

Define

$$\gamma^*(R) = \inf_{\gamma, R_1, R_2, R_3} \{\gamma : (4.20), (4.21) \text{ and } (4.22) \text{ hold.}\}.$$

Notice that $1/\gamma^*(R)$ serves as a lower bound of the stability margin of the feedback system shown in Figure 2.2 with $(K, \mathrm{En}) = (K_0 \circ \mathrm{D}_{\mathrm{pq}}^{\epsilon_1}, \mathrm{E}_{\mathrm{pq}}^{\epsilon_1})$ at the rate R. It is also obvious that $\sup_{R>0} 1/\gamma^*(R) = 1/\gamma_0$, which is equal to 1 in this example. Figure 4.3 shows the trade-off between the stability margin and the data rate constraint. $1/\gamma^*$ is large for a large data rate R. In other words, we need a large data rate in order to robustly stabilize the feedback system against a large uncertainty.

We also see from the figure that the stability margin in the absence of the data rate constraint is almost recovered at a data rate greater than 35.

We next performed the numerical simulations with $(R_1, R_2, R_3) = (10, 10, 10)$. In this case, the left-hand side of (4.22) becomes $\gamma_1 = 1.0143$. Thus, we take $\gamma = 1.2 > \gamma_1$ so that the feedback system is robustly stabilized by the encoder-controller pair $(K, \mathrm{En}) = (K_0 \circ \mathrm{D}_{\mathrm{pq}}^{\epsilon_1}, \mathrm{E}_{\mathrm{pq}}^{\epsilon_1})$.

We also set the upper bound of $\|r\|_{\ell^\infty}$ to $\epsilon = 1$, which leads to $\epsilon_1 = 11.8454$ by (4.13). In view of (3.54) and (3.55) in Theorem 3, upper bounds on $\|z_1\|_{\ell^\infty}$ and $\|z_2\|_{\ell^\infty}$ are given by

$$\|z_1\|_{\ell^\infty} \leq \gamma_1 \epsilon_1 = 12.0145, \tag{4.23}$$

$$\|z_2\|_{\ell^\infty} \leq \frac{\gamma_1 \frac{1}{\gamma} + \frac{1}{\gamma}}{\frac{1}{\gamma} + 1} \epsilon_1 = 10.8454. \tag{4.24}$$

We performed the simulations under the following two scenarios of the uncertainty Δ.

(a) **Linear time-invariant memoryless case:** In this case, Δ is a constant feedback gain such that $|\Delta| \leq 1/\gamma$. 20 values of Δ are randomly generated according to the uniform distribution over the interval $[-1/\gamma, 1/\gamma]$.

(b) **Linear time-varying memoryless case:** In this case, Δ is a time-varying linear feedback gain such that $|\Delta(t)| \leq 1/\gamma$ at each instant t. 20 sequences of Δ: $\mathbb{Z}_+ \to \mathbb{R}$ are randomly generated according to the independently identically distributed (i.i.d.) process with the uniform distribution over $[-1/\gamma, 1/\gamma]$, namely, $\Delta(t) \sim U(-1/\gamma, 1/\gamma)\ \forall t \geq 0$, where $U(a,b)$ is the uniform distribution over $[a,b]$.

In each scenario, 20 different sequences of the exogenous input r were randomly generated according to the i.i.d. process with the uniform distribution over $[-\epsilon, \epsilon] \times [-\epsilon, \epsilon]$. Hence, we simulated $20 \times 20 = 400$ trials for each scenario.

In both scenarios, all the output responses, denoted as $z_1^{(i)}$, $z_1^{(i)}$ $i = 1, \cdots 400$, did not exceed the norm bounds of (4.23) and (4.24). An example of the input-output response of one among the 400 trials is shown in Figures 4.4 and 4.5 for each scenario. The largest output norms among 400 trials were as follows[1].

$$\text{(a)}\ \max_i \|z_1^{(i)}\|_{\ell^\infty} = 4.1747, \qquad \max_i \|z_2^{(i)}\|_{\ell^\infty} = 3.4118,$$
$$\text{(b)}\ \max_i \|z_1^{(i)}\|_{\ell^\infty} = 3.3660, \qquad \max_i \|z_2^{(i)}\|_{\ell^\infty} = 2.5733.$$

From these results, we see that the feedback system is robustly small ℓ^∞ signal ℓ^∞ stable for memoryless uncertainties. On the other hand, the ℓ^∞ norms of z_1 and z_2 are much smaller than the upper bounds of (4.23) and (4.24). This indicates that for this example, there is the conservativeness in the conditions on the upper bounds on signal norms.

4.4 Summary

In this chapter, the robust small ℓ^∞ signal ℓ^∞ stabilization over a rate-limited communication channel was studied by using the framework developed in

[1]The ℓ^∞ norms of z_1 and z_2 are estimated by their peak values over the simulation interval $[0, 300]$.

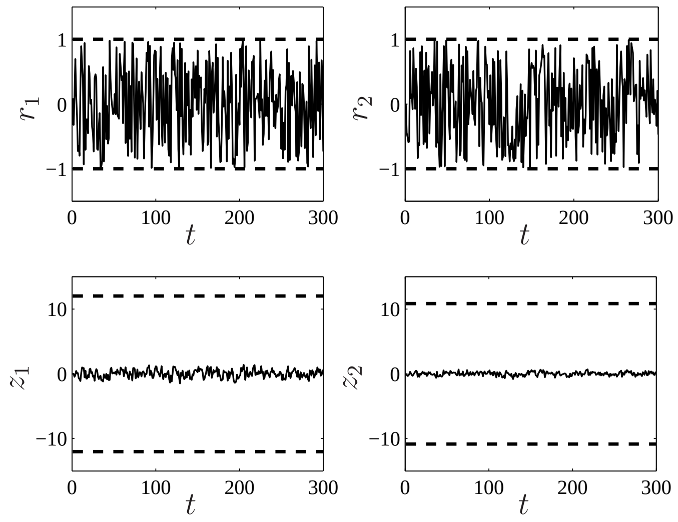

Figure 4.4: Simulation result (scenario (a))

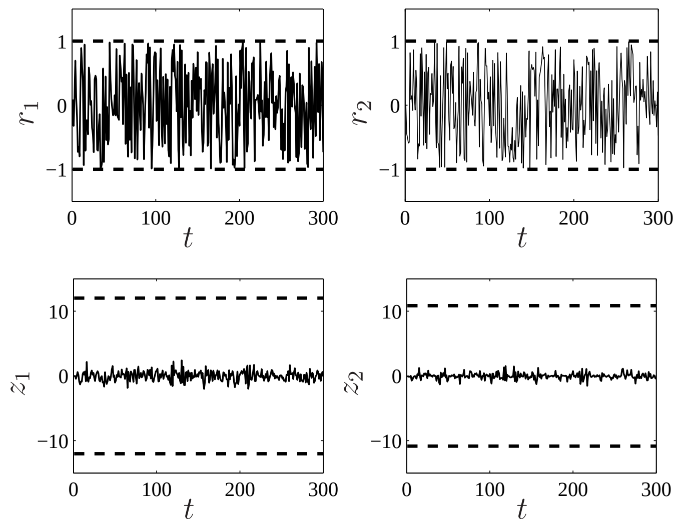

Figure 4.5: Simulation result (scenario (b))

Chapter 3. We explicitly derived a sufficient condition on the data rate R for the existence of an encoder-controller pair that robustly stabilizes the uncertain feedback system in the sense of small ℓ^∞ signal ℓ^∞ stability. This indicates the usefulness of the proposed framework in the thesis.

Appendix 4.A Primitive Quantizer

We introduce the *primitive quantizer* proposed by Tatikonda and Mitter [32] with a slight modification to conform with the ℓ^∞ setting in this chapter.

