Comparison of Optimal Control Strategies for a Generator Model

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Project purpose

- Compare different optimal control strategies
  - Minimum time controller
  - Linear-Quadratic Regulator (LQR)
- Compare performance with respect to
  - Speed of system
  - Disturbance rejection

Generator model

A generator model was achieved from Example 11.2 in Glad and Ljung (2003). The model had the following state space representation

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= u - a x_2 - b \sin(x_1)\end{aligned} \tag{1}$$

and the parameter values $a = 1$ and $b = 2$ has been used.

Linearized generator model

For the LQR calculations a linearized version of the model has been used. If the model is linearized around the stationary point $(x_1^0, x_2^0, u^0) = (n\pi, 0, 0)$ where n is even, the system becomes

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \tag{2}$$

LQR problem formulation

$$\min_u \int_0^\infty x(t)^T Q x(t) + u(t)^T R u(t) \, dt \tag{3}$$

subject to the system equations (2) and

$$\begin{aligned} u(t) &\geq -5 \\ u(t) &\leq 5 \\ x_1(0) &= 1 \\ x_2(0) &= 0. \end{aligned} \tag{4}$$

Solution to the LQR problem

Solution of the Riccati equation

$$Q + PA + A^T P - PBR^{-1}B^T P = 0$$

gives the control law

$$u = -R^{-1}B^T Px + l_r r = -\begin{pmatrix} 29.6860 & 11.6638 \end{pmatrix} x + 31.686r, \quad (5)$$

where l_r is calculated to achieve a static gain of 1, and the weight matrices used were $Q = \begin{pmatrix} 10 & 0 \\ 0 & 1 \end{pmatrix}$ and $R = 0.01$.

Minimum time problem formulation

$$\min_u \int_0^{t_f} 1dt \tag{6}$$

subject to the system equations (1) and

$$\begin{aligned} -5 \leq & u(t) \leq 5 \\ x(0) &= \begin{pmatrix} 1 & 0 \end{pmatrix} \\ x(t_f) &= \begin{pmatrix} 0 & 0 \end{pmatrix} \\ \dot{x}(t_f) &= \begin{pmatrix} 0 & 0 \end{pmatrix}. \end{aligned} \tag{7}$$

Simulations

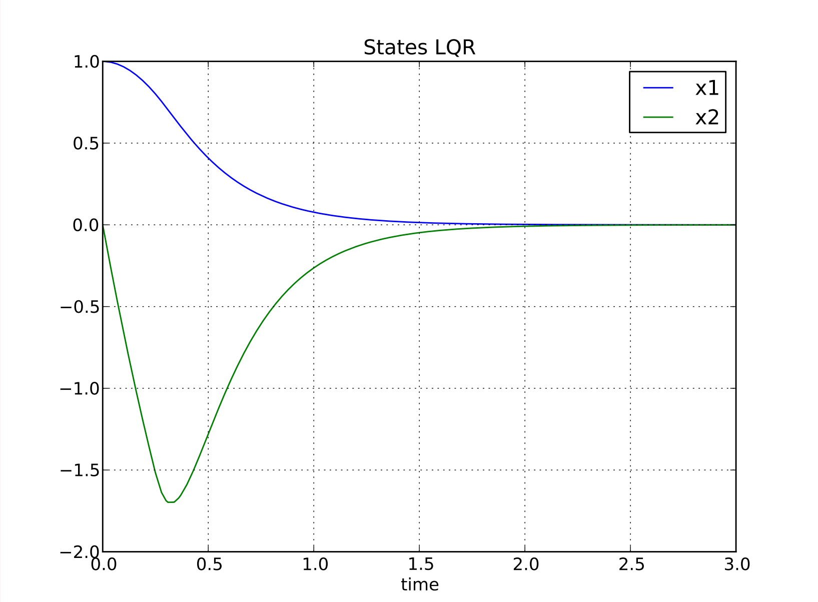
- Problems implemented in JModelica
- Analytical calculations verified

```
optimization LQR(finalTime= 3,
objectiveIntegrand = 10*(x_1)^2+x_2^2+0.01*u^2)
 extends Generator_lin();
constraint
u>=-5;
u<=5;
end LQR;
```

