LECTURE 2

2. Calculus of variations

2.1. Integral constraints. Assume that we add the constraint

$$C(y) := \int_a^b M(x, y(x), y'(x))dx = C_0$$

to the basic calculus of variations problem.

Recall that for finite-dimensional optimization with smooth functions $f : \mathbb{R}^n \to \mathbb{R}$ and $h : \mathbb{R}^n \to \mathbb{R}$ a necessary condition for x^* to solve

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & f(x), \\ s.t. \quad & h(x) = 0 \end{aligned}$$

is that $h(x^*) = 0$ and $(\nabla f(x^*))^\top d = 0$ for all $d \in \mathbb{R}^n$ such that $(\nabla h(x^*))^\top d = 0$, or equivalently that

$$\nabla f(x^*) + \lambda^* \nabla h(x^*), \tag{2.1}$$

for some $\lambda^* \in \mathbb{R}$.

For a perturbation η to be valid we must have that $C(y + \alpha\eta) = C_0 + o(\alpha)$. The infinite-dimensional equivalent to constrained optimization in $\mathbb{R}^n$ is that for y to be a weak extremum, $\delta J|_y(\eta) = 0$ for all perturbations $\eta \in \mathcal{V}$, with $\eta(a) = \eta(b) = 0$, such that $\delta C|_y(\eta) = 0$.

This means that, as elements of a suitable $\mathcal{L}_2$-space, $\left(L_y - \frac{d}{dx}L_{y'}\right)$ is orthogonal to the subspace orthogonal to $\left(M_y - \frac{d}{dx}M_{y'}\right)$, which gives us the following equivalent of (2.1):

$$\left(L_y - \frac{d}{dx}L_{y'}\right) + \lambda^* \left(M_y - \frac{d}{dx}M_{y'}\right) = 0, \tag{2.2}$$

that can be re-written as

$$(L + \lambda^* M)_y = \frac{d}{dx}(L + \lambda^* M)_{y'}.$$

To be an extremum of the constrained problem y should thus be an extremum to $(J + \lambda^* C)(y)$, for some $\lambda^* \in \mathbb{R}$.

Example If we return to the catenary problem were we have

$$J(y) = \int_a^b y(x)\sqrt{1 + (y'(x))^2}dx,$$

which is another "no x" problem with

$$L_{y'} = \frac{yy'}{\sqrt{1 + (y')^2}} \quad \text{and} \quad M_{y'} = \frac{y'}{\sqrt{1 + (y')^2}}$$

Hence,

$$\begin{aligned} c = L_{y'}y' - L + \lambda^*(M_{y'}y' - M) &= (y + \lambda^*)\left(\frac{(y')^2}{\sqrt{1 + (y')^2}} - \sqrt{1 + (y')^2}\right) \\ &= \frac{y + \lambda^*}{\sqrt{1 + (y')^2}}\left((y')^2 - (1 + (y')^2)\right) = -\frac{y + \lambda^*}{\sqrt{1 + (y')^2}}. \end{aligned}$$

That can be written

$$y' = \pm\sqrt{\frac{(\lambda^* + y)^2}{c^2} - 1},$$

the solution of which is $y(x) = \pm(c\cosh\left(\frac{x+d}{c}\right) - \lambda^*)$, where d is another constant. If we assume that $c > 0$ the minus sign can be ruled out since this would correspond to a chain bulging upwards (maximal energy). The constants c, d and λ^* can then be arranged to fit the boundary and length conditions. □

In some situations we have to be careful, however, as can be seen from the following example:

Example Assume that we have the constraints

$$C(y) := \int_0^1 \sqrt{1 + (y'(x))^2}dx = 1, \tag{2.3}$$

and $y(0) = y(1) = 0$, for which the only solution is $y \equiv 0$. We have that

$$M_y - \frac{d}{dx}M_{y'} = -\frac{d}{dx}\frac{y'(x)}{\sqrt{1 + (y'(x))^2}} = 0,$$

since clearly y is a minimizer for C. Hence, the solution to any problem with constraints $y(0) = y(1) = 0$ and (2.3) is $y \equiv 0$, but the solution to (2.2) corresponding to this problem is any extremum for the unconstrained problem. □

In the example there is no allowed perturbation, therefore (2.2) is not valid anymore. An alternative formulation that allow for these type of situations as well is that y solves the Euler-Lagrange equation for $\lambda_0^* L + \lambda^* M$ where $(\lambda_0^*, \lambda^*) \neq (0, 0)$. The so defined λ_0^* is called the *abnormal multiplier*.