The primitive quantizer is an encoding-decoding algorithm defined by the four-tuple $(c(t), \mathcal{R}, L(t), \Phi(t))$. The variable $c \in \mathbb{R}^n$ represents the centroid, $\mathcal{R} = (R_1, \cdots, R_n) \in \mathbb{R}^n$ represents the rate vector, $L(t) = (L_1(t), \cdots, L_n(t)) \in \mathbb{R}^n$ represents the dynamic range, and $\Phi(t)$ is a nonsingular matrix that represents a coordinate transformation at time t.

This encoder partitions the region

$$\Lambda(t) = \{x \in \mathbb{R}^n | \Phi(t)(x - c(t)) \in \{[-L_1(t), L_1(t)] \times \cdots \times [-L_n(t), L_n(t)]\}\}$$

into 2^R boxes with side length $(2L_i(t)/2^{R_i})$. Each box is labeled with a symbol in the alphabet $\mathcal{A} = \{0, 1, \cdots, 2^R - 1\}$. If $x(t)$ lies in one of these 2^R boxes, the encoder sends out the corresponding symbol as $s(t)$. If $x(t) \notin \Lambda(t)$, the encoder sends the special symbol representing overflow.

In turn, the decoder produces the state estimate $\hat{x}(t) = c(t)$, the centroid of the symbol $s(t)$. If $s(t)$ is an overflow symbol, the decoder sets $\hat{x}(t) = 0$. There should be an agreement between the encoder and the decoder on which symbol indicates a box and overflow.

The above algorithm needs $2^R + 1$ symbols including the overflow symbol. In fact, however, we design the parameters of the primitive quantizer so that overflow does not occur. Thus, we need only 2^R symbols in this algorithm.

We prepare several matrices to represent the dynamics of the primitive quantizer, namely, the update rule of $c(t)$, $L(t)$, and $\Phi(t)$. Let T be a nonsingular matrix that transforms A into a real Jordan canonical form $TAT^{-1} = J = \text{diag}(J_1, \cdots, J_n)$. Define block-diagonal matrices $H = \text{diag}(H_1, \cdots, H_m)$ and $\bar{J} = \text{diag}(\bar{J}_1, \cdots, \bar{J}_n)$, where each H_i and $\bar{J}_i$ are associated with the i-th Jordan block J_i:

$$H_i = I, \quad \text{if } \lambda_i \text{ is a real eigenvalue,}$$

$$H_i = \text{diag}[r(\theta_i)^{-1}, \cdots, r(\theta_i)^{-1}],\ r(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix},$$

$$\text{if } \rho_i(\cos\theta_i \pm j\sin\theta_i) \text{ are complex eigenvalues,}$$

$$\bar{J}_i = \begin{bmatrix} |\lambda_i| & 1 & & \\ & |\lambda_i| & \ddots & \\ & & \ddots & 1 \\ & & & |\lambda_i| \end{bmatrix} \quad \text{if } \lambda_i \text{ is a real eigenvalue,}$$

$$\bar{J}_i = \begin{bmatrix} \rho_i I & \bar{r}(-\theta_i) & & \\ & \rho_i I & \ddots & \\ & & \ddots & \bar{r}(-\theta_i) \\ & & & \rho_i I \end{bmatrix},$$

$$\bar{r}(\theta) = \begin{bmatrix} |\cos\theta| & |\sin\theta| \\ |\sin\theta| & |\cos\theta| \end{bmatrix},$$

$$\text{if } \rho_i(\cos\theta_i \pm j\sin\theta_i) \text{ are complex eigenvalues.}$$

Here, we define

$$\hbar = \sup_{t\geq 0} \|H^t\|_1.$$

It is easily seen from the definition of H that $\hbar \leq \sqrt{2}$ generally holds. In particular, we have $\hbar = 1$ if all the eigenvalues of A are real. In addition, because H is an orthogonal matrix, $\|H^t\|_1 = \|H^{-t}\|_1$ holds for all $t \geq 0$. This implies $\hbar = \sup_{t\geq 0} \|H^{-t}\|_1$.

Furthermore, we define

$$F_{\mathcal{R}} = \begin{bmatrix} 2^{-R_1} & & & \\ & 2^{-R_2} & & \\ & & \ddots & \\ & & & 2^{-R_n} \end{bmatrix}.$$

With the above matrices, the quantizer parameters are updated as

$$c(t+1) = A\hat{x}(t) + Bu(t), \tag{4.25}$$

$$\Phi(t+1) = H\Phi(t), \tag{4.26}$$

$$L(t+1) = \bar{J}F_{\mathcal{R}}L(t) + \|\Phi(t+1)E\|_1 \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_1 \end{bmatrix}, \tag{4.27}$$

where the initial conditions are given by $c(0) = 0$, $\Phi(0) = T$, and $L(0) = 0$, respectively.

If $x(t)$ belongs to $\Lambda(t)$, $\hat{x}(t)$ is the centroid of the box that $x(t)$ falls into. Thus, we have by construction,

$$\Phi(t)(x(t)-\hat{x}(t)) \in \left\{ \left[-\frac{L_1(t)}{2^{R_1}}, \frac{L_1(t)}{2^{R_1}} \right], \cdots, \left[-\frac{L_n(t)}{2^{R_n}}, \frac{L_n(t)}{2^{R_n}} \right] \right\}$$

for $x(t) \in \Lambda(t)$.

The following lemmas are useful in the proof of the main result.

Lemma 1. *For all $t \geq 0$, $x(t) \in \Lambda(t)$.*

Proof. It can be shown in the same manner as Lemma 4.3 in [32]. □

Lemma 2. *The following inequalities are true for all $t \geq 0$.*

$$\begin{aligned} \|\Phi(t)E\|_1 &\leq \hbar \|TE\|_1, \\ \|\Phi(t)^{-1}\|_1 &\leq \hbar \|T^{-1}\|_1. \end{aligned}$$

Proof. Recall that $\hbar \geq \|H^t\|_1 = \|H^{-t}\|_1$ for all $t \geq 0$. Because $\Phi(t) = H^t T$ from (4.26), we have

$$\begin{aligned} \|\Phi(t)E\|_1 &= \|H^t TE\|_1 \leq \|H^t\|_1 \|TE\|_1 \leq \hbar \|TE\|_1 \\ \|\Phi(t)^{-1}\|_1 &= \|T^{-1}H^{-t}\|_1 \leq \|T^{-1}\|_1 \|H^{-t}\|_1 \leq \hbar \|T^{-1}\|_1. \end{aligned}$$

□

Appendix 4.B Proof of Theorem 10

Assume that all the conditions in Theorem 10 hold. Let ϵ_1 be an arbitrary positive constant.

(i) We see from (4.6) that

$$\begin{aligned} \|z_1|_{[0,\tau]}\|_{\ell^\infty} &\leq \|P_{11} + P_{12}(I - K_0 P_{22})^{-1} K_0 P_{22}\|_{\ell^\infty\text{-ind}} \|e_1|_{[0,\tau]}\|_{\ell^\infty} \\ &\quad + \|P_{12}(I - K_0 P_{22})^{-1} K_0\|_{\ell^\infty\text{-ind}} \|(x - \hat{x})|_{[0,\tau]}\|_{\ell^\infty} \\ &\leq \gamma_0 \epsilon_1 + \kappa_0 \|(x - \hat{x})|_{[0,\tau]}\|_{\ell^\infty}, \quad \forall \tau \geq 0 \end{aligned} \tag{4.28}$$

holds for any $e_1 \in \ell_e^\infty$ such that $\|e_1|_{[0,\tau]}\|_{\ell^\infty} \leq \epsilon_1$, and for any $\tau \geq 0$. We evaluate $\|(x - \hat{x})|_{[0,\tau]}\|_{\ell^\infty}$ to derive an upper bound on $\|z_1|_{[0,\tau]}\|_{\ell^\infty}$.

