Project Optimal Control: Optimal Control with State Variable Inequality Constraints (SVIC)

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Problem Formulation

Find the maximum of the following cost functional subject to the contraints on states and control signals:

$$J(u) = \int_0^T F(x(t), u(t), t)dt + S(x(T), T)$$

$$\dot{x}(t) = f(x(t), t), \qquad x(0) = x_0$$

$$g(x(t), u(t), t) \geq 0$$

$$h(x(t), t) \geq 0$$

$$a(x(T), T) \geq 0$$

$$b(x(T), T) = 0$$

Outline

- State Constraints
- Direct Adjoining Approach
- Indirect Adjoining Approach
- Example: Double Integrator
  - Numerical Solution by Interior-Point Method
  - Direct Approach
- Conclusion

Constraint Qualification

for terminal constraints:

$$\text{rank}\begin{bmatrix} \partial a/\partial x & \text{diag}(a) \\ \partial b/\partial x & 0 \end{bmatrix} = l + l'$$

$$a : E^n \times E \to E^l, \qquad b : E^n \times E \to E^{l'}$$

for mixed constraints:

$$\text{rank}[\partial g/\partial u \quad \text{diag}(g)] = s$$

$$s : E^n \times E^m \times E \to E^s$$

Order of Pure State Constraints

$$
\begin{aligned}
h^0(x,u,t) &= h = h(x,t) \\
h^1(x,u,t) &= \dot{h} = h_x(x,t)f(x,u,t) + h_t(x,t) \\
h^2(x,u,t) &= \dot{h}^1 = h^1_x(x,t)f(x,u,t) + h^1_t(x,t) \\
&\vdots \\
h^p(x,u,t) &= \dot{h}^{p-1} = h^{p-1}_x(x,t)f(x,u,t) + h^{p-1}_t(x,t)
\end{aligned}
$$

The state constraint is of order p if

$$h^i_u(x,u,t) = 0 \text{ for } 0 \leq i \leq p-1, \quad h^p_u(x,u,t) \neq 0$$

Direct Adjoining approach, Hamiltonian

Hamiltonian and Lagrangian (P-form):

$$H(x, u, \lambda_0, \lambda, t) = \lambda_0 F(x, u, t) + \lambda f(x, u, t)$$
$$L(x, u, \lambda_0, \lambda, \mu, \nu, t) = H(x, u, \lambda_0, \lambda, t) + \mu g(x, u, t) + \nu h(x, u, t)$$

Control region:

$$\Omega(x, t) = \{u \in E^m \,|\, g(x, u, t) \geq 0\}$$

Direct Adjoining approach, Theorem Part 1

- $\{x^*(.), u^*(.)\}$ an optimal pair for the problem on slide 2 over $[0\ T]$ such that
  - $u^*(.)$is right-continuous with left-hand limits
  - constraint qualification holds for every tripple $\{t, x^*(t), u\}, t \in [0\ T], u \in \Omega(x^*(t), u)$
  - Assume $x^*(t)$ has only finitely many junction times
- Then there exist
  - a constant $\lambda_0 \geq 0$
  - a piecewise absolutely continuous costate trajectory $\lambda(.)$ mapping $[0, T]$ into E^n
  - piecewise continuous multiplyer functions $\mu(.)$ and $\nu(.)$ mapping $[0, T]$ into E^s and E^q, respectively
  - a vector $\eta(\tau_i) \in E^q$ for each point τ_i of discontinuity of $\lambda(.)$
  - $\alpha \in E^l$ and $\beta \in E^{l'}$, $\gamma \in E^q$, not all zero,
- such that the following conditions hold almost everywhere:

Direct Adjoining approach, Theorem Part 2

Hamiltonian Maximization

$$u^*(t) = \underset{u \in \Omega(x^*(t),t)}{\arg\max} \; H(x^*(t), u, \lambda_0, \lambda(t), t)$$

$$L_u^*[t] = H_u^*[t] + \mu g_u^*[t] = 0$$

$$\dot{\lambda}(t) = -L_x^*[t]$$

$$\mu(t) \geq 0, \qquad \mu(t) g^*[t] = 0$$

$$\nu(t) \geq 0, \qquad \nu(t) h^*[t] = 0$$

and $\quad dH^*[t]/dt = dL^*[t]/dt = L_t^*[t] \triangleq \partial L^*[t]/\partial t,$

Direct Adjoining approach, Theorem Part 3

At the terminal time T, the transversality conditions hold.

$$\lambda(T^-) = \lambda_0 S_x^* [T] + \alpha a_x^* [T] + \beta b_x^* [T] + \gamma h_x^* [T]$$
$$\alpha \geq 0,\ \gamma \geq 0, \quad \alpha a^* [T] = \gamma h^* [T] = 0$$

