1 a. The pole polynomial is given by the LCD of the minors of G,

$$\frac{1}{s+1}, \quad \frac{-1}{s+1}, \quad \frac{2s}{s+1}, \quad \frac{2s}{s+2}, \quad \frac{2s(2s+3)}{(s+1)^2(s+2)}$$

which is $(s+1)^2(s+2)$. The poles are hence -1 (mult. 2) and -2 (mult. 1). The zero polynomial is given by the GCD of the maximal minor of G normalized with the pole polynomial, which is $\frac{2s(2s+3)}{(s+1)^2(s+2)}$. The zeros are hence 0 and -1.5 (both mult. 1).

b. We rewrite the transfer matrix as (for instance)

$$G(s) = \frac{\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}1 & -1\end{bmatrix}}{s+1} + \frac{\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}-2 & 0\end{bmatrix}}{s+1} + \frac{\begin{bmatrix}0\\1\end{bmatrix}\begin{bmatrix}0 & -4\end{bmatrix}}{s+2} + \begin{bmatrix}0 & 0\\2 & 2\end{bmatrix}$$

from which we get the state-space realization

$$\dot{x} = \begin{bmatrix}-1 & 0 & 0\\0 & -1 & 0\\0 & 0 & -2\end{bmatrix}x + \begin{bmatrix}1 & -1\\-2 & 0\\0 & -4\end{bmatrix}u$$
$$y = \begin{bmatrix}1 & 0 & 0\\0 & 1 & 1\end{bmatrix}x + \begin{bmatrix}0 & 0\\2 & 2\end{bmatrix}u$$

c. The RGA in stationarity is given by $G(0) .* G(0)^{-T}$, which is not possible to compute since $G(0) = \left[\begin{smallmatrix}1 & -1\\0 & 0\end{smallmatrix}\right]$ is singular. The underlying problem is the transmission zero in 0, which makes it impossible to control the system in stationarity.

d. When the input frequency goes to infinity,

$$G(s) \to \begin{bmatrix}0 & 0\\2 & 2\end{bmatrix} = M$$

From this it is immediate that only the second input should be used in order to maximize the gain. For the map $y = Mu$, the input $u = \begin{bmatrix}0 & 1\end{bmatrix}^T$ gives the output $y = \begin{bmatrix}2 & 2\end{bmatrix}$; the gain is hence

$$\|M\| = \sup_u \frac{|y|}{|u|} = \frac{\sqrt{2^2+2^2}}{\sqrt{0^2+1^2}} = 2\sqrt{2}$$

Alternatively, the given SVD $M = U\Sigma V'$ shows that the maximal singular value is $2\sqrt{2}$, and this is by definition the maximum gain. The corresponding input direction is given the corresponding column vector of U, $u = \begin{bmatrix}0\\1\end{bmatrix}$.

2. The system is non-minimum phase and has the poles $s = -2$ and $s = -5$. There are many ways to choose the Q filter for IMC, but we have to respect some fundamental limitations. Here we will use a simple choice of Q. We try to cancel the process dynamics with $Q(s)$, but use the stable counterpart $4+2s$ of the zero instead. We also need to add a pole to $Q(s)$ to make it proper, which we place in $s = -2$. We get

$$Q(s) = \frac{s^2 + 7s + 10}{(4 + 2s)(s + 2)} = \frac{s + 5}{2(s + 2)}$$

The controller becomes

$$C(s) = \frac{Q}{1 - QP} = \frac{s^2 + 7s + 10}{2(s^2 + 5s + 2)}$$

The high-frequency gain is finite and equal to (let $s \to \infty$) 0.5.

3 a. v is a (column) vector of length 2, y is a scalar, and $F(s)$ is a 2×1 system (1 input and 2 outputs).

b. We can use the Small Gain Theorem to guarantee stability if F is stable and $\|F\| < 1/\|G\|$. From the plot we read out $\|G\| \approx 2.12$, which gives the condition $\|F\| < 0.47$.

c. We can take for instance $K = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, which yields the characteristic equation $s + 3 - 4 = 0$ with the unstable root $s = 1$.

4. We see that P_{pF} has a right-half-plane zero in $z = 4$, and a right-half-plane pole in $p = 3$. In the second-to-last slide of lecture 7, we find the following equation that gives a lower bound of the maximum value of the sensitivity function

$$\sup_{\omega} |S(i\omega)| \geq \frac{p + z}{|p - z|} = 7$$

When the sensitivity function is greater than 2–3, the robustness of the controller is too poor to be practically useful. This shows that it is impossible to achieve good control performance.