It follows from Lemmas 1 and 2 that

$$\begin{aligned}\|x(t)-\hat{x}(t)\|_\infty &= \|\Phi^{-1}(t)\Phi(t)(x(t)-\hat{x}(t))\|_\infty \\ &\le \|\Phi^{-1}(t)\|_1\|\Phi(t)(x(t)-\hat{x}(t))\|_\infty \\ &\le \hbar\|T^{-1}\|_1\|F_{\mathcal{R}}L(t)\|_\infty. \end{aligned} \tag{4.29}$$

Now, pre-multiplying (4.27) by $F_{\mathcal{R}}$ yields

$$\begin{aligned} F_{\mathcal{R}}L(t+1) &= F_{\mathcal{R}}\bar{J}F_{\mathcal{R}}L(t) + F_{\mathcal{R}}\|\Phi(t+1)E\|_1 \begin{bmatrix}\epsilon_1\\ \vdots \\ \epsilon_1\end{bmatrix}, \\ L(0) &= 0, \end{aligned}$$

and thus

$$F_{\mathcal{R}}L(t) = \sum_{i=0}^{t-1}\left(F_{\mathcal{R}}\bar{J}\right)^i F_{\mathcal{R}}\|\Phi(t-i)E\|_1 \begin{bmatrix}\epsilon_1\\ \vdots \\ \epsilon_1\end{bmatrix}.$$

This implies by Lemma 2 that

$$\begin{aligned}\|F_{\mathcal{R}}L(t)\|_\infty &\le \sum_{i=0}^{t-1}\left\|\left(F_{\mathcal{R}}\bar{J}\right)^i \|\Phi(t-i)E\|_1 F_{\mathcal{R}} \begin{bmatrix}\epsilon_1\\ \vdots \\ \epsilon_1\end{bmatrix}\right\|_\infty \\ &\le \sum_{i=0}^{t-1}\left\|F_{\mathcal{R}}\bar{J}\right\|_1^i \|\Phi(t-i)E\|_1\|F_{\mathcal{R}}\|_1\epsilon_1 \\ &\le \sum_{i=0}^{t-1}\left\|F_{\mathcal{R}}\bar{J}\right\|_1^i h\|TE\|_1\|F_{\mathcal{R}}\|_1\epsilon_1. \end{aligned}$$

From (4.8), the above inequality results in

$$\begin{aligned}\|F_{\mathcal{R}}L(t)\|_\infty &= \frac{1-\left\|F_{\mathcal{R}}\bar{J}\right\|_1^t}{1-\left\|F_{\mathcal{R}}\bar{J}\right\|_1} h\|TE\|_1\|F_{\mathcal{R}}\|_1\epsilon_1 \\ &\le \frac{\hbar}{1-\left\|F_{\mathcal{R}}\bar{J}\right\|_1}\|TE\|_1\|F_{\mathcal{R}}\|_1\epsilon_1. \end{aligned} \tag{4.30}$$

Substituting (4.30) into (4.29) yields

$$\|x(t)-\hat{x}(t)\|_\infty \le \frac{\hbar^2}{1-\left\|F_{\mathcal{R}}\bar{J}\right\|_1}\|T^{-1}\|_1\|TE\|_1\|F_{\mathcal{R}}\|_1\epsilon_1,$$

and hence,

$$\|(x-\hat{x})|_{[0,\tau]}\|_{\ell^\infty} \le \frac{\hbar^2}{1-\left\|F_{\mathcal{R}}\bar{J}\right\|_1}\|T^{-1}\|_1\|TE\|_1\|F_{\mathcal{R}}\|_1\epsilon_1 \quad \forall\tau \ge 0 \qquad (4.31)$$

It thus follows from (4.28) and (4.31) that

$$\|z_1\|_{\ell^\infty} \le \left(\gamma_0 + \frac{\hbar^2}{1-\left\|F_{\mathcal{R}}\bar{J}\right\|_1}\|T^{-1}\|_1\|TE\|_1\|F_{\mathcal{R}}\|_1\kappa_0\right)\epsilon_1.$$

Because (4.9) holds, we consider that (4.10) is satisfied by setting

$$\gamma_1 = \gamma_0 + \frac{\hbar^2}{1-\left\|F_{\mathcal{R}}\bar{J}\right\|_1}\|T^{-1}\|_1\|TE\|_1\|F_{\mathcal{R}}\|_1\kappa_0. \qquad (4.32)$$

(ii) The robust stability of the feedback system with $(K, \mathrm{En}) = (K_0 \circ \mathrm{D}^{\epsilon_1}_{\mathrm{pq}}, \mathrm{E}^{\epsilon_1}_{\mathrm{pq}})$ is implicit from the strict causality of $\mathcal{F}_\mathrm{l}(P, K\circ\mathrm{En})$, Proposition 1, Theorem 5, and (4.10).

Appendix 4.C Proof of Corollary 1

We first prove the former part of the theorem. Assume that there exists an encoder-controller pair such that $\mathcal{F}_\mathrm{l}(P, K \circ \mathrm{En})$ satisfies (4.10) for some $\epsilon_1 > 0$ and $\gamma_1 \in (0, \gamma)$.

We define $D(t) := \sup_{\|e_1\|_{\ell^\infty}\le\epsilon_1} \|x(t)\|_\infty$. Then, it was shown in [30] that, with any controller K and encoder En, $D(t)$ satisfies

$$\limsup_{t\to\infty} D(t) \ge \begin{cases} \frac{\epsilon_1}{1-\frac{|a|}{2^R}} & (2^R > |a|), \\ \infty & (2^R \le |a|). \end{cases} \qquad (4.33)$$

This implies that if $2^R \le |a|$, for any positive number ϵ_M, there exists a truncation time τ satisfying $\|x|_{[0,\tau]}\|_{\ell^\infty} \ge \epsilon_M$, which contradicts to $\|z_1|_{[0,\tau]}\|_{\ell^\infty} \le \gamma_1\epsilon_1\ \forall\tau \ge 0$. Thus, $2^R > |a|$ must be true.

Furthermore, to obtain (4.16), assume that $|c|/(1-|a|/2^R) > \gamma$. Then, it follows from (4.33) that there exist a positive constant ϵ'_M and a truncation time τ satisfying

$$\frac{\epsilon_1}{1-\frac{|a|}{2^R}} > \epsilon'_M > \frac{\gamma\epsilon_1}{|c|},$$
$$\|x|_{[0,\tau]}\|_{\ell^\infty} > \epsilon'_M.$$

It follows from $z_1(t) = cx(t)$ that $\|z_1|_{[0,\tau]}\|_{\ell^\infty} > |c|\epsilon'_M > \gamma\epsilon_1$, a contradiction. Hence, (4.16) holds.

The latter part is proved in the same manner as Theorem 10 with $n = 1$, $E = 1$, $K_0 = -a$, $F_{\mathcal{R}} = 1/2^R$, $\bar{J} = |a|$, and $H = 1$. We have $\hbar = 1$ in this case. Note also that $K_0 = -a$ implies $\gamma_0 = |c|$ and $\kappa_0 = |c||a|$, and that $-a \in \mathbf{K}_0^\gamma$, i.e., $|c| < \gamma$ follows from (4.18).

Chapter 5

Stability Analysis of Networked Control Systems Subject to Packet Dropouts and Finite-level Quantization

This chapter examines the stability of a networked control system subject to the combined effect of packet dropouts and finite-level quantization. This provides a second example that shows the usefulness of the framework that has been developed in Chapter 3.

For the stabilization of a networked control system with packet dropouts and quantization, Matveev and Savkin have obtained a negative result in a stochastic setting. They have shown that an unstable linear plant subject to arbitrarily and uniformly small external disturbances can never be *almost surely stabilized* over a rate-limited communication channel under the assumption that packet dropouts occur with a certain positive probability.

In contrast, in this chapter we consider the stability analysis of the networked control systems in a different manner. Specifically, we employ a deterministic setting in which the number of consecutive packet dropouts is bounded, and study the small ℓ^∞ signal ℓ^∞ stability of networked control systems. The packet dropouts property of the channel is expressed by the maximum number of consecutive dropouts, instead of the probability of packet dropouts. By employing the framework proposed in Chapter 3, we derive a sufficient condition on quantization for the small ℓ^∞ signal ℓ^∞ stability of the networked control system. This chapter demonstrates the usefulness of the proposed framework for the stability analysis of quantized feedback systems.