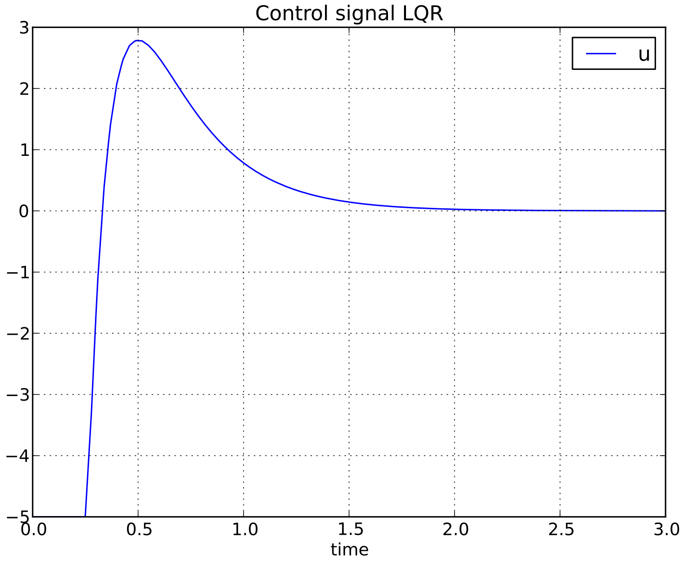
Simulations

```
optimization minTime(finalTime(free=true, min=startTime),
objective=finalTime)
 extends Generator(x_1(start=1), x_2(start=0),
 u(min=-5, max=5));
constraint
x_1(finalTime) = 0;
x_2(finalTime) = 0;
u(finalTime)-a*x_2(finalTime)-b*sin(x_1(finalTime)) = 0;
end minTime;
```

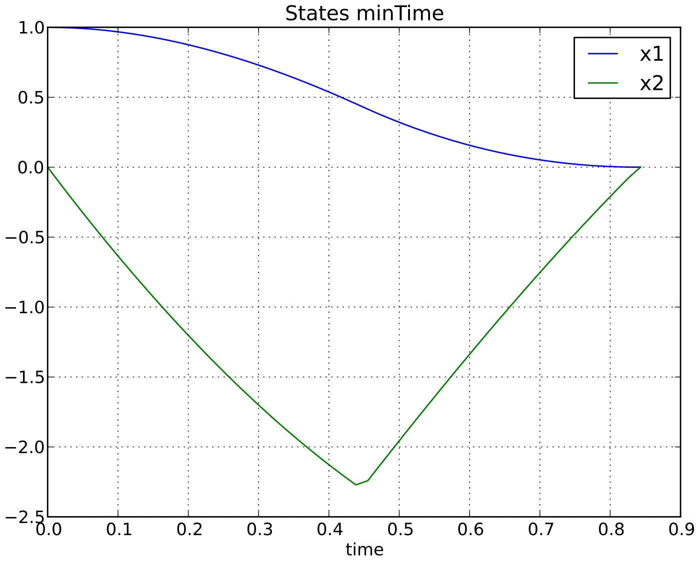
Results



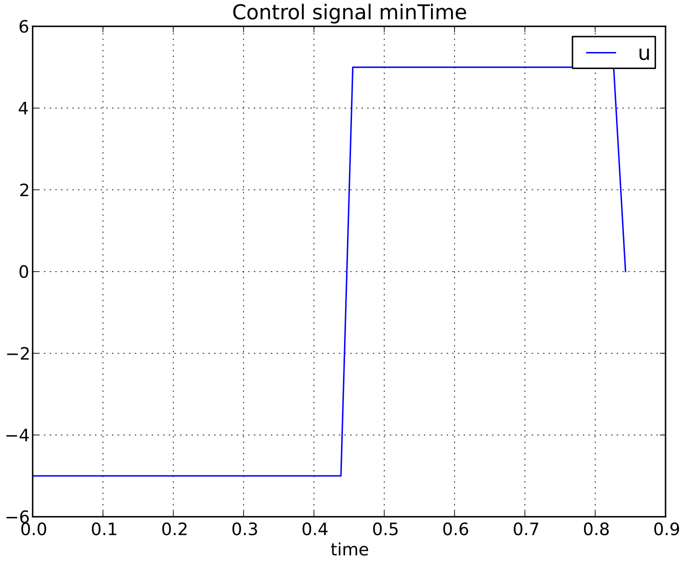
Results



Results


States minTime
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0
−2.5
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
time
x1
x2

