Optimal Controller Synthesis for the Decentralized Two-Player Problem with Output Feedback

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Abstract

In this paper, we present a controller synthesis algorithm for a decentralized control problem. We consider an architecture in which there are two interconnected linear subsystems. Both controllers seek to optimize a global quadratic cost, despite having access to different subsets of the available measurements. Many special cases of this problem have previously been solved, most notably the state-feedback case. The generalization to output-feedback is nontrivial, as the classical separation principle does not hold. Herein, we present the first explicit state-space realization for an optimal controller for the general two-player problem.

1 Introduction

Many large-scale systems such as the internet, power grids, or teams of autonomous vehicles, can be viewed as a network of interconnected subsystems. A common feature of these applications is that subsystems must make control decisions with limited information. The hope is that despite the decentralized nature of the system, global performance criteria can be optimized.

In this paper, we consider a specific information structure in which there are two linear subsystems and the state-space matrices are block-triangular:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + w$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} C_{11} & 0 \\ C_{21} & C_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + v$$

In other words, Player 1's measurements and dynamics only depend on Player 1's inputs, but Player 2's system

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is fully coupled. Our aim is to find an output-feedback law with this same structure; u_1 must depend only on y_1, but u_2 is allowed to depend on both y_1 and y_2.

The controller must be stabilizing, and must also minimize the infinite-horizon quadratic cost

$$\lim_{T \to \infty} \frac{1}{T} \ \mathbb{E} \int_0^T \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^{\mathsf{T}} \begin{bmatrix} Q & S \\ S^{\mathsf{T}} & R \end{bmatrix} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \ \mathrm{d}t$$

The disturbance w and noise v are assumed to be stationary zero-mean Gaussian processes; they may be correlated, and are characterized by the covariance matrix

$$\mathrm{cov} \begin{bmatrix} w \\ v \end{bmatrix} = \begin{bmatrix} W & U \\ U^{\mathsf{T}} & V \end{bmatrix}$$

In this paper, we provide explicit state-space formulae for an optimal controller. These formulae provide tight upper bounds on the minimal state dimension for an optimal controller, which were previously not known.

The paper is organized as follows. In Section 2, we give a brief history of decentralized control and the two-player problem in particular. In Section 3, we review some required background mathematics and notation. In Section 4, we review the solution to the centralized $\mathcal{H}_2$ synthesis problem. In Sections 5 and 6, we construct our main result, given in Theorem 11. In Sections 7 and 8, we discuss the state dimension and estimation structure of the solution, and finally we conclude in Section 9.

2 Prior Work

If we consider the problem of Section 1 but remove the structural constraint on the controller, the problem becomes a classical $\mathcal{H}_2$ synthesis. Such problems are well studied, and are solved for example in [21]. The optimal controller in this centralized case is linear, and has as many states as the original system.

The presence of structural constraints greatly complicates the problem, and the resulting decentralized problem has been outstanding since the 1968 work of Witsenhausen [18]. That paper posed a related problem for which a nonlinear controller strictly outperforms all linear policies [18]. However, this is not always the case. For a broad class of decentralized control problems there exists a linear optimal policy, and finding it amounts

to solving a convex optimization problem [2, 8, 9, 17]. The two-player problem is in this class, and so we may without loss of generality restrict our search to linear controllers. Despite the benefit of convexity, the search space is infinite-dimensional since we must optimize over transfer functions. The standard numerical approach to solving such general problems is to work in a finite dimensional basis and construct a sequence of approximations which converge to an optimal controller.

Several other numerical and analytical approaches for addressing decentralized optimal control exist, including [7, 12, 20]. One particularly relevant numerical approach is to use vectorization, which converts the decentralized problem into an equivalent centralized problem [10]. This conversion process results in a dramatic growth in state dimension, and so the method is extremely computationally intensive and only feasible for small problems. However, it does provide insight into the problem. Namely, it proves that the optimal controller for the two-player problem is rational, and gives an upper bound on the state dimension.

Explicit solutions have also been found, but only for special cases of the problem. Most notably, the state-feedback case admits a clean state-space solution using a spectral factorization approach [15]. This approach was also used to address a case with partial output-feedback, in which there is output-feedback for one player and state-feedback for the other [16]. The work of [13] also provided a solution to the state-feedback case using the Möbius transform associated with the underlying poset. Certain special cases were also solved in [3], which gave a method for splitting decentralized optimal control problems into multiple centralized problems. This splitting approach addresses a broader class of problems, including state-feedback, partial output-feedback, and dynamically decoupled problems. Another important special case appearing recently is the one-timestep-delayed case [4]. All of these problems are overlapping special cases of the general output-feedback problem considered here.

Of the works above, the first solution was to the two-player problem, in [15]. Subsequent work addresses the multi-player state-feedback problem, including [13, 14]. In this paper, we address the two-player output-feedback problem, via a new approach. It is perhaps closest technically to the work of [15] using spectral factorization, but uses the factorization to split the problem in a different way, allowing a solution of the general output-feedback problem. This paper is a more general version of the invited paper [5].