The remainder of this chapter is organized as follows. Section 5.1 de-

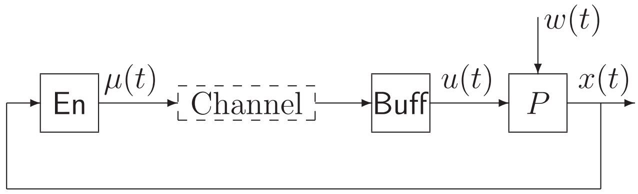

Figure 5.1: Networked control system with quantized control values and packet dropouts

scribes the system and the problem formulation. Section 5.2 discusses the decomposition of the networked control system into a feedback interconnection of appropriate subsystems for stability analysis. Section 5.3 discusses the stability analysis. Section 5.4 presents numerical examples. Section 5.5 concludes this chapter.

5.1 Problem formulation

We consider a networked control system with an unreliable communication channel affected by packet dropouts, as shown in Figure 5.1. To alleviate the effect of packet dropouts, the networked control system incorporates a buffering mechanism in the feedback loop (see, e.g., [25, 24] for the buffering mechanism). However, unlike the setting studied in [25, 24], here, we explicitly take quantization effects. More precisely, the encoder-controller sends out quantized values of current and finite step future control signals at each time instant.

The system is described in detail below.

Plant *P*: The plant P is a discrete-time LTI system whose state-space representation is given by

$$x(t+1) = Ax(t) + Bu(t) + w(t). \tag{5.1}$$

The signals $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}$, and $w(t) \in \mathbb{R}^n$ are the plant state, actuator input, and process disturbance, respectively. The initial state $x(0)$ is assumed to be zero.

Channel: The communication channel is affected by packet dropouts. The packet dropout is characterized in terms of the discrete variable $s(t)$ defined by

$$s(t) = \begin{cases} 1 & \text{if a packet dropout does not occur at time } t, \\ 0 & \text{if a packet dropout occurs at time } t. \end{cases} \tag{5.2}$$

Whenever packet dropouts do not occur, the channel transmits the current control packet to the buffer Buff without errors or delays.

In the sequel, we denote the time instants when the transmission is successfully completed (i.e., a packet is not dropped) with $\{t_0, t_1, t_2, \cdots\}$, and we assume $t_0 = 0$. We thus have

$$s(t) = 1 \Leftrightarrow (\ t = t_i \text{ for some } i \in \mathbb{Z}_+\ ). \tag{5.3}$$

Controller-Encoder En: Throughout this work, we will assume that no acknowledgments of receipt are available. Thus, the controller-encoder En does not know whether previous data packets were successfully received.

To compensate for possible future packet dropouts, at every time $t \in \mathbb{N}$, the controller-encoder transmits a control packet, say, $\mu(t) \in \mathbb{R}^N$, to the buffer. The value $N \in \mathbb{N}$ is given; it represents the packet size as well as the buffer length. The control packet is composed of quantized potential control inputs for the current time instant and $N-1$ future time instants:

$$\mu(t) = \begin{bmatrix} q(\hat{u}(t;t)) \\ q(\hat{u}(t+1;t)) \\ \vdots \\ q(\hat{u}(t+N-1;t)) \end{bmatrix}, \tag{5.4}$$

$$\hat{u}(t+i;t) = K\hat{x}(t+i;t), \quad i \in \{0, \cdots, N-1\}, \tag{5.5}$$

where $\hat{u}(t+i;t) \in \mathbb{R}$ and $\hat{x}(t+i;t) \in \mathbb{R}^n$ are the i-step predictions of the (unquantized) control input and the plant state based on the current state $x(t)$, respectively, and where $K \in \mathbb{R}^{1\times n}$ is a static state-feedback gain.

In (5.4), the function q denotes the uniform static quantizer which is defined by (2.19) in Subsection 2.3.2. The constant d is the step size, and $M := 2m+1$ is the number of quantization levels.[1]

The state predictions $\hat{x}(t+i;t)$ $(i = 1, 2, \cdots, N-1)$ are calculated recursively based on the current state $x(t)$ and the plant dynamics as follows:

$$\hat{x}(t+i;t) = \begin{cases} x(t), & \text{if } i = 0, \\ A\hat{x}(t+i-1;t) + Bq(\hat{u}(t+i-1;t)), & \text{if } i = 1, \cdots, N-1. \end{cases} \tag{5.6}$$

Buffer Buff: The buffer Buff provides the actuator values based on the received channel symbols. The state of the buffer is updated every time it successfully receives the packet μ. To be more precise, the dynamics of Buff is described by

$$b(t) = s(t)\mu(t) + (1 - s(t))Sb(t-1), \quad b(-1) = 0, \tag{5.7}$$

[1] Owing to quantization, each control packet $\mu(t)$ can take only one of M^N different values, and, can thus be expressed via $N \log_2 M$ bits.

where $b(t) \in \mathbb{R}^N$ denotes the state of Buff, and

$$S := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix} \in \mathbb{R}^{N \times N} \tag{5.8}$$

is a shift matrix. Then, the plant input $u(t)$ is given by

$$u(t) = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix} b(t); \tag{5.9}$$

see Figure 5.1.

Remark 2. *The buffering technique adopted here was used, e.g., in [24] to study the input-to-state stability [31] of a related networked control system, where the control packets are designed without quantization by adapting the model predictive control framework. In this chapter, we complement [24] by studying a networked control system subject to packet dropouts as well as finite-level quantization of control signals.*

We make the following assumptions:

Assumption 5. *The number of consecutive packet dropouts is bounded by the buffer length N, i.e., we have*

$$1 \leq t_{i+1} - t_i \leq N, \quad \forall i \in \mathbb{Z}_+. \tag{5.10}$$

Assumption 6. *The matrix $A_K := A + BK$ is Schur stable.*

If Assumption 5 is satisfied, then the buffer length N is equal to the maximal number of consecutive packet dropouts. Assumption 6 implies that the controller gain K in (5.5) stabilizes the plant model (5.1) in the absence of dropouts or quantization constraints.

We recall that in the presence of finite-level quantization, the finite gain ℓ^p stability of the closed-loop system cannot be established (see [16] and Subsection 2.3.1); in this chapter, we are interested in *the small ℓ^∞ signal ℓ^∞ stability* of the closed-map from w to x: [2]

[2] Note that condition (5.11) differs from the small ℓ^∞ signal ℓ^∞ stability of the feedback system shown in Figure 2.1.

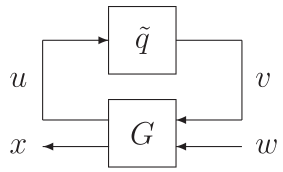

Figure 5.2: Linear fractional representation of the networked control system

Under Assumptions 5 and 6, we wish to derive a sufficient condition for the existence of positive constants ϵ and γ such that

$$\llbracket \; \|w|_{[0,\tau]}\|_{\ell^\infty} \leq \epsilon \Rightarrow \|x|_{[0,\tau]}\|_{\ell^\infty} \leq \gamma\epsilon \; \rrbracket, \quad \forall w \in \ell^\infty, \; \forall \tau \in \mathbb{Z}_+ \tag{5.11}$$

for any sequence of packet dropouts satisfying Assumption 5. Furthermore, if the stability condition is satisfied, we wish to characterize the relationship (trade-off) between the attenuation level γ and the parameters M, d, and N.

We hereafter decompose the networked control system into a feedback interconnection of subsystems that are both small ℓ^∞ signal ℓ^∞ stable. The framework proposed in Chapter 3 is employed to to obtain a sufficient condition for achieving (5.11) for the networked control system shown in Figure 5.1.

5.2 Linear fractional transformation model

The networked control system shown in Figure 5.1 is nonlinear and time-varying due to quantization, packet dropouts, and buffering. To study the stability of the networked control system, we first extract the nonlinearity (denoted by $\tilde{q}$) associated with the quantization error to re-formulate the feedback system shown in Figure 5.1 as the linear fractional transformation shown in Figure 5.2, where the "nominal" subsystem G is linear and time-varying.