For any time τ in the boundary interval and for any contact time τ, the costate trajectory may have a discontinuouty given by the following conditions.

$$\lambda(\tau^-) = \lambda(\tau^+) + \eta(\tau) h_x^* [\tau]$$
$$H^* \left[\tau^-\right] = H^* \left[\tau^+\right] - \eta(\tau) h_t^* [\tau]$$
$$\eta(\tau) \geq 0, \quad \eta(\tau) h^* [\tau] = 0$$

Indirect Adjoining approach, Hamiltonian

Hamiltonian and Lagrangian (P-form):

$$H(x, u, \lambda_0, \lambda, t) = \lambda_0 F(x, u, t) + \lambda f(x, u, t)$$
$$L(x, u, \lambda_0, \lambda, \mu, \nu, t) = H(x, u, \lambda_0, \lambda, t) + \mu g(x, u, t) + \nu h^1(x, u, t)$$

Control region:

$$\Omega(x, t) = \left\{ u \in E^m \,|\, g(x, u, t) \geq 0,\ h^1(x, u, t) \geq 0 \text{ if } h(x, t) = 0 \right\}$$

Indirect Adjoining approach, Theorem Part 1

- $\{x^*(.), u^*(.)\}$ an optimal pair for the problem on slide 2 such that
  - $x^*(.)$ has only finitely many junction times
  - strong constraint qualification holds
- Then there exist
  - a constant $\lambda_0 \geq 0$
  - a piecewise absolutely continuous xostate trajectory λ mapping $[0, T]$ into E^n
  - piecewise continuous multiplyer functions $\mu(.)$ and $\nu(.)$ mapping $[0, T]$ into E^s and E^q, respectively
  - a vector $\eta(\tau_i) \in E^q$ for each point τ_i of discontinuity of $\lambda(.)$
  - $\alpha \in E^l$ and $\beta \in E^{l'}$, not all zero,
- such that the following conditions hold almost everywhere:

Indirect Adjoining approach, Theorem Part 2

Hamiltonian Maximization

$$u^*(t) = \underset{u \in \Omega(x^*(t),t)}{\arg\max}\ H(x^*(t), u, \lambda_0, \lambda(t), t)$$

$$L_u^*[t] = 0$$
$$\dot{\lambda}(t) = -L_x^*[t]$$
$$\mu(t) \geq 0, \qquad \mu(t) g^*[t] = 0$$

ν_i is nondecreasing on boundary intervals of $h_i,\ i = 1, 2, \cdots, q$ with

$$\nu(t) \geq 0,\ \dot{\nu}(t) \leq 0,\ \nu(t) h^*[t] = 0$$
and $$dH^*[t]/dt = dL^*[t]/dt = L_t^*[t],$$

whenever these derivatives exist.

Indirect Adjoining approach, Theorem Part 3

At the terminal time T, the transversality conditions

$$\lambda(T^-) = \lambda_0 S_x^* [T] + \alpha a_x^* [T] + \beta b_x^* [T] + \gamma h_x^* [T]$$
$$\alpha \geq 0,\ \gamma \geq 0, \quad \alpha a^* [T] = \gamma h^* [T] = 0$$

hold. At each entry or contace time, the costate trajectory λ may have a discontinuouty of the form:

$$\lambda(\tau^-) = \lambda(\tau^+) + \eta(\tau) h_x^* [\tau]$$
$$H^* [\tau^-] = H^* \left[\tau^+\right] - \eta(\tau) h_t^* [\tau]$$
$$\eta(\tau) \geq 0, \quad \eta(\tau) h^* [\tau] = 0$$

and $\eta(\tau_1) \geq \nu(\tau_1^+)$ for entry times τ_1.

Example: Double Integrator

minimize $$\int_0^T x^T R x + u^T Q u \, dt$$

subject to $$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(t)$$

$$|u(t)| \leq 1$$

$$|x_2(t)| \leq c$$

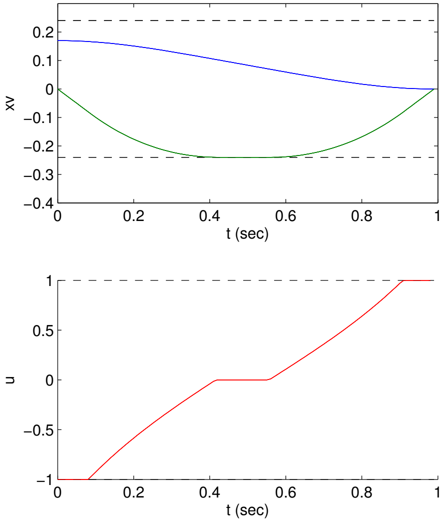
$$x(0) = \begin{pmatrix} x_i & 0 \end{pmatrix}^T \quad x(1) = 0_{2\times 1}$$

