LECTURE 6

6. The Linear Quadratic Regulator

6.1. **Fixed-time, free-endpoint.** Suppose that we have the system

$$\dot{x} = A(t)x + B(t)u, \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,$$

the terminal set $S = \{t_1\} \times \mathbb{R}^n$ and the cost functional

$$J(u) = \int_{t_0}^{t_1} \left(x^\top(t)Q(t)x(t) + u^\top(t)R(t)u(t) \right) dt + x^\top(t_1)Mx(t_1),$$

where Q, R and M are matrixes of appropriate size with $M = M^\top \geq 0$, $Q(t) = Q^\top(t) \geq 0$ and $R(t) = R^\top(t) > 0$.

Hence, J punishes size of both x and u, with

$$L(t, x, u) = x^\top Q(t)x + u^\top R(t)u.$$

We want to find a state-feedback control for this system but we still turn to the maximum principle. We have

$$H(t, x, u, p) = p^\top(A(t)x + B(t)u) - x^\top Q(t)x - u^\top R(t)u.$$

The canonical equations give

$$\dot{p}^*(t) = -H_x\Big|_*(t) = -A^\top(t)p^*(t) + 2Q(t)x^*(t)$$

and by the transversality condition for the free-endpoint problem

$$p^*(t_1) = -2Mx^*(t_1).$$

Since H is continuously differentiable in u, the Hamiltonian maximization condition gives $H_u\Big|_*(t) = (p^*)^\top B(t) - 2(u^*)^\top R(t) = 0$, hence

$$u^*(t) = \frac{1}{2}R^{-1}(t)B^\top(t)p^*(t),$$

which is a maximizer since $H_{uu} = -2R(t)$ is negative definite.

Now, the canonical equations can be written

$$\begin{pmatrix} \dot{x}^* \\ \dot{p}^* \end{pmatrix} = \begin{pmatrix} A(t) & \frac{1}{2}B(t)R^{-1}(t)B^\top(t) \\ 2Q(t) & -A^\top(t) \end{pmatrix} \begin{pmatrix} x^* \\ p^* \end{pmatrix} =: \mathcal{H}(t) \begin{pmatrix} x^* \\ p^* \end{pmatrix}.$$

The matrix $\mathcal{H}(t)$ is sometimes referred to as the Hamiltonian matrix. Let this linear system have state transition matrix $\Phi(\cdot, \cdot)$. Specifically we have

$$\begin{pmatrix} x^*(t) \\ p^*(t) \end{pmatrix} = \Phi(t, t_1) \begin{pmatrix} x^*(t_1) \\ p^*(t_1) \end{pmatrix} = \Phi(t, t_1) \begin{pmatrix} x^*(t_1) \\ -2Mx^*(t_1) \end{pmatrix},$$

where $\Phi(t, t_1) = \Phi^{-1}(t_1, t)$. If we make the division

$$\Phi = \begin{pmatrix} \Phi_{11} & \Phi_{12} \\ \Phi_{21} & \Phi_{22} \end{pmatrix}$$

then this can be written

$$x^*(t) = (\Phi_{11}(t,t_1) - 2\Phi_{12}(t,t_1)M)\, x^*(t_1)$$
$$p^*(t) = (\Phi_{21}(t,t_1) - 2\Phi_{22}(t,t_1)M)\, x^*(t_1).$$

Assuming that $\Phi_{11}(t,t_1) - 2\Phi_{12}(t,t_1)M$ can be inverted we get

$$p^*(t) = (\Phi_{21}(t,t_1) - 2\Phi_{22}(t,t_1)Mx^*(t_1))\, (\Phi_{11}(t,t_1) - 2\Phi_{12}(t,t_1)M)^{-1}\, x^*(t) =: -2P(t)x^*(t),$$

with $P(t_1) = M$, giving us the optimal state-feedback control

$$\boxed{u^*(t) = -R^{-1}(t)B^\top(t)P(t)x^*(t).}$$

6.1.1. *Riccati's differential equation.* Through $p^*(t) = -2P(t)x^*(t)$ we have

$$\dot p^*(t) = -2\dot P(t)x^*(t) - 2P(t)\dot x^*(t).$$

The equation $\dot p^* = -H_x\Big|_*$ now gives

$$\begin{aligned}-2\dot P(t)x^*(t) - 2P(t)A(t)x^*(t) + 2P(t)B(t)R^{-1}(t)B^\top(t)P(t)x^*(t) &= -A^\top(t)p^*(t) + 2Q(t)x^*(t)\\ &= -2A^\top(t)P(t)x^*(t) + 2Q(t)x^*(t).\end{aligned}$$

