LECTURE 4

4. The Maximum Principle

In the previous lecture we proved the maximum principle for the variable-time, fixed-endpoint and the variable-time, variable-endpoint problems. The maximum principle is a necessary condition for local optimality, and assumes existence of the optimal control. In the way the needle perturbations were designed the difference between u and u^* is small in the $\mathcal{L}_1$-sense for small ε while the difference between x and x^* is small in 0-norm. The maximum principle thus gives optimality somewhere between the weak and strong.

4.1. Adapting the maximum principle to other types of problems.

Having derived the maximum principle for the two special cases that we have considered so far, we can easily extend the theory to apply to a large class of problems.

Fixed terminal-time: Define the new variable x_{n+1} with

$$\dot{x}_{n+1} = 1,$$
$$x_{n+1}(t_0) = t_0.$$

Then fixed time means that $x_{n+1}(t_f) = t_1$. Let $\bar{\square}$ denote variables and functions for the augmented system, then

$$\bar{H}\big|_*(t) = \langle \bar{p}^*, \bar{f}\big|_*\rangle + p_0^* L\big|_* = \langle p^*, f\big|_*\rangle + p_{n+1}^* + p_0^* L\big|_* = H\big|_*(t) + p_{n+1}^* \equiv 0.$$

By the canonical equations

$$\dot{p}_{n+1}^* = -H_t\big|_* \equiv 0.$$

Hence,

$$H_t\big|_*(t) = -p_{n+1}^* = \text{const}.$$

We added one constraint but in return got one more free variable.

Time dependence in f and L: The same principle as in the previous case still applies but we no longer have that $H_t\big|_* \equiv 0$. Instead we get

$$\frac{d}{dt} H\big|_* = H_t\big|_*,$$

with the end-point condition $H\big|_*(t_f) = -p_{n+1}^*(t_f)$. If the terminal time is free then the transversality-condition implies that $p_{n+1}^*(t_f) = 0$, so that $H\big|_*(t) = -\int_t^{t_f} H_t\big|_*(s)ds$.

Example Let t_f be a free variable and solve

$$\min_{u(\cdot),\, t_f > 0} \frac{1}{2}\int_0^{t_f} (t^4 + (u(t))^2)dt \quad \text{subj. to} \quad \begin{cases} \dot{x}(t) = u(t), & \\ x(0) = x_0, & x(t_f) = 0. \end{cases}$$

The Hamiltonian for this problem is

$$H(t, x, u, p) = up - \frac{1}{2}(t^4 + u^2).$$

Pointwise maximization gives

$$u^*(t) = \arg\max_{u \in \mathbb{R}} H(t, x^*(t), u, p^*(t)) = p^*(t).$$

The canonical equations give $\dot{p}^* = 0$. Hence $u^* = c$ is constant. Solving the differential equation we get

$$\begin{aligned} & x(t) = x_0 + ct, \\ \implies \quad & x(t_f) = x_0 + ct_f = 0, \\ \implies \quad & u^* = -\frac{x_0}{t_f} = 0. \end{aligned}$$

Furthermore, since we are dealing with a free-time problem $H(t_f, x^*(t_f), u^*(t_f), p^*(t_f)) = 0$, *i.e.*

$$\frac{x_0^2}{t_f^2} - \frac{1}{2}(t_f^4 + \frac{x_0^2}{t_f^2}) = \frac{1}{2}(\frac{x_0^2}{t_f^2} - t_f^4) = 0.$$

Solving this equation we find that $t_f = |x_0|^{1/3}$ and thus

$$u^* = -|x_0|^{2/3}\mathrm{sign}(x_0).$$

Terminal cost: Assume that we only have a terminal cost (Mayer form). Now, $x^*(t^*)$ should be chosen such that K does not decrease along any direction in the terminal cone (here we do not need any x^0-axis since $L \equiv 0$), hence,

$$\langle -K_x(x^*(t^*)), \delta\rangle \leq 0, \quad \forall \delta \text{ such that } x^*(t^*) + \delta \in C_{t^*}.$$

A comparison to the result obtained in Step 7 then suggests that we can choose $p^*(t^*) = -K_x(x^*(t^*))$.

4.2. **The bang-bang principle for optimal control problems.** A control that jumps between extremes in the control set is called a *bang-bang control.* When the control set is an interval $U = [a, b]$, with $a < b$, this means that $u^*(t) \in \{a, b\}$ for all $t \in [t_0, t_f]$.