5 a. From lecture slide we get the following Riccati equation for LQ problem and Kalman gain

$$\begin{aligned} K &= (PC^T + NR_{12})R_2^{-1} \qquad L = Q_2^{-1}(SB + Q_{12})^T \\ 0 &= Q_1 + A^T S + SA - (SB + Q_{12})Q_2^{-1}(SB + Q_{12})^T \\ 0 &= NR_1N^T + AP + PA^T - (PC^T + NR_{12})R_2^{-1}(PC^T + NR_{12})^T \end{aligned}$$

where in this problem we have

$$A = 1 \quad B = 1 \quad M = 1 \quad N = 1 \quad C = 1 \quad Q_{12} = 0 \quad R_{12} = 0$$

from LQ Riccati equation one can write

$$\begin{aligned} S + S + Q_1 - S^2/Q_2 = 0 \quad &=> \quad S = Q_2(1 + \sqrt{1 + Q_1/Q_2}) \\ &=> \quad L = Q_2^{-1}S = 1 + \sqrt{1 + q} \end{aligned}$$

similarly for Kalman gain we have

$$P + P + R_1 - P^2/R_2 = 0 \quad => \quad P = R_2(1 + \sqrt{1 + R_1/R_2})$$
$$=> \quad K = PR_2^{-1} = 1 + \sqrt{1 + r}$$

Note that P and S can not take negative values. Since controllers have the form $u = -L\widehat{x}$ with $\frac{d}{dt}\widehat{x} = \widehat{x} + u + K[y - \widehat{x}]$, and L is a function of q and K is a function of r, the controller is only function of r and q.

b. From Lecture slides, the closed loop dynamics is

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ \widetilde{x}(t) \end{bmatrix} = \begin{bmatrix} A - BL & BL \\ 0 & A - KC \end{bmatrix} \begin{bmatrix} x(t) \\ \widetilde{x}(t) \end{bmatrix} + \begin{bmatrix} Nv_1(t) \\ Nv_1(t) - Kv_2(t) \end{bmatrix}$$

in this problem, both the plant and the observer have first order dynamics. We know that poles of the closed loop system are eigenvalues of $A - BL$ and $A - KC$. Here we have

$$A - KC = 1 - (1 + \sqrt{1 + r}) = -\sqrt{1 + r}$$
$$A - BL = 1 - (1 + \sqrt{1 + q}) = -\sqrt{1 + q}$$

c. If we put more trusts on measurements we get $R_1 >> R_2$, hence $r >> 1$, then the observer pole goes farther to the left, that is we get a faster convergence rate for the filter. On the other hand, if we trust the process more than the measurements, we have $R_1 << R_2$ then $r << 1$, so the observer pole approaches -1, that is we get slower convergence rate.

d. Putting larger penalty on the state variable than the control signal means $Q_1 >> Q_2$, as a result we have $q >> 1$, then the proces pole goes farther to the left (to large negative values), that is the process response is faster (goes faster to the origin). In the other words, since we get a large q, L would be larger and it means that we get a larger control signal (In another words, since Q_2 is small, the control signal is allowed to have larger values, so it results in a faster response). On the other hand, if $Q_1 << Q_2$ then $q << 1$, we are not allowed to have large control signals (since $q << 1$, L would be small) and the process pole approaches -1, that is the process response would be slower compared to the other case.

6.

a. In the balanced realization the observability Gramian and the controllability Gramian are equal so $S_\xi = O_\xi$.

b. Consider a balanced realization

$$\begin{bmatrix} \dot{\hat{\xi}}_1 \\ \dot{\hat{\xi}}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \quad , \qquad \Sigma = \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix}$$
$$y = [\, C_1 \quad C_2 \,] \begin{bmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \end{bmatrix} + Du$$

where Σ_2 corresponds to the small Hankel singular values. Replacing the second state equation by $\dot{\hat{\xi}}_2 = 0$, gives the reduced system below

$$\begin{aligned}\dot{\hat{\xi}}_1 &= (A_{11} - A_{12}A_{22}^{-1}A_{21})\hat{\xi}_1 + (B_1 - A_{12}A_{22}^{-1}B_2)u \\ y_{red} &= (C_1 - C_2A_{22}^{-1}A_{21})\hat{\xi}_1 + (D - C_2A_{22}^{-1}B_2)u\end{aligned}$$

after carrying out some calculations, we get

$$\begin{aligned}\dot{\hat{\xi}}_1 &= -0.4480\,\hat{\xi}_1 + \begin{bmatrix}-0.8738 & -2.4549\end{bmatrix}u \\ y_{red} &= \begin{bmatrix}-0.3502 \\ -2.5822\end{bmatrix}\hat{\xi}_1 + \begin{bmatrix}0.3171 & 0.3315 \\ -0.0357 & -0.8980\end{bmatrix}u\end{aligned}$$

c. No, since ξ_2 and ξ_3 has almost the same influence which is small in comparison to the influence of the first state it should not make a big difference if you remove both.