Results

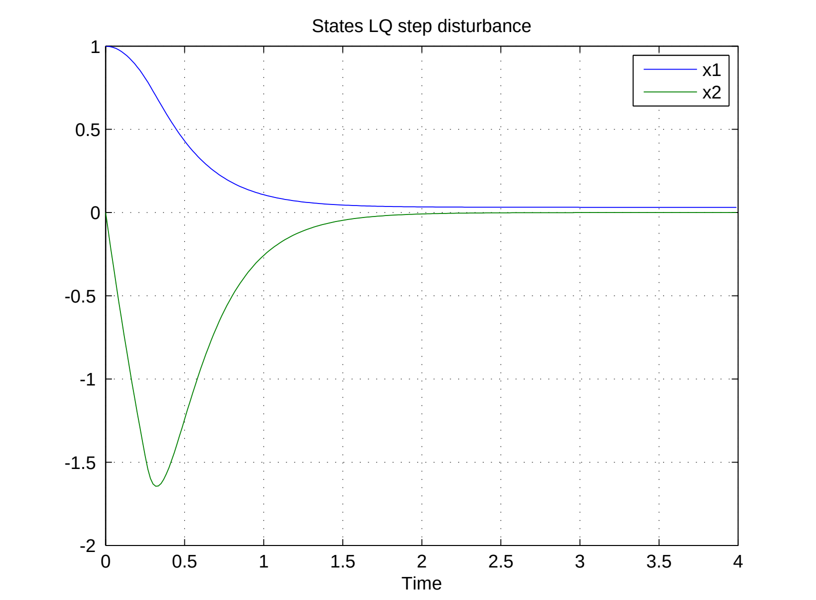


Control signal minTime
u
6
4
2
0
−2
−4
−6
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
time

Disturbance handling

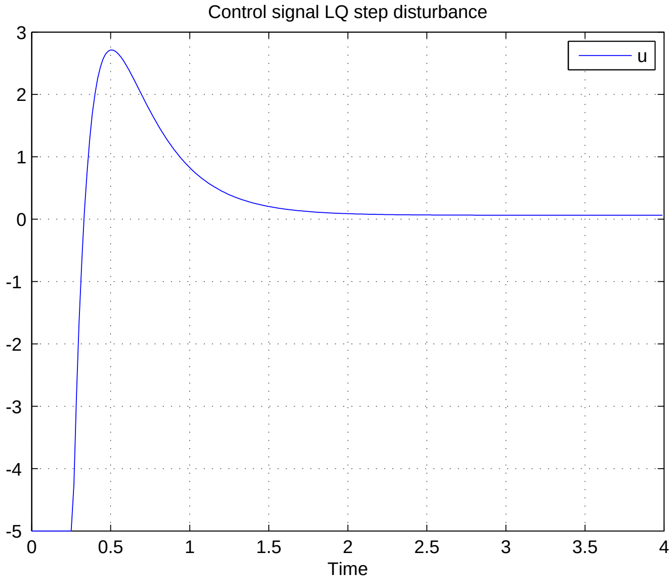
- Interesting to investigate how optimal controller handle disturbances
- Unit step disturbance introduced on control signal at time 0.2
- Tested for both LQR and minimum time controller
- Tests performed in Matlab/Simulink using the results from the JModelica optimization

Results



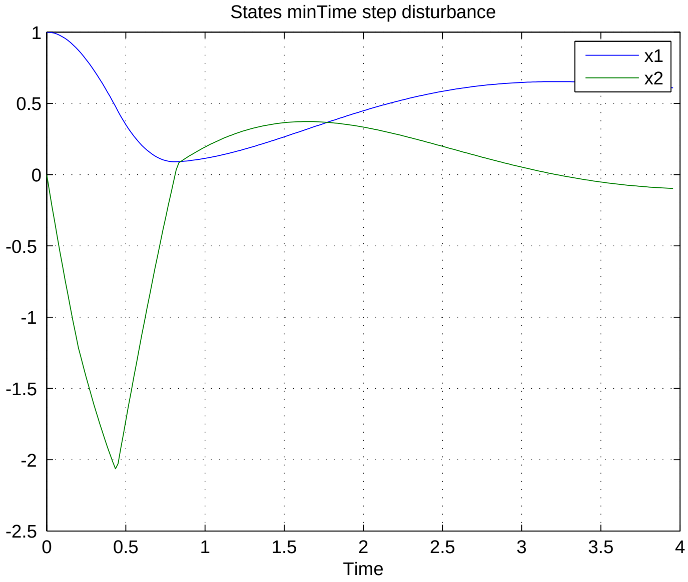
States LQ step disturbance
x1
x2
1
0.5
0
-0.5
-1
-1.5
-2
0
0.5
1
1.5
2
2.5
3
3.5
4
Time