2.2. **Non-Integral constraints.** In the case of non-integral constraints of the type

$$M(x, y(x), y'(x)) = 0$$

We look for solutions of the Euler-Lagrange equation for

$$L + \lambda^*(x)M.$$

This can be realized by noting that the non-integral constraint is similar to the integral constraint except that instead of holding for the integral over the entire interval it holds for every $x \in [a, b]$. Thus we get the same type of equation but with a different multiplier for each $x \in [a, b]$.

3. From Calculus of Variations to Optimal Control

Compared to the calculus of variations, optimal control deals with stronger local optima over less regular curves and can also take into account constraints on the control actions (y' in the CV-setting).

We will first try to loosen the regularity constraints and consider functions that are only piece-wise $\mathcal{C}^1$.

3.1. **Corner Points.** A Corner Point (CP) is a point $c \in (a, b)$ such that $\lim_{x \nearrow c} y'(x)$ and $\lim_{x \searrow c} y'(x)$ both exist, but are different.

Example If we try to minimize

$$J(y) = \int_{-1}^1 y^2(x)(y'(x) - 1)^2 dx$$

over all $y \in \mathcal{C}^1([-1, 1] \to \mathbb{R})$ with $y(-1) = 0$ and $y(1) = 0$, we can get infinitely close to the global optimum 0. But to get zero we need insert a CP to get

$$y(x) = \begin{cases} 0, & \text{for } x \in [-1, 0], \\ x, & \text{for } x \in (0, 1]. \end{cases}$$

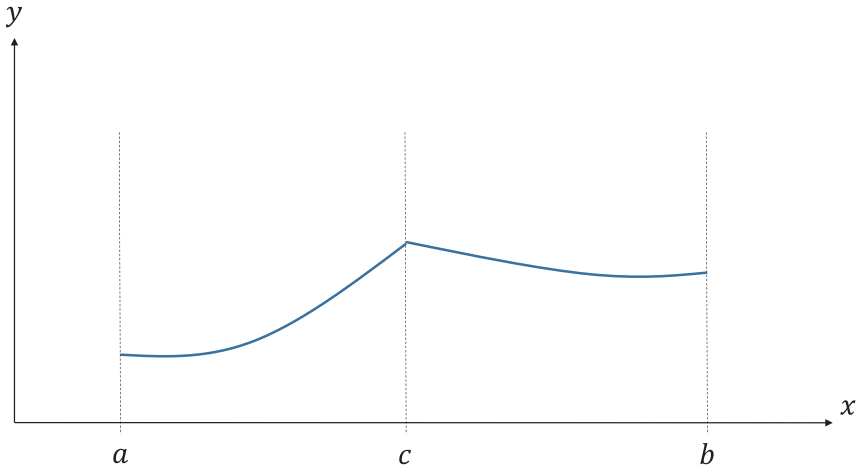

FIGURE 1. A trajectory with a CP at $x = c$.

□

We consider functions y that are piecewise-$\mathcal{C}^1$ and thus have a finite number of corner points. To find extremals of this type we have to generalize the 1-norm to

1-norm: $\|y\|_1 = \max_{x\in[a,b]} |y(x)| + \max_{x\in[a,b]} \max\{|y'(x^-)|, |y'(x^+)|\}.$

With this definition, strong minima are also weak minima, as in the $\mathcal{C}^1$-case. The piecewise extremals are sometimes also referred to as *broken extremals*.

Assume first that y only has one CP in $c \in (a, b)$. We then divide y into two different curves $y_1 : [a, c] \to \mathbb{R}$ and $y_2 : [c, b] \to \mathbb{R}$. To add a perturbation to y we add perturbations η_1 to y_1 and η_2 to y_2, with $\eta_1(a) = \eta_2(b) = 0$. Now, we must allow the perturbation to move the CP an amount proportional to α, say $\alpha\Delta x$. Here we run into a problem since y_1 is not defined on $[c, b]$ and wise versa. To remedy this we use linear extrapolation of the curves y_1 and y_2 at the point $x = c$. In order for the perturbed curve $y(\cdot, \alpha)$ to be continuous at $c + \alpha\Delta x$, we must have

$$\begin{aligned} y_1(c) + \alpha\Delta x y_1'(c) + \alpha n_1(c + \alpha\Delta x) &= y_2(c) + \alpha\Delta x y_2'(c) + \alpha n_2(c + \alpha\Delta x) \\ \Rightarrow \alpha\Delta x y_1'(c) + \alpha n_1(c + \alpha\Delta x) &= \alpha\Delta x y_2'(c) + \alpha n_2(c + \alpha\Delta x). \end{aligned}$$