It follows from (5.3), (5.4), (5.7), (5.8), and (5.9) that the plant inputs at the time instants

$$\mathbb{Z}_+ = \cup_{i \in \mathbb{Z}_+} \{t_i, t_i + 1, \ldots, t_{i+1} - 1\}$$

are given by

$$u(t) = q(\hat{u}(t; t_i)), \quad t \in \{t_i, t_i + 1, \ldots, t_{i+1} - 1\}. \tag{5.12}$$

Thus, the plant model (5.1) can be rewritten as

$$x(t+1) = Ax(t) + Bq(\hat{u}(t;t_i)) + w(t),\ t \in \{t_i, t_i+1, \ldots, t_{i+1}-1\}. \quad (5.13)$$

If we denote the quantization error at each time instant $t \in \mathbb{Z}_+$ via

$$v(t) := q(\hat{u}(t;t_i)) - \hat{u}(t;t_i), \quad t \in \{t_i, t_i+1, \ldots, t_{i+1}-1\}, \quad (5.14)$$

then it follows from (5.6) and (5.14) that

$$\begin{aligned} \hat{x}(t+1;t_i) &= A\hat{x}(t;t_i) + B\big(\hat{u}(t;t_i) + v(t)\big) \\ &= A_K\hat{x}(t;t_i) + Bv(t), \quad t \in \{t_i, t_i+1, \ldots, t_{i+1}-2\}. \end{aligned} \quad (5.15)$$

Moreover, (5.6) and (5.13) give

$$x(t) = \hat{x}(t;t_i) + \sum_{l=1}^{t-t_i} A^{l-1}w(t-l), \quad t \in \{t_i, t_i+1, \ldots, t_{i+1}-1\}, \quad (5.16)$$

and thus,

$$\begin{aligned} \hat{x}(t_{i+1};t_{i+1}) &= x(t_{i+1}) \\ &= \hat{x}(t_{i+1};t_i) + \sum_{l=1}^{t_{i+1}-t_i} A^{l-1}w(t_{i+1}-l) \\ &= A_K\hat{x}(t_{i+1}-1;t_i) + Bv(t_{i+1}-1;t_i) \\ &\quad + \sum_{l=1}^{t_{i+1}-t_i} A^{l-1}w(t_{i+1}-l), \quad i \in \mathbb{Z}_+. \end{aligned} \quad (5.17)$$

It follows from (5.5), (5.15), (5.16), and (5.17) that the networked control system shown in Figure 5.1 can be described in terms of the linear fractional transformation of a linear subsystem G and the static nonlinear function $\tilde{q}$, as shown in Figure 5.2.

The nonlinearity $\tilde{q}$ is a static map defined by

$$\tilde{q}(\hat{u}) = q(\hat{u}) - \hat{u}, \quad (5.18)$$

and thus, it represents the quantization error, as shown in Figure 5.3. It can be easily verified from (2.19) that $\tilde{q}$ satisfies

$$\left[\!\left[\|\hat{u}|_{[0,\tau]}\|_{\ell^\infty} \le \frac{Md}{2} \Rightarrow \|\tilde{q}(\hat{u})|_{[0,\tau]}\|_{\ell^\infty} \le \frac{d}{2} \right]\!\right], \quad \forall \hat{u} \in \ell_e,\ \forall \tau \in \mathbb{Z}_+. \quad (5.19)$$

In other words, $\tilde{q}$ has small ℓ^∞ signal ℓ^∞ stability with attenuation level $1/M$ and input bound $Md/2$.

To characterize the subsystem G shown in Figure 5.2, it is convenient to introduce a state vector $\xi(t)$ and control signal $\hat{u}(t)$ via

$$\xi(t) := \hat{x}(t;t_i), \quad \hat{u}(t) := \hat{u}(t;t_i), \quad t \in \{t_i, t_i+1, \ldots, t_{i+1}-1\},\ i \in \mathbb{Z}_+ \tag{5.20}$$

This allows us to describe G in the state-space form via

$$G: \begin{cases} \xi(t+1) = A_K\xi(t) + Bv(t) + F_1 w(t), \\ \hat{u}(t) = K\xi(t), \\ x(t) = \xi(t) + F_2 w(t), \end{cases} \tag{5.21}$$

where F_1 and F_2 are linear time-varying maps defined by

$$F_1 : w \mapsto \begin{cases} 0, & \text{if } t \in \{t_i, t_i+1, \ldots, t_{i+1}-2\}, \\ \displaystyle\sum_{l=1}^{t_{i+1}-t_i} A^{l-1} w(t_{i+1}-l), & \text{if } t = t_{i+1}-1, \end{cases} \tag{5.22}$$

$$F_2 : w \mapsto \begin{cases} 0, & \text{if } t = t_i, \\ \displaystyle\sum_{l=1}^{t-t_i} A^{l-1} w(t-l), & \text{if } t \in \{t_i+1, t_i+2, \ldots, t_{i+1}-1\}. \end{cases} \tag{5.23}$$

Interestingly, the map G: $(v,w) \mapsto (u,x)$ can be decomposed as

$$G = \begin{bmatrix} G_{00} & G_{01} \circ F_1 \\ G_{10} & G_{11} \circ F_1 + F_2 \end{bmatrix}, \tag{5.24}$$

where G_{00}, G_{01}, G_{10}, and G_{11} are LTI maps with impulse responses

$$g_{00}(t) = \begin{cases} 0, & \text{if } t = 0, \\ KA_K^{t-1}B, & \text{if } t \in \mathbb{N}, \end{cases} \qquad g_{01}(t) = \begin{cases} 0, & \text{if } t = 0, \\ KA_K^{t-1}, & \text{if } t \in \mathbb{N}, \end{cases}$$

$$g_{10}(t) = \begin{cases} 0, & \text{if } t = 0, \\ A_K^{t-1}B, & \text{if } t \in \mathbb{N}, \end{cases} \qquad g_{11}(t) = \begin{cases} 0, & \text{if } t = 0, \\ A_K^{t-1}, & \text{if } t \in \mathbb{N}, \end{cases}$$

respectively. It is easily seen from these equations that the effect of packet dropouts is confined in the time-varying maps F_1 and F_2.

5.3 Stability analysis

With the previous section as a background, we will study the small ℓ^∞ signal ℓ^p stability of the closed-map from w to x.

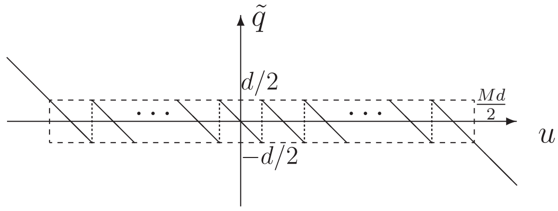

Figure 5.3: Nonlinearity associated with quantization error

5.3.1 Preliminaries

A key property of the linear fractional transformation model presented in Section 5.2 is that if Assumption 6 holds, then the LTI maps G_{00}, G_{01}, G_{10}, and G_{11} are stable and have finite ℓ^∞ gains. Furthermore, it is easy to see that F_1 and F_2 also have finite ℓ^∞ gains:

Lemma 3. *Suppose that Assumption 5 holds. Then, the maps F_1 and F_2 defined in* (5.22) *and* (5.23) *are unbiased finite gain ℓ^∞ stable, and*

$$\|F_1\|_{\ell^\infty\text{-ind}} \leq \kappa, \quad \|F_2\|_{\ell^\infty\text{-ind}} \leq \kappa \tag{5.25}$$

hold for

$$\kappa := \sum_{l=0}^{N-1} \|A^l\|_1 < \infty. \tag{5.26}$$

Proof. See Appendix 5.A. □

The linear fractional transformation model provides important information for the stability analysis of the networked control system shown in Figure 5.1. By virtue of the buffering technique, the adverse effect of packet dropouts is isolated from the feedback loop. Indeed, the maximal length N of consecutive packet dropouts is contained only in the stable feedforward maps F_1 and F_2. This implies that packet dropouts play no role in the deteriorating stability of the networked control system.