Numerical Solution

No of discretization points: 100

$$Q = 0.25, R = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$c = 0.24, x_I = \begin{pmatrix} 0.17 \\ 0 \end{pmatrix}$$



Direct Approach: Parameters, Hamiltonian and Lagrangian

$$F(x(t), u(t), t) = x^T R x + u^T Q u$$
$$S(T) = 0$$
$$f_1(x(t), u(t), t) = x_2(t)$$
$$f_2(x(t), u(t), t) = u(t)$$

$$H = -\lambda_0(r_1 x_1(t)^2 + r_2 x_2(t)^2 + q u(t)^2) + \lambda_1(t) x_2(t) + \lambda_2(t) u(t)$$
$$L = H + \mu_1(t)(1 - u(t)) + \mu_2(t)(1 + u(t))$$
$$+ \nu_1(t)(c - x_2(t)) + \nu_2(t)(c + x_2(t))$$

Direct Approach: Constraints

$$h(x(t)) = \begin{pmatrix} c - x_2(t) \\ c + x_2(t) \end{pmatrix} \; g(u(t)) = \begin{pmatrix} 1 - u(t) \\ 1 + u(t) \end{pmatrix} \;\; b(x(t)) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$$

Direct Approach: Hamiltonian Maximization

Without constraints, the Hamiltonian is maximized for:

$$\hat{u}(t) = \frac{1}{2}\frac{1}{q}\frac{\lambda_2(t)}{\lambda_0}$$

Considering the constraints, the optimal solution is:

$$u^*(t) = \begin{cases} -1 & \text{if } \lambda_2(t) < -2q \\ \frac{1}{2}\frac{1}{q}\frac{\lambda_2(t)}{\lambda_0} & \text{if } -2q < \lambda_2(t) < 2q \\ 1 & \text{if } \lambda_2(t) > 2q \end{cases}$$

Looking at the numerical solution $\Rightarrow$ time dependent u^*:

$$u^*(t) = \begin{cases} -1 & \text{for } t < \tau_1 \\ \frac{1}{2}\frac{1}{q}\frac{\lambda_2(t)}{\lambda_0} & \text{for } \tau_1 < t < \tau_2 \\ 0 & \text{for } \tau_2 < t < \tau_3 \\ \frac{1}{2}\frac{1}{q}\frac{\lambda_2(t)}{\lambda_0} & \text{for } \tau_3 < t < \tau_4 \\ 1 & \text{for } \tau_4 < t \end{cases}$$

Direct Approach: The Conditions

$$L_u^*[t] = H_u^*[t] + \mu g_u^*[t] = 0$$
$$\Rightarrow -2qu(t) + \lambda_2(t) - \mu_1(t) + \mu_2(t) = 0 \qquad (1)$$

$$\dot{\lambda}(t) = -L_x^*[t]$$
$$\Rightarrow \begin{cases} \dot{\lambda}_1(t) = 2\lambda_0 r_1 x_1(t) \\ \dot{\lambda}_2(t) = 2\lambda_0 r_2 x_2(t) - \lambda_1(t) + \nu_1(t) - \nu_2(t) \end{cases} \qquad (2)$$

$$\mu(t) = \begin{pmatrix} \mu_1(t) & \mu_2(t) \end{pmatrix}^T \geq 0$$
$$\mu(t) g^*[t] = 0 \Rightarrow (\mu_1(t) + \mu_2(t)) - (\mu_1(t) - \mu_2(t)) u^*(t) = 0 \qquad (3)$$

$$\nu(t) = \begin{pmatrix} \nu_1(t) & \nu_2(t) \end{pmatrix}^T \geq 0, \dot{\nu}(t) \leq 0,$$
$$\nu(t) h^*(t) = 0 \Rightarrow (\nu_1(t) + \nu_2(t)) c - (\nu_1(t) - \nu_2(t)) x_2(t) = 0 \qquad (4)$$

Direct Approach: The Conditions

$$\frac{dH^*[t]}{dt} = \frac{dL^*}{dt} = L_t^*[t] \Rightarrow$$
$$- 2r_1 x_1(t)x_2(t) - 2r_2 x_2(t)u(t) - 2qu(t)\dot{u}(t) + \lambda_1(t)u(t)$$
$$+ \dot{\lambda}_1(t)x_2(t) + \lambda_2(t)\dot{u}(t) + \dot{\lambda}_2(t)u(t) = 0 \quad (5)$$
$$\dot{u}(t)(\mu_1(t) - \mu_2(t)) + (\dot{\mu}_1(t) - \dot{\mu}_2(t))u(t)$$
$$- (\dot{\mu}_1(t) + \dot{\mu}_2(t)) + (\nu_1(t) - \nu_2(t))u(t)$$
$$+ (\dot{\nu}_1(t) - \dot{\nu}_2(t))x_2(t) - c(\dot{\nu}_1(t) + \dot{\nu}_2(t)) = 0 \quad (6)$$