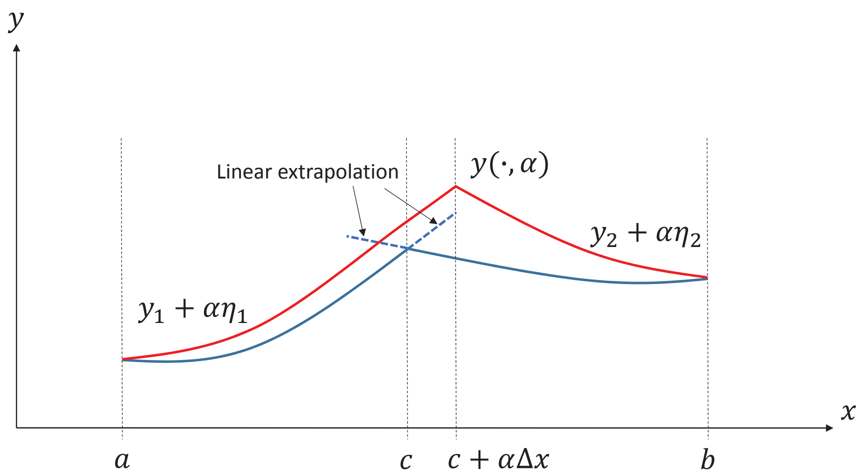

FIGURE 2. Adding perturbations to a trajectory with a CP.

Evaluating the derivative w.r.t. α at $\alpha = 0$ we get

$$\Delta x y_1'(c) + n_1(c) = \Delta x y_2'(c) + n_2(c).$$

Hence,

$$\Delta x = \frac{n_1(c) - n_2(c)}{y_2'(c) - y_1'(c)} = \frac{n_1(c) - n_2(c)}{y'(c^+) - y'(c^-)}.$$

The perturbed cost functionals are

$$J_1(y_1 + \alpha\eta_1) := \int_a^{c+\alpha\Delta x} L(x, y_1(x) + \alpha\eta_1(x), y_1'(x) + \alpha\eta_1'(x))dx$$

and

$$J_2(y_2 + \alpha\eta_2) := \int_{c+\alpha\Delta x}^{b} L(x, y_2(x) + \alpha\eta_2(x), y_2'(x) + \alpha\eta_2'(x))dx.$$

Hence,

$$\begin{aligned} \delta J_1\big|_{y_1}(\eta_1) = &\int_a^c \left(L_y(x, y_1(x), y_1'(x))\eta_1 + L_{y'}(x, y_1(x), y_1'(x))\eta_1'(x)\right) dx \\ &+ L(c, y_1(c), y_1'(c))\Delta x. \end{aligned}$$

Using integration by parts in the usual manner and noting that $y_1 = y$ on $[a, c]$ we get

$$\begin{aligned} \delta J_1\big|_{y_1}(\eta_1) = &\int_a^c \left(L_y(x, y(x), y'(x)) - \frac{d}{dx} L_{y'}(x, y(x), y'(x))\right) \eta_1(x)dx \\ &+ L_{y'}(c, y(c), y'(c^-))\eta_1(c) + L(c, y(c), y'(c^-))\Delta x, \end{aligned}$$

and similarly

$$\begin{aligned} \delta J_2\big|_{y_2}(\eta_2) = &\int_b^c \left(L_y(x, y(x), y'(x)) - \frac{d}{dx} L_{y'}(x, y(x), y'(x))\right) \eta_2(x)dx \\ &- L_{y'}(c, y(c), y'(c^+))\eta_2(c) - L(c, y(c), y'(c^+))\Delta x. \end{aligned}$$

Now, for y to be an extremum we must have that

$$\delta J_1\big|_{y_1}(\eta_1) + \delta J_2\big|_{y_2}(\eta_2) = 0,$$

for all perturbations η_1 and η_2. Letting $\eta_1(c) = \eta_2(c) = 0$ we find that the Euler-Lagrange equation must hold on $[a, b] \setminus \{c\}$.