5.3.2 Main results

Lemma 4 stated below shows that because all 4 components of G are finite gain ℓ^∞ stable, the stability analysis of the overall networked control system reduces to investigating the stability of the feedback interconnection between G_{00} and $\tilde{q}$ shown in Figure 5.4. To state the result, we denote the ℓ^∞ gains

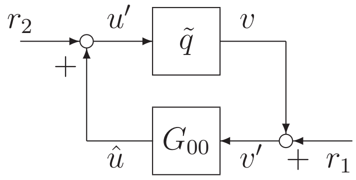

Figure 5.4: Feedback interconnection of G_{00} and $\tilde{q}$

of the components of G via

$$\begin{aligned}\gamma_{00} &:= \|G_{00}\|_{\ell^\infty\text{-ind}}, \quad \gamma_{01} := \|G_{01}\|_{\ell^\infty\text{-ind}},\\ \gamma_{10} &:= \|G_{10}\|_{\ell^\infty\text{-ind}}, \quad \gamma_{11} := \|G_{11}\|_{\ell^\infty\text{-ind}}.\end{aligned} \tag{5.27}$$

Lemma 4. *Assume that the feedback interconnection of G_{00} and $\tilde{q}$ shown in Figure 5.4 is small ℓ^∞ signal ℓ^∞ stable, i.e., there exist positive constants ϵ_r, δ_u, and δ_v such that*

$$\left[\!\!\left[\left\| \begin{bmatrix} r_1 \\ r_2 \end{bmatrix}\Big|_{[0,\tau]} \right\|_{\ell^\infty} \leq \epsilon_r \Rightarrow \left(\|\hat{u}|_{[0,\tau]}\|_{\ell^\infty} \leq \delta_u \text{ and } \|v|_{[0,\tau]}\|_{\ell^\infty} \leq \delta_v\right) \right]\!\!\right], \quad \forall r_1, r_2 \in \ell_e,\ \forall \tau \in \mathbb{Z}_+. \tag{5.28}$$

Then, the feedback system shown in Figure 5.2 satisfies

$$[\![\ \|w|_{[0,\tau]}\|_{\ell^\infty} \leq \epsilon \Rightarrow \|x|_{[0,\tau]}\|_{\ell^\infty} \leq \gamma\epsilon\ ]\!], \quad \forall w \in \ell_e,\ \forall \tau \in \mathbb{Z}_+. \tag{5.29}$$

for any sequences of packet dropouts satisfying Assumption 5, where

$$\epsilon = \frac{\epsilon_r}{\gamma_{01}\kappa}, \quad \gamma = \frac{\gamma_{01}\kappa\left((\gamma_{11}+1)\epsilon_r\kappa + \gamma_{10}\delta_v\right)}{\epsilon_r}, \tag{5.30}$$

and κ is defined in (5.26).

Proof. See Appendix 5.B. □

Having established Lemma 4, we will next present the main result of this chapter, namely, a sufficient condition for the small ℓ^∞ signal ℓ^∞ stability of the closed-loop map from w to x, as shown shown in Figure 5.2. This also quantifies the disturbance attenuation level in terms of the parameters of the quantizer and the buffer length.

Theorem 11. *Suppose that Assumptions 5 and 6 hold. If*

$$\gamma_{00} < M, \tag{5.31}$$

then the networked control system in Figure 5.1 satisfies

$$[\![\ \|w|_{[0,\tau]}\|_{\ell^\infty} \leq \epsilon \Rightarrow \|x|_{[0,\tau]}\|_{\ell^\infty} \leq \gamma\epsilon \ ]\!] \quad \forall w \in \ell_e^\infty, \ \forall \tau \in \mathbb{Z}_+ \tag{5.32}$$

for any sequence of packet dropouts satisfying Assumption 5, where

$$\epsilon = \frac{(M - \gamma_{00})d}{2\gamma_{01}(\gamma_{00}+1)\kappa}, \tag{5.33}$$

$$\gamma = \frac{\gamma_{01}\kappa\left(\gamma_{10}(\gamma_{00}+1) + (\gamma_{11}+1)(M-\gamma_{00})\kappa\right)}{M - \gamma_{00}}. \tag{5.34}$$

Proof. See Appendix 5.C. □

Theorem 11 establishes that the small ℓ^∞ signal ℓ^∞ stability of the closed-loop map from w to x in the presence of bounded packet dropouts and finite-level quantization can be guaranteed if a sufficiently large number of quantization levels M is available. While condition (5.31) does not depend on the step size d, this quantizer parameter does affect the input bound ϵ in (5.33). Because the input bound ϵ is monotonically increasing with respect to d, compensating for a large disturbance $w(t)$ requires a large step size d. This observation is rather intuitive, because with large d, the control signal is allowed to take large values, as shown in (2.19), in this uniform quantizer. Note also that the upper bound on $\|x\|_{\ell^\infty}$ depends on d through the input bound ϵ in (5.29).

The stability condition (5.31) also indicates that the stability of the closed-loop map from w to x is independent of the maximum number of consecutive packet dropouts N, provided sufficient control inputs are contained in each control packet (see Assumption 5). However, the disturbance attenuation level of the closed-loop map from w to x is strongly affected by N. In fact, it can be seen from (5.26) and (5.34) that for open-loop unstable plants, the attenuation level γ exponentially increases with respect to N. This suggests that the magnitude of the state x may become extremely large if the network introduces too many consecutive packet dropouts. The latter observation is hardly surprising, because, during periods of consecutive dropouts, the plant is unavoidably left in an open loop.

5.4 Numerical examples

5.4.1 Example 1 (scalar plant)

Consider a scalar plant described by

$$x(t+1) = ax(t) + u(t) + w(t), \quad a > 1.$$

We choose the stabilizing (deadbeat) feedback gain $K = -a$, providing $A_K = A + BK = 0$. In this case, the ℓ^∞ gains in (5.27) are given by

$$\gamma_{00} = a, \quad \gamma_{01} = a, \quad \gamma_{10} = 1, \quad \gamma_{11} = 1,$$

and the sufficient stability condition (5.31) becomes

$$M > a. \tag{5.35}$$

It then follows from (5.26), (5.33), and (5.34) that the parameters κ, ϵ, and γ are given by

$$\kappa = \sum_{l=0}^{N-1} a^l = \frac{a^N - 1}{a - 1}, \tag{5.36}$$

$$\epsilon = \frac{(M-a)d}{2(a+1)\kappa} = \frac{(M-a)d}{2(a+1)} \frac{a-1}{a^N - 1}, \tag{5.37}$$

$$\gamma = \frac{a\kappa((a+1) + 2(M-a)\kappa)}{M-a} = \frac{a(a+1)}{M-a}\frac{a^N-1}{a-1} + 2a\left(\frac{a^N-1}{a-1}\right)^2. \tag{5.38}$$

Interestingly, it turns out that for scalar plant models, stability condition (5.35) is tight. In fact, it is known that $M \geq a$ is a necessary condition to stabilize the scalar system over a rate-limited channel when there are no packet dropouts ($N = 1$); see, e.g., [32].

As a special case, suppose that $a = 2.99$. Then, the stability condition (5.35) becomes $M > 2.99$, which corresponds to a quantizer with at least three levels. For this fixed a, the associated trade-off between the attenuation level γ in (5.38) and the maximal number of consecutive packet dropouts N for a quantizer with $M = 11$ quantization levels is shown in Figure 5.5. As can be seen from this figure and (5.38), for a fixed M, γ exponentially increases in N.

Figure 5.6 shows the relationship between γ and M for fixed $N = 2$. It can be seen from this figure that when M is close to $\gamma_{00} = a = 2.99$, the attenuation level γ becomes very large. On the other hand, the attenuation level monotonically decreases with respect to M, and we have $\lim_{M\to\infty} \gamma = 2a(a^2-1)^2/(a-1)^2 = 95.20$ as a lower bound of γ in (5.38).