Direct Approach: The Conditions

Transversality conditions:

$$\lambda(T^-) = \beta b_x^*[T] + \gamma h_x^*[T]$$
$$\begin{pmatrix} \lambda_1[1^-] \\ \lambda_2[1^-] \end{pmatrix} = I_{2\times 2} \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \end{pmatrix} \tag{7}$$
$$\gamma \geq 0, \gamma h[T] = 0 \Rightarrow (\gamma_1 + \gamma_2)c - (\gamma_1 - \gamma_2)x_2(T) = 0$$

Since $x_2(T) = 0$, we conclude

$$\gamma_1 = \gamma_2 = 0. \tag{8}$$

Direct Approach: $u^*(t) = -1,\ t < \tau_1$

Assuming that $x_i(0) > 0$

$$x_1(t) = -\frac{1}{2}t^2 + x_i$$
$$x_2(t) = -t \tag{9}$$

From (1) follows:

$$2q + \lambda_2(t) - \mu_1(t) + \mu_2(t) = 0 \tag{10}$$

From (3) follows:

$$\mu_1(t) = 0,\ \mu_2(t) \text{ free} \tag{11}$$

From (4) follows:

$$\nu_1(t) = \frac{t-c}{t+c}\nu_2(t) \tag{12}$$

From (6) follows:

$$2\dot{\mu}_1 + (t+c)\dot{\nu}_1 - (t-c)\dot{\nu}_2 + (\nu_1 - \nu_2) = 0 \tag{13}$$

Direct Approach: $u^*(t) = -1,\ t < \tau_1$

From (2) follows:

$$\begin{cases} \dot{\lambda}_1(t) = -r_1 t^2 + 2r_1 x_i \\ \dot{\lambda}_2(t) = -2r_2 t - \lambda_1(t) + \nu_1(t) - \nu_2(t) \end{cases} \tag{14}$$

Combined with (5) follows:

$$\nu_1(t) = \nu_2(t) \tag{15}$$

From (12) and (15) follows:

$$\nu_1(t) = \nu_2(t) = 0$$

From (14) follows:

$$\begin{cases} \lambda_1(t) = -\frac{1}{3} r_1 t^3 - r_1 x_i t - K_1 \\ \lambda_2(t) = \frac{1}{12} r_1 t^4 - (r_1 x_i + r_2) t^2 + tK_1 + K_2 \end{cases} \tag{16}$$

Direct Approach: $u^*(t) = 0,\ \tau_2 < t < \tau_3$

$$\begin{aligned} x_1(t) &= -ct + K_6 \\ x_2(t) &= -c \end{aligned} \tag{17}$$

From (1) follows:

$$\lambda_2(t) - \mu_1(t) + \mu_2(t) = 0 \tag{18}$$

From (3) follows:

$$\mu_1(t) = \mu_2(t) = 0 \tag{19}$$

From (4) follows:

$$\nu_1(t) = 0,\ \nu_2(t), \text{free} \tag{20}$$

(6) provides no new information.

Direct Approach: $u^*(t) = 0,\ \tau_2 < t < \tau_3$

From (2) follows:

$$\dot{\lambda}_1(t) = -2r_1 ct + 2r_1 K_6$$
$$\dot{\lambda}_2(t) = -\lambda_1(t) - 2r_2 c - \nu_2(t) \qquad (21)$$

(5) gives the same equation for $\dot{\lambda}_1(t)$

From (18) and (19) we conclude $\lambda_2(t) = 0$. Considering this, (21) simplifies to:

$$\nu_2(t) = -\lambda_1(t) - 2r_2 c \qquad (22)$$

$$\lambda_1(t) = -r_1 ct^2 + K_6 t + K_7 \qquad (23)$$

Direct Approach: $u^*(t) = 1,\ t > \tau_4$

$$x_1(t) = \frac{1}{2}t^2 + K_9 t + K_8$$
$$x_2(t) = t + K_9 \tag{24}$$

From (1) follows:

$$-2q + \lambda_2(t) - \mu_1(t) + \mu_2(t) = 0 \tag{25}$$

From (3) follows:

$$\mu_1(t) \text{ free},\ \mu_2(t) = 0 \tag{26}$$

From (4) follows:

$$(c-t)\nu_1(t) + (c+t)\nu_2(t) - (\nu_1(t) - \nu_2(t))K_9 = 0 \tag{27}$$

Direct Approach: $u^*(t) = 1,\ t > \tau_4$

From (6) follows:

$$\nu_1(t) - \nu_2(t) + (\dot{\nu}_1(t) - \dot{\nu}_2(t))(t - c + K_9) = 0 \qquad (28)$$