A zero first variation is thus obtained by having

$$0 = L_{y'}(c, y(c), y'(c^-))\eta_1(c) - L_{y'}(c, y(c), y'(c^+))\eta_2(c) + (L(c, y(c), y'(c^-)) - L(c, y(c), y'(c^+)))\Delta x,$$

first note that if we let $\eta_1(c) = \eta_2(c) \neq 0$ we get $\Delta x = 0$, so that

$$L_{y'}(c, y(c), y'(c^-)) = L_{y'}(c, y(c), y'(c^+)).$$

Hence, $L_{y'}$ is continuous at $x = c$. Plugging this in and using the relation for Δx we get

$$\begin{aligned} 0 = &L_{y'}(c, y(c), y'(c^-))(\eta_1(c) - \eta_2(c)) + (L(c, y(c), y'(c^-)) - L(c, y(c), y'(c^+)))\frac{n_1(c) - n_2(c)}{y'(c^+) - y'(c^-)} \\ &\Rightarrow L_{y'}(c, y(c), y'(c^-))(y'(c^+) - y'(c^-)) + (L(c, y(c), y'(c^-)) - L(c, y(c), y'(c^+))) = 0 \end{aligned}$$

Hence, also $y'L_{y'} - L$ is continuous at x_c.

This leads us to the Weierstrass-Erdmann corner conditions:
If a curve y is a strong extremum then $L_{y'}$ and $y'L_{y'} - L$ must be continuous in every CP of y.

3.2. **Optimal Control formulation and assumptions.** Assume that we have a control system

$$\dot{x} = f(t, x, u), \quad x(t_0) = x_0,$$

where $x \in \mathbb{R}^n$ is now the state vector and $u \in U \subset \mathbb{R}^m$ is the control vector.

We will often assume that u is piecewise continuous, but keep in mind that measurability and local boundedness (local integrability) is enough.

We define the cost functional

$$J(u) := \int_{t_0}^{t_f} L(t, x(t), u(t))dt + K(t_f, x_f),$$

where $x_f = x(t_f)$. This form with a running cost and a final cost is called the Bolza form. A problem is in Lagrange form if $K \equiv 0$ and in Mayer form if $L \equiv 0$.

Note that we can always move from Bolza form to Mayer form by introducing the additional state x^0, with

$$\dot{x}^0 = L(t, x(t), u(t))$$

and $x^0(t_0) = 0$. Then the terminal cost is $\tilde{K}(t_f, x_f) = x^0(t_f) + K(t_f, x_f)$.

We can also move from Bolza to Lagrange form since

$$J(u) = \int_{t_0}^{t_f} \left(L(t, x(t), u(t)) + \frac{d}{dt} K(t, x(t)) \right) dt + K(t_0, x_0),$$

where the last part is a constant and can be removed from the optimization.

We can also represent time as one of the state variables by introducing the extra state x_{n+1} with $\dot{x}_{n+1} = 1$ and $x_{n+1}(t_0) = t_0$.

We have the *target set* $S \subset [t_0, \infty) \times \mathbb{R}^n$, such that $(t_f, x_f) \in S$. A few possibilities target sets are

Free-time, fixed-endpoint: $S = [t_0, \infty) \times \{x_1\}$.
Fixed-time, free-endpoint: $S = t_1 \times \mathbb{R}^n$.
Fixed-time, fixed-endpoint: $S = t_1 \times \{x_1\}$.

3.3. **Variational approach to the fixed-time, free-endpoint problem.** We can write the cost function for this case as

$$J(u) = \int_{t_0}^{t_1} L(t, x(t), u(t))dt + K(x(t_1)).$$

Let $u^*(\cdot)$ be an optimal control for this problem, so that $J(u^*) \leq J(u)$ for all u that are piecewise $\mathcal{C}^0$, and let $x^*(\cdot)$ be the corresponding state trajectory.

In Calculus of Variations we consider perturbations of the form

$$x = x^* + \alpha\eta, \tag{3.1}$$

but this is not practical for the problem at hand since it is not obvious how this perturbation would translate to the control. Instead we let the perturbation ξ act on u and get

$$u = u^* + \alpha\xi, \tag{3.2}$$

where ξ is a piecewise continuous function from $[t_0, t_1]$ to $\mathbb{R}^m$.