We next performed a numerical simulation for a critical case where[3]

$$w(t) = \epsilon, \quad \forall t \in \{0, 1, \ldots 300\}$$
$$t_{i+1} - t_i = N = 2, \quad \forall i \in \{0, 1, \ldots 150\}.$$

[3] Namely, the disturbance takes the extreme value at each time instant, and the network periodically drops packets every other time instant.

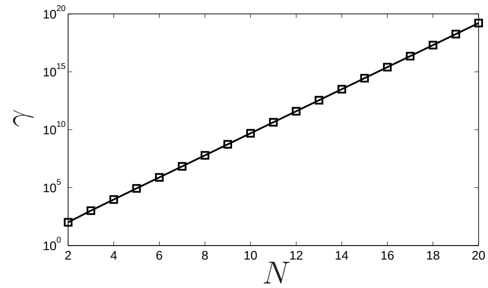

Figure 5.5: Effect of the maximal number of consecutive dropouts N on the attenuation level γ

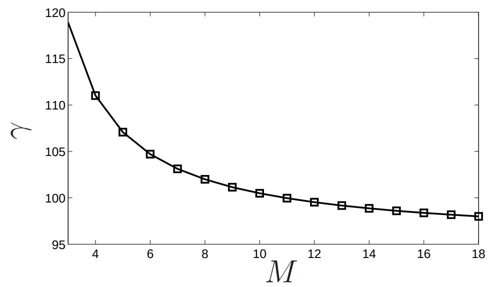

Figure 5.6: Trade-off between the attenuation level γ and the number of quantization levels M

In this simulation, we choose $M = 3$, $d = 2$, and $N = 2$. Then, it follows from (5.36) that $\kappa = 3.99$. Theorem 3 then guarantees that the closed-loop map from w to x is small ℓ^∞ signal ℓ^∞ stable, and that (5.32) holds with

$$\epsilon = 2.1008 \times 10^{-4}, \quad \gamma = 4.8553 \times 10^3.$$

The simulation result of $x(t)$ for $t \in \{0, 1, \ldots 300\}$ is shown in Figure 5.7, confirming (5.32), namely, that $\|x(t)\|_\infty$ is less than $\delta := \gamma\epsilon = 1.02$ for all $t \in \{0, 1, \ldots 300\}$.

5.4.2 Example 2 (third-order plant)

To verify the effectiveness of the buffering scheme considered in this chapter, we next carry out simulations for the following three control schemes:

(a) Buffering scheme:
The buffering scheme considered in the previous sections is applied to compensate for packet dropouts.

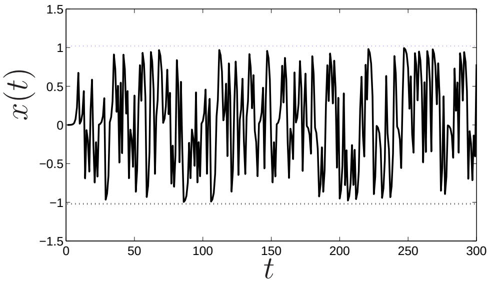

Figure 5.7: Plant state trajectory in the presence of dropouts and disturbances

(b) Zero input scheme ($s(t) = 0 \Rightarrow u(t) = 0$):
If a packet is dropped, the actuator applies zero input to the plant.

(c) Previous input scheme ($s(t) = 0 \Rightarrow u(t) = u(t-1)$):
If a packet is dropped, the actuator applies the previous input to the plant.

Consider the third-order plant given by

$$x(t+1) = \begin{bmatrix} 0.1 & 2.3 & 1.4 \\ 0 & 2 & 1.5 \\ 0 & 0.9 & 1.6 \end{bmatrix} x(t) + \begin{bmatrix} 1.1 \\ 0.9 \\ 1 \end{bmatrix} u(t) + w(t),$$

and the communication channel with $N = 3$. We choose the nominally stabilizing feedback gain as

$$K = \begin{bmatrix} 0.0013 & -1.6591 & -1.5782 \end{bmatrix}.$$

For this networked control system, we have $\kappa = 42.63$ and

$$\gamma_{00} = 3.7545, \quad \gamma_{01} = 4.3528, \quad \gamma_{10} = 6.4171, \quad \gamma_{11} = 7.4988.$$

We choose $M = 5$ so that the stability condition (5.31) is satisfied. We carried out the simulations with random disturbances satisfying $\|w(t)\|_\infty \leq \epsilon = 5.8918 \times 10^{-4}$ and periodic packet dropout sequences consisting of two consecutive dropouts and one success.

The simulation results are shown in Figure 5.8. It is clearly seen from this figure that the buffering scheme (a) succeeds in keeping $\|x(t)\|_\infty$ much smaller as compared to the zero input schemes (b) and (c). Furthermore, Figure 5.8 shows that the previous input scheme (c) does not stabilize this plant with this particular choice of K and M because the plant state is unbounded.

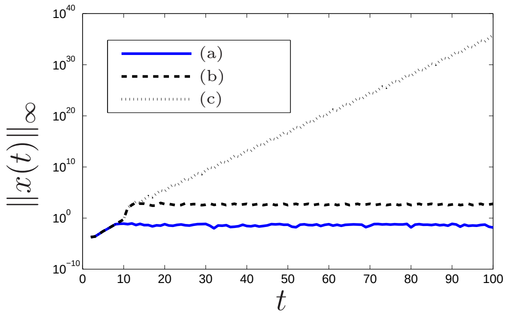

Figure 5.8: Simulation results for the control schemes (a), (b), and (c)

Note also that in the proposed scheme, it is sufficient to choose the feedback gain K such that the matrix $A + BK$ is stable. This stands in stark contrast to the zero input scheme, where in the presence of packet dropouts, a necessary condition for the stability of the networked control system is that all finite products

$$(A + BK)A^i, \quad i \in \{1, \cdots, N-1\}$$

are stable. The class of admissible feedback gains is therefore quite restricted in the zero input case. On the other hand, in the proposed scheme, the constraint on the feedback gain is much milder. This is useful when designing the networked control system.

5.5 Summary

This chapter discusses the small ℓ^∞ signal ℓ^∞ stability of a networked control system subject to disturbances, packet dropouts, and finite-level quantization. We have employed a deterministic setting and the small ℓ^∞ signal ℓ^∞ stability analysis framework for the stability analysis. Within this framework, we have shown that by incorporating a buffering mechanism at the receiving end of the channel, the adverse effect of packet dropouts on closed loop stability can be canceled as long as the number of consecutive dropouts is smaller than the buffer length. A sufficient condition for stability, which is stated in terms of the number of quantization levels, has been derived. We have also elucidated the effect of the quantizer step size and the maximal number of consecutive packet dropouts on the disturbance attenuation level. Through the stability analysis of this networked control system, this chapter has shown the usefulness of the framework developed in this thesis.

Appendix 5.A Proof of Lemma 3

We give the proof of only $\|F_1\|_{\ell^\infty\text{-ind}} \leq \kappa$, because $\|F_2\|_{\ell^\infty\text{-ind}} \leq \kappa$ can be proved in the same manner.

Define $f_1 = F_1 w$ for $w \in \ell_e$. There always exists $i \in \mathbb{Z}_+$ such that $t \in \{t_i, t_i+1, \cdots, t_{i+1}-1\}$, and $F_1 w$ is given by (5.22). Clearly, $\|f_1(t)\|_\infty = 0 \leq \kappa\|w\|_{\ell^\infty}$ holds for $t \in \{t_i, \cdots, t_{i+1}-2\}$. In the case of $t = t_{i+1}-1$, we obtain

$$\begin{aligned}\|f_1(t)\|_\infty &\leq \sum_{l=1}^{t_{i+1}-t_i} \|A^{l-1}w(t_{i+1}-l)\|_\infty \leq \sum_{l=1}^{t_{i+1}-t_i} \|A^{l-1}\|_1\|w(t_{i+1}-l)\|_\infty \\ &\leq \sum_{l=1}^{t_{i+1}-t_i} \|A^{l-1}\|_1\|w\|_{\ell^\infty} \leq \sum_{l=1}^{N} \|A^{l-1}\|_1\|w\|_{\ell^\infty} = \kappa\|w\|_{\ell^\infty}.\end{aligned}$$

In the last inequality, we have used Assumption 5.

