HOME ASSIGNMENT 2

Problem 1. A train should travel the distance l in time T. The speed $v(t)$ during the journey should be adjusted so that the total discomfort of the passengers is minimized. The total discomfort is related to the acceleration $a(t)$

$$\int_0^T c_1(a(t))^2 dt$$

but also, due to shaking, to the speed as

$$\int_0^T c_2(v(t))^2 dt$$

where c_1 and c_2 are positive constants. To avoid claims for damages the train should be at rest at embarkation and alighting at the start and end of the journey. Decide the speed of the train $v(t)$.

Problem 2. When the water in a straight circular cylinder is rotated with angular velocity ω around the circular axis, the surface of the water takes the shape that minimizes the potential energy ϕ in the rotating system. The contribution $d\phi$ from the volume element dV is

$$d\phi = \rho\left(gz - \frac{\omega^2 r^2}{2}\right) dV.$$

Decide the height of the water surface $h(r)$ above the bottom if the total volume of water is assumed to be V_0.

Problem 3. Let $y : [a, b] \to \mathbb{R}$ be a weak (but not strong) extremum with one corner point which is located at $x = c \in (a, b)$.

a) Explain where the proof of the Weierstrass-Erdmann condition breaks down for weak extremals.

b) Show the $L_{y'}$ is still continuous on $[a, b]$ by considering perturbations that do not change the x-location of the corner point.

Problem 4. Consider the problem of minimizing the functional

$$J(y) = \int_{-1}^{1} (y'(x))^3 dx$$

subject to the boundary conditions $y(-1) = y(1) = 0$.

a) Characterize all piecewise $\mathcal{C}^1$ extremals of J. Do any of them satisfy the Weierstrass-Erdmann corner conditions?

b) For each one that does, check if it is a minimum (weak- or strong).

Problem 5. Consider the problem of minimizing

$$J(y) = \int_a^b L(x, y(x), y'(x), y''(x))dx,$$

over $\mathcal{C}^2([a, b] \to \mathbb{R})$, with boundary conditions $y(a) = y_0$ and $y(b) = y_1$. Derive the Euler-Lagrange equation and all boundary conditions for this problem.