HOME ASSIGNMENT 1

Problem 1. Consider the problem of minimizing the functional

$$J(y) = \int_0^1 (y'(x))^2(1 - (y'(x))^2)dx,$$

subject to the boundary conditions $y(0) = y(1) = 0$. Is the curve $y \equiv 0$ a weak minimum over $\mathcal{C}^1$ curves? Is it a strong minimum over piecewise $\mathcal{C}^1$ curves? Is there another curve that is a strong minimum?

Problem 2. An exhausted desert wanderer observes an inviting oasis in direction arctan(5/12) over the horizon. Fermat's principle stipulates that light travels along the path that minimizes $\int n ds$, where n is the index of refraction and s is arc-length. During hot days the index of refraction of the air is given by

$$n = n_0(1 - ay)$$

where y is the altitude. Decide the path of the light and the distance to the oasis. The ground can be approximated as a flat surface in this problem.

Problem 3. When climbing a conic volcano sir Edmund Hillary is surprised by a threatening stream of hot lava. From an earlier expedition he knows that there is a cave a bit further on the volcano side at the same altitude that he is presently at. He wishes to seek shelter in the cave and wants to get there as soon as possible by taking the shortest path. Show that he should not follow his first impulse of walking along the contour line. In cylindrical coordinates the cone is given by $r = -az$. Sir Edmund and the caves coordinates are given by $(z, \theta) = (-h, 0)$ and $(z, \theta) = (-h, \pi/2)$, respectively.

Problem 4. You want to minimize the energy (fuel) that is consumed when an airplane takes off. Assume that you start on the ground at $x = 0$ and wish to take off while traveling a distance l horizontally.

First you have to overcome the resistance from the air. The density of the air decreases linearly with altitude, so we assume that the energy required to overcome the air resistance is

$$\int_0^l a(h - y(x))\sqrt{1 + (y'(x))^2}dx,$$

where a and h are two constants.

We also increase the potential energy of the airplane by

$$\int_0^l mg\, dy = \int_0^l mgy'(x)dx,$$

where m is the mass of the airplane. The final altitude $y(l)$ is not decided in advance, but will be decided from the optimal solution. Decide the trajectory y along which the airplane moves.

Problem 5. Consider the problem of minimizing

$$J(y) = \int_a^{t_f} L(x, y(x), y'(x))dx,$$

over $\mathcal{C}^1([a, t_f] \to \mathbb{R})$, with constraints $y(a) = y_0$ and $y(t_f) = \varphi(t_f)$, where $\varphi : \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}^1$ function. Note that this is a free-time problem since t_f is not explicitly constrained. Hence, the perturbations have to be adapted to account for variations in t_f as well.

a) Give a first order necessary condition for y to be a weak extremum.

b) What does the solution to this problem tell us about the Hamiltonian when φ is constant?

c) How do the curves y and φ meet when L is arc-length?