Results



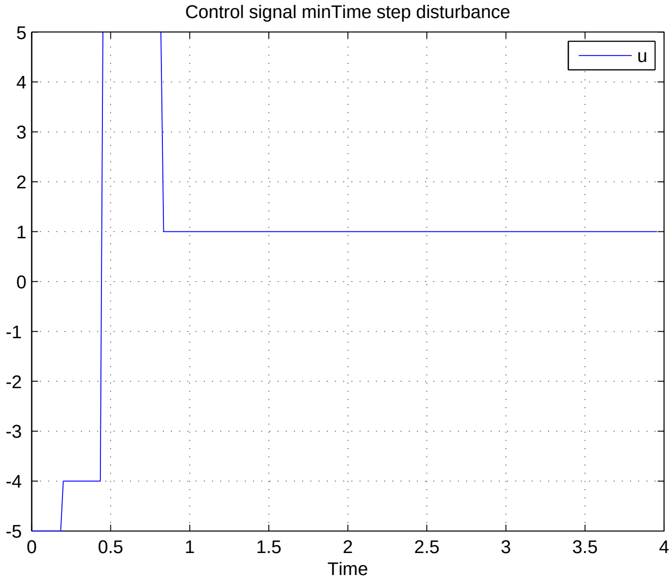
Control signal LQ step disturbance
u
3
2
1
0
-1
-2
-3
-4
-5
0
0.5
1
1.5
2
2.5
3
3.5
4
Time

Results



States minTime step disturbance
x1
x2
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
0
0.5
1
1.5
2
2.5
3
3.5
4
Time

Results



Control signal minTime step disturbance
u
5
4
3
2
1
0
-1
-2
-3
-4
-5
0
0.5
1
1.5
2
2.5
3
3.5
4
Time

Improving disturbance rejection

- Desirable to improve disturbance rejection
- Our idea is to combine the two optimal control strategies
- First design an LQR for the system
- Subsequently design a minimum time reference trajectory for the closed-loop system.

The closed-loop system is given by

$$\dot{x} = (A - (-BR^{-1}B^T P))x + Bl_r r \tag{8}$$

Minimum time for LQR

New minimum time problem given by

$$\min_{r} \int_0^{t_f} 1dt \tag{9}$$

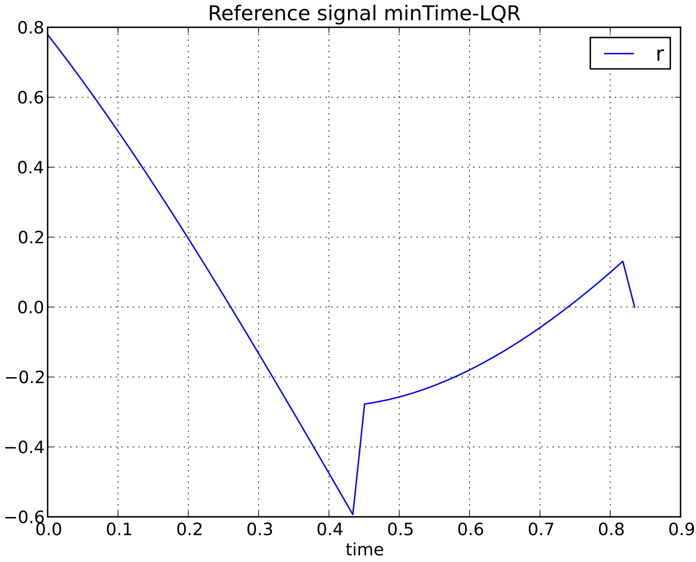
subject to the system equation (8) and

$$\begin{aligned} r(t) &\text{ free} \\ -5 \leq -R^{-1}B^T Px + l_r r &\leq 5 \\ x(0) &= \begin{pmatrix} 1 & 0 \end{pmatrix} \\ x(t_f) &= \begin{pmatrix} 0 & 0 \end{pmatrix} \\ \dot{x}(t_f) &= \begin{pmatrix} 0 & 0 \end{pmatrix}. \end{aligned} \tag{10}$$