From (2) follows:

$$\begin{cases} \dot{\lambda}_1(t) = 2r_1(\frac{1}{2}t^2 + K_9 t + K_8) \\ \dot{\lambda}_2(t) = 2r_2(t + K_9) - \lambda_1(t) + \nu_1(t) - \nu_2(t) \end{cases} \qquad (29)$$

Combined with (5) follows:

$$\nu_1(t) = \nu_2(t) \qquad (30)$$

From (27) and (30) follows $\nu_1(t) = \nu_2(t) = 0$. From (29) follows:

$$\begin{cases} \lambda_1(t) = -\frac{1}{3}r_1 t^3 + r_1 K_9 t^2 + 2r_1 K_8 t + K_{10} \\ \lambda_2(t) = \frac{1}{12}r_1 t^4 - \frac{1}{3}r_1 K_9 t^3 + (r_2 - r_1 K_8)t^2 \\ \qquad\qquad +(2r_2 K_9 - K_{10})t + K_{11} \end{cases} \qquad (31)$$

Direct Approach: $u^*(t) = \frac{1}{2q}\lambda_2(t)$, $\tau_1 < t < \tau_2,\ \tau_3 < t < \tau_4$

From (1) follows:

$$u(t) = \frac{\lambda_2(t) - \mu_1(t) + \mu_2(t)}{2q} \tag{32}$$

$$\Rightarrow \mu_1(t) = \mu_2(t) \tag{33}$$

From (3) follows:

$$\mu_1(t) + \mu_2(t) = 0 \tag{34}$$

Therefore, $\mu_1(t) = \mu_2(t) = 0$.
From (4) follows:

$$c(\nu_1(t) + \nu_2(t)) - (\nu_1(t) - \nu_2(t))x_2(t) = 0 \tag{35}$$

From (6) follows:

$$(\nu_1(t) - \nu_2(t))\frac{\lambda_2(t)}{2q} + (\dot{\nu}_1(t) - \dot{\nu}_2(t))x_2(t) - c(\dot{\nu}_1(t) + \dot{\nu}_2(t)) = 0 \tag{36}$$

Direct Approach: $u^*(t) = \frac{1}{2q}\lambda_2(t)$, $\tau_1 < t < \tau_2,\ \tau_3 < t < \tau_4$

From (5) follows:

$$\left(-2r_2x_2(t) + \lambda_1(t) + \dot{\lambda}_2(t)\right)\lambda_2(t) = 0 \tag{37}$$

With $\lambda_2(t) \neq 0$,

$$-2r_2x_2(t) + \lambda_1(t) + \dot{\lambda}_2(t) = 0. \tag{38}$$

Comparing with (2) and using (4), we conclude that $\nu_1(t) = \nu_2(t) = 0$.

Using (38), system dynamics and $u^*(t) = \frac{1}{2q}\lambda_2(t)$, we can conclude:

$$\lambda_2^{(4)}(t) = \frac{1}{q}(r_2\ddot{\lambda}_2(t) - r_1\lambda_2(t)) \tag{39}$$

Direct Approach: $u^*(t) = \frac{1}{2q}\lambda_2(t)$, $\tau_1 < t < \tau_2,\ \tau_3 < t < \tau_4$

From this follows:

$$\lambda_2(t) = C_1 e^{-\sigma_1 t} + C_2 e^{\sigma_1 t} + C_3 e^{-\sigma_2 t} + C_4 e^{\sigma_2 t} \tag{40}$$

with

$$\sigma_1 = \sqrt{\frac{r_2 + \sqrt{r_2^2 - 4qr_1}}{2q}}$$

$$\sigma_2 = \sqrt{\frac{r_2 - s\sqrt{r_2^2 - 4qr_1}}{2q}}$$

Direct Approach: $u^*(t) = \frac{1}{2q}\lambda_2(t),$ $\tau_1 < t < \tau_2,\ \tau_3 < t < \tau_4$