How does then (3.2) translate to (3.1) or rather to

$$x(\cdot, \alpha) = x^* + \alpha\eta + o(\alpha). \tag{3.3}$$

We want $x_\alpha(t,0) = \eta(t)$. Then we have

$$\begin{aligned}
\dot\eta(t) &= \frac{d}{dt} x_\alpha(t,0) = x_{\alpha t}(t,0) = x_{t\alpha}(t,0) \\
&= \frac{d}{d\alpha}\Big|_{\alpha=0} \dot x(t,\alpha) \\
&= \frac{d}{d\alpha}\Big|_{\alpha=0} f(t, x(t,\alpha), u^*(t) + \alpha\xi(t)) \\
&= f_x(t, x(t,0), u^*(t)) x_\alpha(t,0) + f_u(t, x(t,0), u^*(t))\xi(t) \\
&= f_x\big|_* \eta + f_u\big|_* \xi, \quad \eta(t_0) = 0.
\end{aligned}$$

We thus get a linearization of the original system around the optimal trajectory.

To be able to use Calculus of variations we apply the differential-equation constraint

$$\dot x(t) - f(t, x(t), u(t)) = 0$$

and get the augmented cost function

$$J(u) = \int_{t_0}^{t_1} \left(L(t, x(t), u(t)) + \langle p(t), \dot x(t) - f(t, x(t), u(t)) \rangle \right) dt + K(x(t_1)),$$

for some $\mathcal{C}^1$ function $p : [t_1, t_1] \to \mathbb{R}^n$.

By defining the Hamiltonian $H(t,x,u,p) := \langle p(t), f(t,x,u) \rangle - L(t,x,u)$, we can rewrite the augmented cost as

$$J(u) = \int_{t_0}^{t_1} \left(\langle p(t), \dot x(t) \rangle - H(t, x(t), u(t), p(t)) \right) dt + K(x(t_1)).$$

The first variation is defined by

$$J(u) - J(u^*) = \delta J\big|_{u^*}(\xi)\alpha + o(\alpha)$$

We have

$$K(x(t_1)) - K(x^*(t_1)) \approx \alpha \left\langle K_x(x^*(t_1)), \eta(t_1) \right\rangle,$$

and

$$\begin{aligned}
H(t,x,u,p) - H(t,x^*,u^*,p) &= H(t, x^* + \alpha\eta + o(\alpha), u^* + \alpha\xi, p) - H(t, x^*, u^*, p) \\
&\approx \alpha \left\langle H_x\big|_*(t), \eta(t) \right\rangle + \alpha \left\langle H_u\big|_*(t), \xi(t) \right\rangle.
\end{aligned}$$

Using integration by parts we get

$$\begin{aligned}
\int_{t_0}^{t_1} \langle p(t), \dot x(t) - \dot x^*(t) \rangle \, dt &= \langle p(t), x(t) - x^*(t) \rangle \big|_{t_0}^{t_1} - \int_{t_0}^{t_1} \langle \dot p(t), x(t) - x^*(t) \rangle \, dt \\
&\approx \alpha \langle p(t_1), \eta(t_1) \rangle - \alpha \int_{t_0}^{t_1} \langle \dot p(t), \eta(t) \rangle \, dt.
\end{aligned}$$

Putting this together we get

$$\begin{aligned}
\delta J\big|_{u^*}(\xi) = &- \int_{t_0}^{t_1} \left(\left\langle \dot p(t) + H_x\big|_*(t), \eta(t) \right\rangle + \left\langle H_u\big|_*(t), \xi(t) \right\rangle \right) dt \\
&+ \left\langle K_x(x^*(t_1)) + p(t_1), \eta(t_1) \right\rangle.
\end{aligned}$$

If we let $\dot p^* = -H_x\big|_*$, with boundary condition $p^*(t_1) = -K_x(x^*(t_1))$ we get

$$\delta J\big|_{u^*}(\xi) = -\int_{t_0}^{t_1} \left\langle H_u\big|_*(t), \xi(t) \right\rangle dt = 0,$$

for all ξ that are piecewise-$\mathcal{C}^0$ on $[t_0, t_1]$. Hence, $H_u\big|_*(t) \equiv 0$ implying that the function $H(t, x^*(t), \cdot, p^*(t))$ has a stationary point in $u^*(t)$, for all $t \in [t_0, t_1]$. The vector (x^*, p^*) solves the *canonical equations*

$$\boxed{\begin{aligned} \dot{x}^* &= H_p\big|_* \\ \dot{p}^* &= -H_x\big|_* \end{aligned}}$$

Written out in terms of f and L we have

$$\dot{p}^* = -(f_x)^\top p^* + L_x\big|_*.$$

The two linear systems $\dot{x} = Ax$ and $\dot{z} = -A^\top z$ are called *adjoint*. Therefore, p^* is sometimes referred to as the *adjoint vector*.