Using the fact that this should be true for all initial points we get the Riccati differential equation (RDE) for P:

$$\boxed{\dot P(t) = -P(t)A(t) - A^\top(t)P(t) - Q(t) + P(t)B(t)R^{-1}(t)B^\top(t)P(t).}$$

6.1.2. *Value function and optimality.* So far the maximum principle has given us a candidate for an optimal control. But as we remember the maximum principle is only a necessary condition for optimality. To make sure that the found control is in fact a global optimum we use dynamic programming.

The equation for the adjoint vector $p^*(t) = -2P(t)x^*(t)$ leads us to the guess $V(t,x) = x^\top P(t)x$.

Lecture assignment Show that the guess $V(t,x) = x^\top P(t)x$ solves the HJB equations when $P(t)$ solves the Riccati differential equation.

Solution The HJB-equation gives

$$-x^\top \dot P(t)x = \inf_u \{x^\top Q(t)x + u^\top R(t)u + \langle 2P(t)x, A(t)x + B(t)u\rangle\}.$$

The infimum is attained with $2R(t)u^* + 2B^\top(t)P(t)x = 0$ or

$$u^* = -R^{-1}(t)B^\top(t)P(t)x.$$

Putting this into the HJB-equation gives

$$\begin{aligned}-x^\top \dot P(t)x =& x^\top Q(t)x + (-R^{-1}(t)B^\top(t)P(t)x)^\top R(t)(-R^{-1}(t)B^\top(t)P(t)x)\\ &+ 2x^\top P^\top(t)(A(t)x + B(t)(-R^{-1}(t)B^\top(t)P(t)x))\\ =& x^\top Q(t)x + x^\top P(t)B(t)R^{-1}(t)R(t)R^{-1}(t)B^\top(t)P(t)x + 2x^\top P(t)A(t)x\\ &- 2x^\top P(t)B(t)R^{-1}(t)B^\top(t)P(t)x.\end{aligned}$$

This holds for all $x \in \mathbb{R}^n$ if

$$\begin{aligned}\dot P(t) &= -Q(t) - P(t)B(t)R^{-1}B^\top(t)P(t) - P(t)A(t) - A^\top(t)P(t) + 2P(t)B(t)R^{-1}(t)B^\top(t)P(t)\\ &= -P(t)A(t) - A^\top(t)P(t) - Q(t) + P(t)B(t)R^{-1}(t)B^\top(t)P(t).\end{aligned}$$

Furthermore the terminal constraints give

$$V(t_1, x) = x^\top P(t_1)x = x^\top Mx,$$

for all $x \in \mathbb{R}^n$. Hence,

$$P(t_1) = M.$$

$V(t,x) = x^\top P(t)x$ thus solves the HJB equations if $P(t)$ solves the Riccati differential equation with boundary condition $P(t_1) = M$. □

Since $V(t,x) \geq 0$ for all (t,x), we must have that $P(t)$ is non-negative definite.

Example. Consider the linear-quadratic control problem

$$\min_u \int_{t_0}^{t_1} (x^2(t) + u^2(t))dt \quad \text{subj. to} \quad \dot{x}(t) = u(t).$$

We have the scalar matrices $A = 0$, $B = 1$, $Q = 1$, $R = 1$ and $M = 0$. The RDE then reads

$$\begin{aligned}\dot{P}(t) &= -P(t)A - A^\top P(t) - Q + P(t)BR^{-1}B^\top P(t) \\ &= -1 + P^2(t),\end{aligned}$$

with $P(t_1) = 0$. Separation of variables now gives,

$$\int_{P(t)}^{0} \frac{dP}{P^2 - 1} = \int_{t}^{t_1} ds,$$

with solution $P(t) = \tanh(t_1 - t)$. Hence, the optimal feedback law is given by $u^*(t) = -R^{-1}B^\top P(t)x(t) = \tanh(t_1 - t)x(t)$. □

