HOME ASSIGNMENT 4

Problem 1. Consider the more general optimal control problem with the system

$$\dot{x} = f(t, x, u),$$

the cost

$$J(u) = \int_{t_0}^{t_f} L(t, x(t), u(t))dt + K(x_f),$$

and the target set $S = [t_0, \infty) \times S_1$ where S_1 is a k-dimensional surface in $\mathbb{R}^n$. Reduce this problem by a change of variables to the Basic Variable-Endpoint Control Problem and arrive at a precise statement of the maximum principle for this more general case.

Problem 2. Suppose that we have a optimal control problem where the boundary conditions are that

$$\begin{pmatrix} x_0 \\ x_f \end{pmatrix} \in S_2 \subset \mathbb{R}^{2n}.$$

Give a heuristic argument in support of the transversality condition

$$\left\langle \begin{pmatrix} p^*(t_0) \\ p^*(t_f) \end{pmatrix}, d \right\rangle = 0, \quad \forall d \in T_{\begin{pmatrix} x^*(t_0) \\ x^*(t_f) \end{pmatrix}} S_2,$$

based on an appropriate modification of the proof of the maximum principle. Hint: propagate perturbations in the initial point to the terminal time.

Problem 3. Solve the problem

$$\min_u \int_0^1 \left((x(t))^2 - 2x(t)\right) dt \quad \text{subj. to} \quad \begin{cases} \dot{x}(t) = u(t), & u(t) \in [-1, 1], \\ x(0) = 0, & x(1) = 0. \end{cases}$$

Problem 4. Let $t_f \geq 0$ be a free variable and solve

$$\min_{u(\cdot),\, t_f > 0} \int_0^{t_f} \left(\frac{1}{2}(u(t))^2 - x(t) + t^3\right) dt \quad \text{subj. to} \quad \begin{cases} \dot{x}(t) = u(t), & u(t) \in \mathbb{R} \\ x(0) = 0, & x(t_f) \text{ free,} \end{cases}$$

assuming an optimal control u^* along with a minimizing t_f exists.

Problem 5. Let $t_f \geq 0$ be a free variable and solve

$$\min_{u(\cdot),\, t_f > 0} t_f + \int_0^{t_f} |u(t)| dt \quad \text{subj. to} \quad \begin{cases} \dot{x}(t) = -ax(t) + u(t), & u(t) \in [-1, 1], \\ x(0) = x_0, & x(t_f) = 0, \end{cases}$$

where $a > 0$ and $x_0 \neq 0$.

Problem 6. A boat travels in a river with a strong current c. The current makes the boat drift in the x_1-direction, see Fig. 1. You would like to determine the steering angle such that the boat reaches the shoreline of an island in shortest possible time. The shoreline is described by the circle $S_1 = x : (x_1 - \xi_1)^2 + (x_2 - \xi_2)^2 = r^2$, where $\xi = (\xi_1, \xi_2)$ denotes the center of this island and r is the radius of the island. Assume that the boat starts in the origin $x = (0, 0)$ and assume that its speed relative the water is v and thus $\dot{x}_1(t) = v\cos(\theta(t)) + c$ and $\dot{x}_2(t) = v\sin(\theta(t))$, where $\theta(t)$ is the heading angle indicated in Fig. 1.

a) Formulate the problem as an optimal control problem on standard form.

b) Show that the optimal trajectory is obtained with a constant θ.

c) Determine a system of equations from which the optimal heading can be calculated.

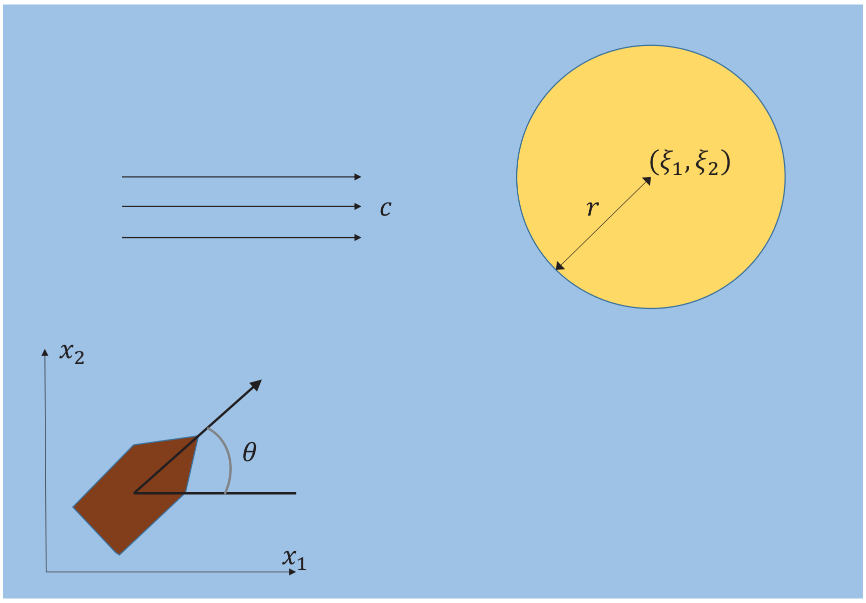

FIGURE 1. The boat and the island in Problem 6.