Linear time-optimal control problems in normal form is one class of problems that always have bang-bang optimal controls. Time-optimal control problems are free-time problems where the cost functional take the specific form $J(u) = \int_{t_0}^{t_f} 1dt$.

Example We consider the double integrator

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u, \end{aligned}$$

with $u \in [-1, 1]$. Let the initial condition be $x(0) = x_0$, let the final state be $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and let the running cost be $L \equiv 1$ so that we have a time-optimal control problem. This is a free-time, fixed-endpoint problem which corresponds to parking a car at $x_1(t_f) = 0$ in the least possible time.

We get the Hamiltonian

$$\begin{aligned} H(x, u, p, p_0) &= \langle p, f(x, u)\rangle + p_0 \\ &= p_1 x_2 + p_2 u + p_0. \end{aligned}$$

By the Hamiltonian maximization property we thus have that

$$u^*(t) = \begin{cases} -1, & \text{if } p_2^* < 0, \\ ?, & \text{if } p_2^* = 0, \\ 1, & \text{if } p_2^* > 0. \end{cases}$$

The canonical equations give

$$\begin{aligned} \dot{p}_1^* &= 0 \\ \dot{p}_2^* &= -p_1. \end{aligned}$$

Hence,

$$H\big|_*(t) = c_1 x_2^* + (-c_1 t + c_2)u^* - p_0^* = 0.$$

We will only have that $p_2^*(t) = 0$ on an interval if $p_1^* = 0$ which means that $p_0^* = 0$ thus violating the non-triviality constraint. Instead $c_1 \neq 0$ and we will have one switching. To satisfy the end-point constraints this switch will have to be such that the system reaches $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

Assume first that $u = 1$, then

$$\begin{aligned} & \dot{x} = t + a \\ \implies \quad & x = \tfrac{1}{2}t^2 + at + b \\ & \quad = \tfrac{1}{2}\dot{x}^2 + b - \tfrac{1}{2}a^2 \\ & \quad = \dot{x}^2 + C. \end{aligned}$$

For $C = 0$ we thus get a trajectory that ends in $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. For $u = -1$, we get

$$\begin{aligned} & \dot{x} = -t + a \\ \implies \quad & x = -\tfrac{1}{2}t^2 + at + b \\ & \quad = -\tfrac{1}{2}\dot{x}^2 + b + \tfrac{1}{2}a^2 \\ & \quad = -\dot{x}^2 + C'. \end{aligned}$$

So that $C' = 0$ gives a trajectory ending in $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$. These curves together form something called a *switching curve*, denoted by Λ.

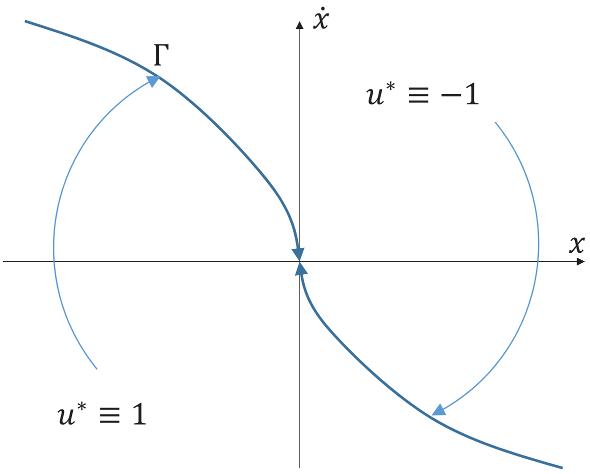

FIGURE 1. The optimal control where areas with $u = -1$ and $u = 1$ are separated by the the switching curve.

We have thus found a state feedback control by eliminating p^* to get $u(x)$.

However, time-optimal control problems are not the only problems where bang-bang controls are optimal:

Example Solve the problem

$$\min_u \int_0^1 2(1-u)x dt \quad \text{subj. to} \quad \begin{cases} \dot{x} = (2u-1)x, & u(t) \in [0,1], \\ x(0) = 1, & x(1) = 2. \end{cases}$$

Hint: Assume that $x(t) > 0$. Determine the number of possible zero-crossings of the switching function by considering its derivative.