From system equations:

$$\dot{x}_2(t) = u(t) \Rightarrow$$

$$x_2(t) = \frac{1}{2q}\left(-\frac{C_1}{\sigma_1}e^{-\sigma_1 t} + \frac{C_2}{\sigma_1}e^{\sigma_1 t} - \frac{C_3}{\sigma_2}e^{-\sigma_2 t} + \frac{C_4}{\sigma_2}e^{\sigma_2 t}\right) + \kappa_1 \tag{41}$$

$$\dot{x}_1(t) = x_2(t) \Rightarrow$$

$$x_1(t) = \frac{1}{2q}\left(\frac{C_1}{\sigma_1^2}e^{-\sigma_1 t} + \frac{C_2}{\sigma_1^2}e^{\sigma_1 t} + \frac{C_3}{\sigma_2^2}e^{-\sigma_2 t} + \frac{C_4}{\sigma_2^2}e^{\sigma_2 t}\right) + \kappa_1 t + \kappa_2 \tag{42}$$

Direct Approach: $u^*(t) = \frac{1}{2q}\lambda_2(t)$, $\tau_1 < t < \tau_2,\ \tau_3 < t < \tau_4$

With (38) it follows:

$$\lambda_1(t) = \left(\sigma_1 - \frac{r_2}{\sigma_1 q}\right)\left(C_1 e^{-\sigma_1 t} - C_2 e^{\sigma_1 t}\right) + \left(\sigma_2 - \frac{r_2}{\sigma_2 q}\right)\left(C_3 e^{-\sigma_2 t} - C_4 e^{\sigma_2 t}\right) + 2 r_2 K_{12} \qquad (43)$$

Direct Approach: Initial and Final Conditions

Initial condition on x used earlier.

Final conditions for x:

$$x_1(1) = 0, x_2(1) = 0 \tag{44}$$

From system equations when $u = 1$ follows:

$$K_8 = \frac{1}{2},\ K_9 = -1 \tag{45}$$

Continuity u:

$$u(\tau_1^+) = -1 \Rightarrow \lambda_2(\tau_1^+) = -2q \Rightarrow$$
$$C_1 e^{-\sigma_1 \tau_1} + C_2 e^{\sigma_1 \tau_1} + C_3 e^{-\sigma_2 \tau_1} + C_4 e^{\sigma_2 \tau_1} = -2q \tag{46}$$

$$u(\tau_2^-) = 0 \Rightarrow \lambda_2(\tau_2^-) = 0 \Rightarrow$$
$$C_1 e^{-\sigma_1 \tau_2} + C_2 e^{\sigma_1 \tau_2} + C_3 e^{-\sigma_2 \tau_2} + C_4 e^{\sigma_2 \tau_2} = 0 \tag{47}$$

Direct Approach: Initial and Final Conditions

$$u(\tau_3^+) = 0 \Rightarrow \lambda_2(\tau_3^+) = 0 \Rightarrow$$
$$D_1 e^{-\sigma_1 \tau_3} + D_2 e^{\sigma_1 \tau_3} + D_3 e^{-\sigma_2 \tau_3} + D_4 e^{\sigma_2 \tau_3} = 0$$

$$u(\tau_4^-) = 1 \Rightarrow \lambda_2(\tau_4^-) = 2q \Rightarrow$$
$$D_1 e^{-\sigma_1 \tau_4} + D_2 e^{\sigma_1 \tau_4} + D_3 e^{-\sigma_2 \tau_4} + D_4 e^{\sigma_2 \tau_4} = 2q$$

Continuity of x:

$$x_1(\tau_1^-) = x_1(\tau_1^+) \Rightarrow$$
$$-\frac{1}{2}\tau_1^2 + x_i = \frac{1}{2q}\left(\frac{C_1}{\sigma_1^2} e^{-\sigma_1 \tau_1} + \frac{C_2}{\sigma_1^2} e^{\sigma_1 \tau_1} + \frac{C_3}{\sigma_2^2} e^{-\sigma_2 \tau_1} + \frac{C_4}{\sigma_2^2} e^{\sigma_2 \tau_1}\right)$$
$$+ K_{12}\tau_1 + K_{13}$$

Direct Approach: Initial and Final Conditions

$$x_2(\tau_1^-) = x_2(\tau_1^+) \Rightarrow$$
$$-\tau_1 = \frac{1}{2q}\left(-\frac{C_1}{\sigma_1}e^{-\sigma_1\tau_1} + \frac{C_2}{\sigma_1}e^{\sigma_1\tau_1} - \frac{C_3}{\sigma_2}e^{-\sigma_2\tau_1} + \frac{C_4}{\sigma_2}e^{\sigma_2\tau_1}\right) + K_{12}$$

$$x_1(\tau_2^-) = x_1(\tau_2^+) \Rightarrow$$
$$\frac{1}{2q}\left(\frac{C_1}{\sigma_1^2}e^{-\sigma_1\tau_2} + \frac{C_2}{\sigma_1^2}e^{\sigma_1\tau_2} + \frac{C_3}{\sigma_2^2}e^{-\sigma_2\tau_2} + \frac{C_4}{\sigma_2^2}e^{\sigma_2\tau_2}\right) + K_{12}\tau_1 + K_{13}$$
$$= -c\tau_2 + K_6$$

$$x_2(\tau_2^-) = x_2(\tau_2^+) \Rightarrow$$
$$\frac{1}{2q}\left(-\frac{C_1}{\sigma_1}e^{-\sigma_1\tau_2} + \frac{C_2}{\sigma_1}e^{\sigma_1\tau_2} - \frac{C_3}{\sigma_2}e^{-\sigma_2\tau_2} + \frac{C_4}{\sigma_2}e^{\sigma_2\tau_2}\right) + K_{12} = -c$$