6.2. **Infinite horizon LQR.** We move to the infinite horizon case where

$$\dot{x} = Ax + Bu, \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^m,$$

with (A, B) a controllable pair and

$$J(u) = \int_{t_0}^{\infty} \Big(x^\top(t)Qx(t) + u^\top(t)Ru(t)\Big)\, dt.$$

By the controllability property of the system there is a control $\bar{u}$ and a $\bar{t} \geq t_0$ (both depending on the initial state) that takes the system to $x(\bar{t}) = 0$ after which the optimal control is $u \equiv 0$. Hence, if an optimal control u^* exists, then

$$J(u^*) \leq \int_{t_0}^{\bar{t}} \Big(x^\top(t)Qx(t) + \bar{u}^\top(t)R\bar{u}(t)\Big)\, dt < \infty.$$

In finite horizon we had

$$-V_t(t,x) = \inf_{u \in U} \Big\{L(t,x,u) + \langle V_x(t,x), f(t,x,u)\rangle\Big\}.$$

In the infinite horizon case we look for a value function that is independent of t and get

$$0 = \inf_{u \in U} \Big\{L(x,u) + \langle V_x(x), f(x,u)\rangle\Big\}.$$

The anzats $V(x) = x^\top Px$ leads to

$$0 = \inf_{u \in \mathbb{R}^m} \Big\{x^\top Qx + u^\top Ru + \langle 2Px, Ax + Bu\rangle\Big\}.$$

which gives

$$\boxed{u^* = -R^{-1}B^\top Px.}$$

Putting this into the HJB equation we get

$$\begin{aligned} 0 &= x^\top Qx + (R^{-1}B^\top Px)^\top RR^{-1}B^\top Px + \langle 2Px, Ax - BR^{-1}B^\top Px\rangle \\ &= x^\top Qx + x^\top PBR^{-1}B^\top Px + x^\top PAx + x^\top A^\top Px - 2x^\top PBR^{-1}B^\top Px. \end{aligned}$$

Since this should be valid for all x we get

$$\boxed{Q + PA + A^\top P - PBR^{-1}B^\top P = 0}$$

which is called the algebraic Riccati equation (ARE).

Example. Consider the infinite horizon LQR problem

$$\min_u \int_0^\infty (x_1^2(t) + u^2(t))dt \quad \text{subj. to} \quad \begin{cases} \dot{x}_1(t) = x_2(t), & x_1(0) = x_{10}, \\ \dot{x}_2(t) = u(t), & x_2(0) = x_{20}. \end{cases}$$

This problem can be written in standard form with

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad R = 1, \quad A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad x(0) = \begin{bmatrix} x_{10} \\ x_{20} \end{bmatrix}.$$

The algebraic Riccati equation becomes

$$\begin{aligned} &Q + PA + A^\top P - PBR^{-1}B^\top P \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \\ &\quad - \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & p_{11} \\ 0 & p_{21} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ p_{11} & p_{12} \end{bmatrix} - \begin{bmatrix} p_{12} \\ p_{22} \end{bmatrix}\begin{bmatrix} p_{21} & p_{22} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & p_{11} \\ 0 & p_{21} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ p_{11} & p_{12} \end{bmatrix} - \begin{bmatrix} p_{12}p_{21} & p_{12}p_{22} \\ p_{22}p_{21} & p_{22}p_{22} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

giving the system of equations

$$\begin{cases} -p_{12}^2 + 1 = 0 \\ -p_{22}^2 + 2p_{12} = 0 \\ p_{11} - p_{12}p_{22} = 0 \end{cases} \quad \Rightarrow \quad \begin{cases} p_{12} = \pm 1 \\ p_{22} = \pm\sqrt{2p_{12}} \\ p_{11} = p_{12}p_{22} \end{cases}$$

The unique positive definite solution to this system is

$$P = \begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix}.$$

This gives the optimal feedback control

$$u^* = -R^{-1}B^\top Px = -\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{bmatrix}x = -\begin{bmatrix} 1 & \sqrt{2} \end{bmatrix}x = -x_1 - \sqrt{2}x_2.$$

The closed loop system is

$$\dot{x} = (A - BR^{-1}B^T P)x = \begin{bmatrix} 0 & 1 \\ -1 & -\sqrt{2} \end{bmatrix}x.$$

The system matrix has both eigenvalues in the left half plane and is thus stable, showing that u^* is a stabilizing feedback control. $\square$