HOME ASSIGNMENT 3

Problem 1. We consider the brachistochrone problem as an optimal control problem. With suitably chosen units we have $v^2 = y$, so that $\dot{x}^2 + \dot{y}^2 = y$. Let $u_1 = \dot{x}/\sqrt{y}$, $u_2 = \dot{y}/\sqrt{y}$. Hence,

$$\begin{aligned}\dot{x}_1 &= u_1\sqrt{x_2},\\ \dot{x}_2 &= u_2\sqrt{x_2},\end{aligned}$$

where $u_1^2 + u_2^2 = 1$ ($u \in U = S^1$). We have changed the driving variable to time and, since the brachistochrone problem is a time-optimal problem, we get the cost functional

$$J(u) = \int_{t_0}^{t_1} 1dt.$$

The Euler-Lagrange equation for the Brachistochrone problem can be written $1+(y')^2+2yy'' = 0$. Derive this relation using the maximum principle of the optimal control version of the problem.

Problem 2. Consider the double integrator

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= u,\end{aligned}$$

with $u \in [-1, 1]$. Let the initial condition be $x(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, let the final state be $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$, and let the running cost be $L \equiv 1$ so that we have a time-optimal control problem. It is easy to check that the control $u^*(t) = 1$, $t \in [0, 2]$ is optimal. Specialize the main steps of the proof of the maximum principle to this problem and this control. Bring the relevant concepts and conclusions to the level of expicit numerical formulas, and draw figures wherever appropriate.

Problem 3. Explain the geometric interpretation of having $p_0^* = 0$.