Direct Approach: Initial and Final Conditions

$$x_1(\tau_3^-) = x_1(\tau_3^+) \Rightarrow$$

$$-c\tau_3 + K_6 = \frac{1}{2q}\left(\frac{D_1}{\sigma_1^2}e^{-\sigma_1\tau_3} + \frac{D_2}{\sigma_1^2}e^{\sigma_1\tau_3} + \frac{D_3}{\sigma_2^2}e^{-\sigma_2\tau_3} + \frac{D_4}{\sigma_2^2}e^{\sigma_2\tau_3}\right)$$

$$+ K_{14}\tau_3 + K_{15}$$

$$x_2(\tau_3^-) = x_2(\tau_3^+) \Rightarrow$$

$$-c = \frac{1}{2q}\left(-\frac{D_1}{\sigma_1}e^{-\sigma_1\tau_3} + \frac{D_2}{\sigma_1}e^{\sigma_1\tau_3} - \frac{D_3}{\sigma_2}e^{-\sigma_2\tau_3} + \frac{D_4}{\sigma_2}e^{\sigma_2\tau_3}\right) + K_{14}$$

Direct Approach: Initial and Final Conditions

$$x_1(\tau_4^-) = x_1(\tau_4^+) \Rightarrow$$
$$\frac{1}{2q}\left(\frac{D_1}{\sigma_1^2}e^{-\sigma_1\tau_4} + \frac{D_2}{\sigma_1^2}e^{\sigma_1\tau_4} + \frac{D_3}{\sigma_2^2}e^{-\sigma_2\tau_4} + \frac{D_4}{\sigma_2^2}e^{\sigma_2\tau_4}\right) + K_{14}\tau_4 + K_{15}$$
$$= \frac{1}{2}\tau_4^2 + \frac{1}{2} - \tau_4$$

$$x_2(\tau_4^-) = x_2(\tau_4^+) \Rightarrow$$
$$\frac{1}{2q}\left(-\frac{D_1}{\sigma_1}e^{-\sigma_1\tau_4} + \frac{D_2}{\sigma_1}e^{\sigma_1\tau_4} - \frac{D_3}{\sigma_2}e^{-\sigma_2\tau_4} + \frac{D_4}{\sigma_2}e^{\sigma_2\tau_4}\right) + K_{14} = \tau_4 - 1$$

Direct Approach: Continuity of λ

$$\lambda_1(\tau_1^-) = \lambda_1(\tau_1^+) \Rightarrow$$

$$-\frac{1}{3}r_1\tau_1^3 - r_1 x_i \tau_1 - K_1 = \left(\sigma_1 - \frac{r_2}{\sigma_1 q}\right)\left(C_1 e^{-\sigma_1\tau_1} - C_2 e^{\sigma_1\tau_1}\right)$$

$$+\left(\sigma_2 - \frac{r_2}{\sigma_2 q}\right)\left(C_3 e^{-\sigma_2\tau_1} - C_4 e^{\sigma_2\tau_1}\right) + 2r_2 K_{12}$$

$$\lambda_2(\tau_1^-) = \lambda_2(\tau_1^+) \Rightarrow$$

$$\frac{1}{12}r_1\tau_1^4 - (r_1 x_i + r_2)\tau_1^2 + \tau_1 K_1 + K_2$$

$$= C_1 e^{-\sigma_1\tau_1} + C_2 e^{\sigma_1\tau_1} + C_3 e^{-\sigma_2\tau_1} + C_4 e^{\sigma_2\tau_1}$$

Direct Approach: Continuity of λ

$$\lambda_1(\tau_2^-) = \lambda_1(\tau_2^+) \Rightarrow$$

$$\left(\sigma_1 - \frac{r_2}{\sigma_1 q}\right)\left(C_1 e^{-\sigma_1\tau_2} - C_2 e^{\sigma_1\tau_2}\right)$$

$$+ \left(\sigma_2 - \frac{r_2}{\sigma_2 q}\right)\left(C_3 e^{-\sigma_2\tau_2} - C_4 e^{\sigma_2\tau_2}\right) + 2r_2K_{12}$$

$$= -r_1 c\tau_2^2 + K_6\tau_2 + K_7$$

$$\lambda_2(\tau_2^-) = \lambda_2(\tau_2^+) \Rightarrow$$

$$C_1 e^{-\sigma_1\tau_2} + C_2 e^{\sigma_1\tau_2} + C_3 e^{-\sigma_2\tau_2} + C_4 e^{\sigma_2\tau_2} = 0$$