Appendix 5.B Proof of Lemma 4

From (5.24), the input-output relationships shown in Figure 5.2 are described by

$$v = \tilde{q}(\hat{u}), \tag{5.39}$$
$$\hat{u} = G_{00}v + (G_{01} \circ F_1)w, \tag{5.40}$$
$$x = G_{10}v + (G_{11} \circ F_1 + F_2)w. \tag{5.41}$$

On the other hand, Lemma 3 gives

$$\|G_{01} \circ F_1\|_{\ell^\infty\text{-ind}} \leq \|G_{01}\|_{\ell^\infty\text{-ind}}\|F_1\|_{\ell^\infty\text{-ind}} \leq \gamma_{01}\kappa, \tag{5.42}$$
$$\|G_{11} \circ F_1 + F_2\|_{\ell^\infty\text{-ind}} \leq \|G_{11}\|_{\ell^\infty\text{-ind}}\|F_1\|_{\ell^\infty\text{-ind}} + \|F_2\|_{\ell^\infty\text{-ind}} \leq \gamma_{11}\kappa + \kappa. \tag{5.43}$$

It is easily seen by taking $r_1 := 0$, $r_2 := (G_{01} \circ F_1)w$, $v' := v$ in Figure 5.4 that the small ℓ^∞ signal ℓ^∞ stability of the feedback interconnection $(G_{00}, \tilde{q})$ implies that

$$\|(G_{01} \circ F_1 w)|_{[0,\tau]}\|_{\ell^\infty} \leq \epsilon_r \Rightarrow (\|\hat{u}|_{[0,\tau]}\|_{\ell^\infty} \leq \delta_u \text{ and } \|v|_{[0,\tau]}\|_{\ell^\infty} \leq \delta_v), \quad \forall \tau \in \mathbb{Z}_+ \tag{5.44}$$

holds for the feedback system of (5.39) - (5.41) shown in Figure 5.2.

Now, we assume that $\|w|_{[0,\tau]}\|_{\ell^\infty} \leq \epsilon$, where ϵ is defined by (5.30). To complete the proof, we only need to establish the boundedness of $\|x|_{[0,\tau]}\|_{\ell^\infty}$.

It follows from (5.42) that

$$\|(G_{01}\circ F_1 w)|_{[0,\tau]}\|_{\ell^\infty} \le \|G_{01}\circ F_1\|_{\ell^\infty\text{-ind}}\|w|_{[0,\tau]}\|_{\ell^\infty} \le \gamma_{01}\kappa\epsilon = \epsilon_r. \tag{5.45}$$

We then have $\|v|_{[0,\tau]}\|_{\ell^\infty} \le \delta_v$ from (5.44) and (5.45). Therefore, we conclude from (5.41) and (5.43) that

$$\begin{aligned}\|x|_{[0,\tau]}\|_{\ell^\infty} &\le \|G_{00}\|_{\ell^\infty\text{-ind}}\|v|_{[0,\tau]}\|_{\ell^\infty} + \|G_{11}\circ F_1 + F_2\|_{\ell^\infty\text{-ind}}\|w|_{[0,\tau]}\|_{\ell^\infty}\\ &\le \gamma_{00}\delta_v + (\gamma_{11}+1)\kappa\epsilon\\ &= \gamma\epsilon < +\infty,\end{aligned} \tag{5.46}$$

where $\gamma := \gamma_{00}\delta_v/\epsilon + (\gamma_{11}+1)\kappa$.

Appendix 5.C Proof of Theorem 11

Suppose that (5.31) holds. Because G_{00} is strictly causal, it is easily verified that $\gamma_{00} = \|G_{00}\|_{\ell^\infty\text{-ind}}$ implies

$$\|(G_{00}v)|_{[0,\tau]}\|_{\ell^\infty} \le \gamma_{00}\|v|_{[0,\tau-1]}\|_{\ell^\infty}, \quad \forall v \in \ell_e,\ \forall \tau \in \mathbb{Z}_+ \tag{5.47}$$

(see Theorem 5). It then follows from (5.47), (5.31), (5.19), and Theorem 4 that the feedback interconnection of G_{00} and $\tilde{q}$ shown in Figure 5.4 is small ℓ^∞ signal ℓ^∞ stable. In particular, we have

$$[\![\ \|r|_{[0,\tau]}\|_{\ell^\infty} \le \epsilon_r \Rightarrow (\|u|_{[0,\tau]}\|_{\ell^\infty} \le \delta_u \text{ and } \|v|_{[0,\tau]}\|_{\ell^\infty} \le \delta_v)\ ]\!],\quad \forall r \in \ell_e,\ \forall \tau \in \mathbb{Z}_+$$

where the constants

$$\epsilon_r = \frac{(M-\gamma_{00})d}{2(1+\gamma_{00})}, \quad \delta_u = \frac{(M+1)\gamma_{00}d}{2(1+\gamma_{00})}, \quad \delta_v = \frac{d}{2} \tag{5.48}$$

are obtained by substituting $\gamma_2 := \gamma_{00}$, $\gamma_1 := 1/M$, and $\epsilon_1 := Md/2$ into (3.11).

Consequently, we conclude from Lemma 4 that the networked control system shown in Figure 5.1, or equivalently, in Figure 5.2, satisfies (5.32), where ϵ and γ in (5.33) and (5.34) are obtained by substituting (5.48) into (5.30).

Chapter 6

Conclusion

In this thesis, we have

- provided the motivation for developing a new framework for the stability analysis of quantized feedback systems,
- developed a new framework based on a new stability notion,
- demonstrated its usefulness through examples of quantized feedback systems.

As a concluding chapter, we now summarize the contributions of this thesis.

In Chapter 2, we have revealed the difficulties involved in applying the traditional small gain theorem to the stability analysis of quantized feedback systems. Specifically, by utilizing two examples of quantized feedback systems, we have provided the motivation for introducing a new stability notion and a new framework for stability analysis that are applicable to quantized feedback systems. Stability analyses of an uncertain networked control system with a rate-limited communication channel and a feedback system involving a uniform quantizer have been discussed.

In Chapter 3, a new framework for the stability analysis of quantized feedback systems has been developed. We have introduced a new notion of stability, *small ℓ^p signal ℓ^p stability*, that is achievable in a wide class of quantized feedback systems. Then, we have prepared mathematical tools to establish the stability of a feedback system in the sense of the new stability. In particular, *the small level theorem* has been derived as a key theorem for stability analysis. It provides a sufficient condition for a feedback system to be small ℓ^p signal ℓ^p stable. A new class of uncertainty, *level bounded uncertainty*, has also been introduced. This uncertainty effectively approximates some classes of nonlinearities that include quantization errors. Robust stability against the level bounded uncertainty and the traditional gain bounded uncertainty have been investigated.

Chapter 4 provides the first example that shows the usefulness of the proposed framework for the stability analysis of quantized feedback systems. We have studied the robust stabilization of an uncertain networked control system over a rate-limited communication channel. By using the proposed framework, we have elucidated the combined effect of quantization at the communication channel and the model uncertainty in the plant dynamics, on the robust stabilizability of the entire networked control system. A design method for a stabilizing encoder-controller pair has also been presented.

Finally, Chapter 5 provides the second example that demonstrates the usefulness of the proposed framework. In this chapter, we have examined the stability of a networked control system subject to the combined effect of finite-level quantization and packet dropouts. A uniform quantizer has been employed as the quantizer in this example. In this example, we have shown that the class of level bounded uncertainty effectively approximates the nonlinearity associated with the quantization error. We have elucidated the effect of quantization and packet dropouts on the stability of the overall networked control system.

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