# Riccati Equations and Inequalities in Robust Control

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# Riccati (in)equalities in Robust Control

- Riccati (in)equalities are not only used to express optimality conditions (Libertzon chapter 6). They also play a central role in finding robust controllers that can ensure a bound on the "worst-case" disturbance gain: $H_\infty$ - synthesis.
- This presentation will cover the material in Libertzon section 7.3.

# Presentation Content

- Present the result (theorem?) that connects a Riccati equation to the $H_\infty$ norm of a system.
- How this can be used in closed loop synthesis.
- How to practically solve Riccati inequalities using LMI solvers.

# Solve a Riccati equation and ensure a bound on the $u$ to $y$ $L_2$ gain.

### Theorem

*Given the system:* $\dot{x} = Ax + Bu, \quad y = Cx$

*The two following statements are equivalent:*

$\exists$ $P$ *that fulfills 1,2:*

*1)* $P \geq 0$
*2)* $PA + A^T P + \frac{1}{\gamma} C^T C + \frac{1}{\gamma} PBB^T P \leq 0$

$$\sup_{u \in L_2 \setminus \{0\}} \frac{\sqrt{\int_0^\infty |y(t)|^2 dt}}{\sqrt{\int_0^\infty |u(t)|^2 dt}} = \sup_{\omega \in \mathbb{R}} |G(j\omega)| \leq \gamma.$$

# Proof Explanation

Proof (Not Complete).

Solve the following optimal control problem:

Minimize: $J(u) = \int_0^\infty \left(\gamma u^T(t)u(t) - \frac{1}{\gamma} y^T(t)y(t)\right) dt$

Subject to: $\dot{x} = Ax + Bu, y = Cx$.

It is straightforward to show that if there exists a $P$ fulfilling 1,2 (equality) and if $A + \frac{1}{\gamma}BB^TP$ is Hurwitz.

Then the optimal cost is given by:

$V(x_0) = -x_0^T P x_0$

obtained by:

$u^*(t) = \frac{1}{\gamma} B^T P x^*(t)$.

□

## Proof (Cont.)

The optimal control result implies that:

$$-x_0^T P x_0 \leq \int_0^\infty \left(\gamma u^T(t)u(t) - \frac{1}{\gamma} y^T(t)y(t)\right) dt$$

Then if $x_0 = 0 \implies$

$$\sup_{u \in L_2 \setminus \{0\}} \frac{\sqrt{\int_0^\infty |y(t)|^2 dt}}{\sqrt{\int_0^\infty |u(t)|^2 dt}} = \sup_{\omega \in \mathbb{R}} |G(j\omega)| \leq \gamma.$$

Then it can be shown that the Hurwitz condition could be relaxed and the Riccati equality could be relaxed to an inequality.

□

# This result can be used to find robust controllers.

Consider the system:

$$\dot{x} = Ax + Bu + D\omega$$
$$y = Cx$$
$$z = Ex$$

With the assumptions that:
$C = I$ (all the states are known.)
$u = Kx$ (Static state feedback control.)

# The $H_\infty$ control problem

The design problem of finding $K$ so that:

1. $A + BK$ is Hurwitz.
2. The $H_\infty$ norm of the closed loop system from $\omega$ to $z$ should be less than a predetermined value $\gamma$.

Since we have the closed loop system:

$$\dot{x} = (A + BK)x + D\omega$$
$$z = Ex$$

We can use the previously presented theorem to fulfill 2 by finding $P \geq 0$ such that:

$$PA_{cl} + A_{cl}^T P + \frac{1}{\gamma} E^T E + \frac{1}{\gamma} PDD^T P \leq 0.$$

# The $H_\infty$ control problem

To also encode the stability criterion into the Riccati equation, one can introduce $Q > 0$ and $\epsilon > 0$ and instead solve the equality:

$$PA_{cl} + A_{cl}^T P + \frac{1}{\gamma} E^T E + \frac{1}{\gamma} PDD^T P + \epsilon Q = 0.$$

Since this implies that:

$$PA_{cl} + A_{cl}^T P < 0, \quad \implies A_{cl} \text{ is Hurwitz.}$$

# The $H_\infty$ control problem

Now we have an equation with $K$ and $P$ unknown:

$$PA_{cl} + A_{cl}^T P + \frac{1}{\gamma} E^T E + \frac{1}{\gamma} PDD^T P + \epsilon Q = 0.$$

To make the process of finding $K$ and $P$ easier one can introduce a tuning matrix $R > 0$, and instead solve:

$$PA + A^T P + \frac{1}{\gamma} E^T E + \frac{1}{\gamma} PDD^T P + \epsilon Q - \frac{1}{\epsilon} PBR^{-1} B^T P = 0.$$

and then let

$$K = -\frac{1}{2\epsilon} R^{-1} B^T P.$$

# If the states are not measured..

There is a result saying that a dynamic controller satisfying 1 and 2 can be found by solving the two Riccati equalities:

$$P_1A + A^TP_1 + \frac{1}{\gamma}E^TE + \frac{1}{\gamma}P_1DD^TP_1 + \epsilon Q - \frac{1}{\epsilon}P_1BR^{-1}B^TP_1 = 0.$$

$$P_2A + A^TP_2 + \frac{1}{\gamma}D^TD + \frac{1}{\gamma}P_2EE^TP_2 + \epsilon Q - \frac{1}{\epsilon}P_2CR^{-1}C^TP_2 = 0.$$

where the largest singular value of $P_1P_2 \leq \gamma^2$

The found controller can be interpreted as a state feedback law combined with an observer.

# Solving a Riccati inequality

To ensure that the $L_2$ gain is less than or equal to $\gamma$:

$$P = P^T \geq 0$$

$$PA + A^T P + \frac{1}{\gamma} C^T C + \frac{1}{\gamma} PBB^T P \leq 0$$

# Convert it to an LMI (Linear Matrix Inequality)

$PA + A^T P + \frac{1}{\gamma} C^T C + \frac{1}{\gamma} PBB^T P \leq 0$ (Unfortunately quadratic in $P$)

$\Leftrightarrow$

$$\begin{pmatrix} PA + A^T P + \frac{1}{\gamma} C^T C & \sqrt{\frac{1}{\gamma}} PB \\ \sqrt{\frac{1}{\gamma}} B^T P & -I \end{pmatrix} \leq 0 \text{ (The Schur complement)}$$

This LMI can be solved numerically, using e.g. the Robust Control Toolbox in Matlab.

# Solving an example

Consider the system

$$G(s) = \frac{\omega_0^2}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

where $\omega_0 = 1$ and $\zeta = 0.3$

# Solving an example

1) Find a state-space representation of the system

2) Formulate the Linear Matrix Inequality in Matlab

$$\begin{pmatrix} PA + A^T P + \frac{1}{\gamma} C^T C & \sqrt{\frac{1}{\gamma}} PB \\ \sqrt{\frac{1}{\gamma}} B^T P & -I \end{pmatrix} \leq 0$$

3) Solve the inequality for $P$ using feasp

4) Verify the solution

# Solving an example using feasp

Solver for LMI feasibility problems L(x) < R(x) This solver minimizes t subject to L(x) < R(x) + t*I
The best value of t should be negative for feasibility

$$P = \begin{pmatrix} 1.7697 & 0.3246 \\ 0.3246 & 1.7699 \end{pmatrix}$$

Success!

Sometimes, however:
Result: best value of t: 0.407204
These LMI constraints were found infeasible

# The End