Direct Approach: Continuity of λ

$$\lambda_1(\tau_3^-) = \lambda_1(\tau_3^+) \Rightarrow$$
$$-r_1 c\tau_3^2 + K_6\tau_3 + K_7 = \left(\sigma_1 - \frac{r_2}{\sigma_1 q}\right)\left(D_1 e^{-\sigma_1\tau_3} - D_2 e^{\sigma_1\tau_3}\right)$$
$$+\left(\sigma_2 - \frac{r_2}{\sigma_2 q}\right)\left(D_3 e^{-\sigma_2\tau_3} - D_4 e^{\sigma_2\tau_3}\right) + 2r_2 K_{14}$$

$$\lambda_2(\tau_3^-) = \lambda_2(\tau_3^+) \Rightarrow$$
$$0 = D_1 e^{-\sigma_1\tau_2} + D_2 e^{\sigma_1\tau_3} + D_3 e^{-\sigma_2\tau_3} + D_4 e^{\sigma_2\tau_3}$$

Direct Approach: Continuity of λ

$$\lambda_1(\tau_4^-) = \lambda_1(\tau_4^+) \Rightarrow$$

$$\left(\sigma_1 - \frac{r_2}{\sigma_1 q}\right)\left(D_1 e^{-\sigma_1\tau_4} - D_2 e^{\sigma_1\tau_4}\right)$$

$$+ \left(\sigma_2 - \frac{r_2}{\sigma_2 q}\right)\left(D_3 e^{-\sigma_2\tau_4} - D_4 e^{\sigma_2\tau_4}\right)$$

$$+ 2r_2K_{14} = -\frac{1}{3}r_1\tau_4^3 - r_1\tau_4^2 + K_{10} + r_1\tau_4$$

$$\lambda_2(\tau_4^-) = \lambda_2(\tau_4^+) \Rightarrow$$

$$D_1 e^{-\sigma_1\tau_2} + D_2 e^{\sigma_1\tau_4} + D_3 e^{-\sigma_2\tau_4} + D_4 e^{\sigma_2\tau_4}$$

$$= \frac{1}{12}r_1\tau_4^4 + \frac{1}{3}r_1\tau_4^3 + (r_2 - \frac{1}{2}r_1)\tau_4^2 + (-2r_2 - K_{10})\tau_4 + K_{11}$$

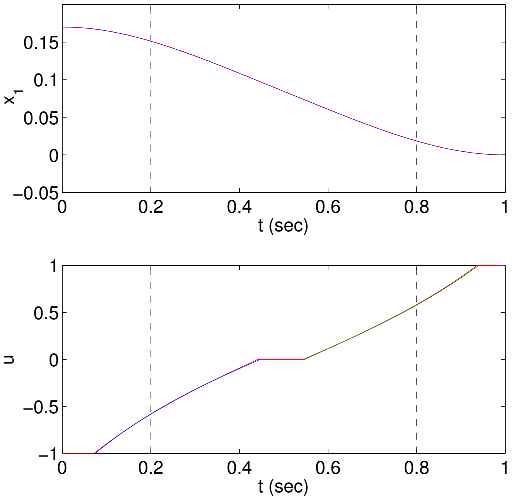
Direct Approach: Results

These conditions result in 24 unkowns and 20 equations, most of them nonlinear.

To determine a solution, we need four more equations. Hence, the solution is calculated by measuring values at the following four points:

$$
\begin{aligned}
u(0.2) &= -0.5840, \\
x_2(0.2) &= -0.1724 \\
u(0.8) &= 0.5841, \\
x_2(0.8) &= -0.1692
\end{aligned}
$$

Direct Approach: Results



Conclusion

- Since H is concave, we can conclude that the solution to the example found by the direct approach is optimal, according to the Mangasarian-Type sufficient condition.
- The necessary condition by itself does not provide enough information to calculate the solution for the example.

References

This work is based on the following references:

- Richard F. Hartl, Suresh P. Sethi and Raymond G. Vickson, "A Survey of the Maximum Principles for Optimal Control Problems with State Constraints," SIAM Review, Vol. 37, No. 2 (Jun., 1995), pp. 181-218
- Seierstad, Atle, and Knut Sydsæter. *Optimal Control Theory with Economic Applications.* Amsterdam: North-Holland, 1987.