The Hamiltonian for this problem is (with $p_0 = -1$)

$$H(x,u,p) = (2u-1)xp - 2(1-u)x.$$

Hamiltonian maximization gives

$$u^* = \arg\max_{u\in[0,1]} H(x^*,u,p^*) = \arg\max_{u\in[0,1]} 2(p^*+1)ux^* = \begin{cases} 1, & p^* > -1, \\ 0, & p^* < -1. \end{cases}$$

We can thus define a switching function $\sigma = p^* + 1$ such that we switch control when σ switches sign. To determine the number of switches we consider the adjoint equation

$$\dot{p}^* = -H_x\big|_* = -(2u^*-1)p^* + 2(1-u^*).$$

At $\sigma = 0$ we get

$$\dot{\sigma}\big|_{\sigma=0} = \dot{p}^*\big|_{p^*=-1} = (2u^*-1) + 2(1-u^*) = 1.$$

Hence, we can at most do one switch, from $u = 0$ to $u = 1$.

First, if $u \equiv 0$ on the entire interval $[0,1]$, then $\dot{x} = -x$ and we cannot fulfill the boundary conditions. If instead $u \equiv 1$ on the entire interval $[0,1]$ then $\dot{x} = x$ and $x(1) > 2$. Hence, we must make one switch from $u = 0$ to $u = 1$. Let τ denote the time of the switch, so that

$$u^*(t) = \begin{cases} 0, & t < \tau, \\ 1, & t \geq \tau. \end{cases}$$

Since $u^* = 0$ on $[0,\tau)$ we have

$$x(\tau) = e^{-\tau}.$$

With $u^* = 1$ on $[\tau, 1]$ we thus get

$$x(1) = e^{-\tau}e^{1-\tau} = e^{1-2\tau} = 2.$$

Hence,

$$\tau = \tfrac{1}{2}(1 - \ln(2)).$$

For control problems with optimal controls of bang-bang type we can thus identify a procedure of finding the solution that generally works:

(1) Identify the switching function σ by considering the Hamiltonian maximization property.
(2) Solve the adjoint equation to compute the number of switches.
(3) Solve the differential equation to find the switching boundary Λ.

4.3. **Existence of optimal controls.** In the statement of the maximum principle we assumed that an optimal control exists and derived a set of constraints for the Hamiltonian in this case. Up until now nothing has been said about existence of optimal controls. In general the existence of optimal controls is a difficult matter, but in some special cases it is possible to prove existence.

Let $R^t(x_0)$ be the set of points that can be reached from x_0 at time t.

Theorem 4.1. *(Filippov's theorem) Given a control system on standard form with $u \in U$, assume that the solution x to the differential equation $\dot{x} = f(t, x, u)$ exists on a time interval $[t_0, t_f]$ for all controls $u(\cdot)$ and that for every pair (t, x) the set $\{f(t, x, u) : u \in U\}$ is compact and convex. Then the reachable set $R^t(x_0)$ is compact for each $t \in [t_0, t_f]$.*

Hence, for linear systems $R^t(x_0)$ exists and is compact (and convex) whenever U is compact and convex.

Now assume a system in Mayer form (*i.e.* $J(u) = K(x_f)$), where t_f is fixed. If $R^{t_f}(x_0)$ is compact then we have a minimization problem over a compact set, which clearly has a solution.

One important problem that we have already seen examples of is time optimal control for linear systems. The following theorem assures that under certain conditions optimal controls to such problems exists.

Theorem 4.2. *(Existence of time-optimal controls for linear systems) Consider a linear control system with a compact and convex control set U. Let the objective be to steer x from a given initial state $x(t_0) = x_0$ to a given final state x_1 in minimal time. Assume that $x_1 \in R^t(x_0)$ for some $t \geq t_0$. Then there exists a time-optimal control.*

The outline of a proof would be to let $t^* = \inf\{t \geq t_0 : x_1 \in R^t(x_0)\}$ and realize that the theorem is true if the inf is attained. By the properties of inf there is a sequence t^k and a corresponding set of controls u^k such that $x^k(t^k) = x_1$.

Since $x^k(t^*) \in R^{t^*}(x_0)$ we only need to show that $\lim_{k\to\infty} x^k(t^*) = x_1$ since compactness of $R^{t^*}(x_0)$ then implies that $x_1 \in R^{t^*}(x_0)$.