Code

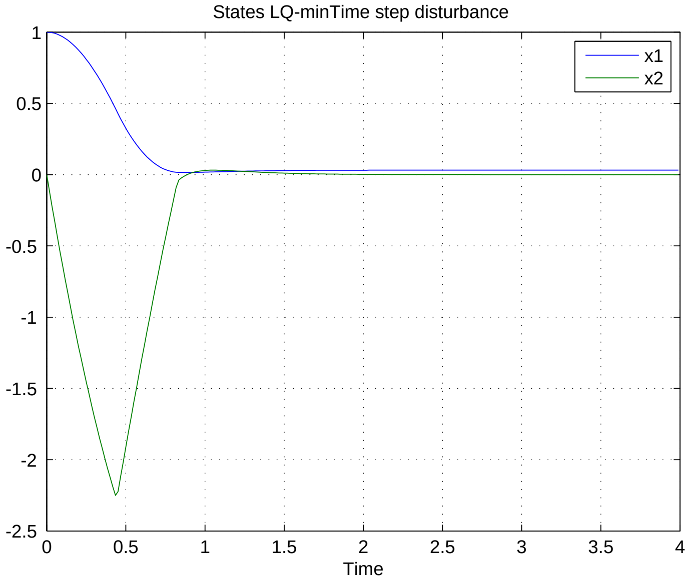
```
optimization minTime_lq(finalTime(free=true,min=startTime),
 objective=finalTime)
 extends Generator_lq(x_1(start=1), x_2(start=0));
constraint
x_1(finalTime) = 0;
x_2(finalTime) = 0;
lr*r(finalTime)+(-b-l1)*x_1(finalTime)+
 (-a-l2)*x_2(finalTime) = 0;
-l1*x_1-l2*x_2+lr*r <=5;
-l1*x_1-l2*x_2+lr*r >=-5;
end minTime_lq;
```

Results



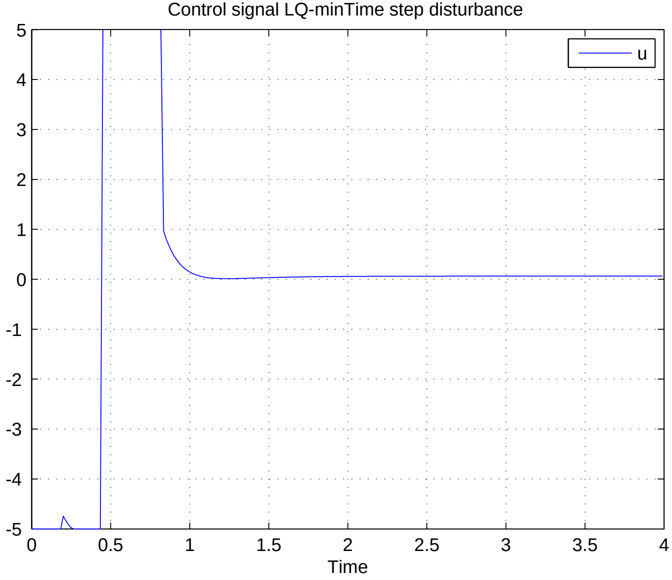
Reference signal minTime-LQR
r
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
−0.6
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
time

Results



States LQ-minTime step disturbance
x1
x2
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
0
0.5
1
1.5
2
2.5
3
3.5
4
Time

Results



Control signal LQ-minTime step disturbance
u
5
4
3
2
1
0
-1
-2
-3
-4
-5
0
0.5
1
1.5
2
2.5
3
3.5
4
Time

Conclusions

- Without disturbances the minimum time controller is clearly faster ($\sim$ 0.8 s) than the LQR ($\sim$ 2 s).
- The minimum time controller however does not handle disturbances in a good manner in contrast to the LQR.
- By combining the strategies the disturbance rejection is improved and the system is faster ($\sim$ 1 s).
- Static errors can be removed by introducing integral action to the LQR.