3   Preliminaries

Transfer functions.   We use the following notation in this paper. The real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$ respectively. We use the notation $\mathcal{L}_2$ to denote the Hilbert space of matrix-valued functions on the imaginary axis, so that $\mathcal{F} \in \mathcal{L}_2$ if $\mathcal{F} : j\mathbb{R} \to \mathbb{C}$ and the following norm is bounded:

$$\|\mathcal{F}\|^2 \triangleq \frac{1}{2\pi} \int_{-\infty}^{\infty} \|\mathcal{F}(j\omega)\|_F^2 \, \mathrm{d}\omega$$

where $\|\cdot\|_F$ is the Frobenius norm. Occasionally we may need to refer to other $\mathcal{L}_2$ spaces also. We use $\mathcal{H}_2$ to denote the well-known corresponding Hardy space, which is a subspace of $\mathcal{L}_2$. This space may be identified with the set of Laplace transforms of $\mathcal{L}_2[0, \infty)$; see for example [1] for details. We append the symbol $\mathcal{R}$ to denote rational functions with real coefficients. The orthogonal complement of $\mathcal{H}_2$ in $\mathcal{L}_2$ is written as $\mathcal{H}_2^{\perp}$.

State-space.   In this paper, all systems are linear and time-invariant (LTI), rational, and continuous-time. Given a state-space representation (A, B, C, D) for such a system, we can describe the input-output map as a matrix of proper rational functions

$$\mathcal{F} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right] \triangleq D + C(sI - A)^{-1}B$$

If the realization is minimal, $\mathcal{F}$ having stable poles is equivalent to A being Hurwitz, and $\mathcal{F}$ being strictly proper is equivalent to $D = 0$. The conjugate transpose $\mathcal{F}^{\sim}(j\omega) = [\mathcal{F}(j\omega)]^*$ satisfies

$$\mathcal{F}^{\sim} = \left[\begin{array}{c|c} -A^{\mathsf{T}} & C^{\mathsf{T}} \\ \hline -B^{\mathsf{T}} & D^{\mathsf{T}} \end{array} \right]$$

Of particular interest is $\mathcal{R}\mathcal{H}_2$, the set of strictly proper rational transfer functions with stable poles. If $z = \mathcal{G}w$ where $\mathcal{G} \in \mathcal{R}\mathcal{H}_2$ and w is white Gaussian noise with unit variance, the average infinite-horizon cost is finite, and equal to the square of the $\mathcal{L}_2$-norm:

$$\lim_{T \to \infty} \frac{1}{T} \, \mathbb{E} \int_0^T \|z(t)\|^2 \, \mathrm{d}t \, = \, \|\mathcal{G}\|^2$$

In this case, the $\mathcal{L}_2$-norm is also called the $\mathcal{H}_2$-norm.

Sylvester Equations.   A Sylvester equation is a matrix equation of the form

$$AX + XB + C = 0$$

where A and B are square matrices, possibly of different sizes. Here, we must solve for X and all other parameters are known. We write $X = \mathrm{LYAP}(A, B, C)$ to denote a solution when it exists.

Riccati Equations.   A continuous-time algebraic Riccati equation (CARE) is a matrix equation of the form

$$A^{\mathsf{T}}X + XA - (XB + S)R^{-1}(XB + S)^{\mathsf{T}} + Q = 0$$

Again, we must solve for X and all other parameters are known. We say $X \geq 0$ is a *stabilizing* solution if $(A + BK)$ is stable, where $K = -R^{-1}(XB + S)^{\mathsf{T}}$ is the associated gain matrix. We write $X = \mathrm{CARE}(A, B, Q, R, S)$ to denote a stabilizing solution when it exists.

Projection. A proper rational matrix transfer function $\mathcal{G}$ may be split into a sum $\mathcal{G} = \mathcal{G}_1 + D + \mathcal{G}_2$ where D is a constant, $\mathcal{G}_1 \in \mathcal{RH}_2$, and $\mathcal{G}_2 \in \mathcal{RH}_2^\perp$.

Lemma 1. *Suppose $\mathcal{G}_1$ and $\mathcal{G}_2$ are stable proper rationals*

$$\mathcal{G}_1 = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array} \right] \quad and \quad \mathcal{G}_2 = \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array} \right]$$

Then $Z = \mathrm{LYAP}(A_1, A_2^\mathsf{T}, B_1 B_2^\mathsf{T})$ has a unique solution and $\mathcal{G}_1 \mathcal{G}_2^\sim$ may be split up as

$$\mathcal{G}_1 \mathcal{G}_2^\sim = \left[\begin{array}{c|c} A_1 & B_1 D_2^\mathsf{T} + Z C_2^\mathsf{T} \\ \hline C_1 & 0 \end{array} \right] + D_1 D_2^\mathsf{T}$$
$$+ \left[\begin{array}{c|c} A_2 & B_2 D_1^\mathsf{T} + Z^\mathsf{T} C_1^\mathsf{T} \\ \hline C_2 & 0 \end{array} \right]^\sim$$

Proof. The identity is easily verified algebraically. Existence and uniqueness of Z follows from the stability of A_1 and A_2. See for example [21, §2]. ∎

We use $\mathbb{P}$ to denote the projection operator $\mathcal{L}_2 \to \mathcal{H}_2$.

Stabilization. For simplicity, we assume throughout this paper that the plant dynamics are stable. In the centralized case, no generality is lost by this assumption. The celebrated Youla parametrization explicitly parametrizes all stabilizing controllers [19, 21].

More care is needed in the decentralized case because coprime factorizations do not preserve sparsity structure in general. In recent work by Sabău and Martins [11], a structure-preserving coprime factorization is found that yields a Youla-like parametrization for all quadratically invariant structural constraints. In particular, it would apply to the problem considered herein.

4 The Centralized Problem

In this section, we review the spectral factorization approach to solving the centralized $\mathcal{H}_2$ synthesis problem. In Section 5, these ideas will be applied to the two-player case. The state-space equations are

$$\dot{x} = Ax + Bu + Mw, \tag{1}$$
$$z = Fx + Hu \tag{2}$$
$$y = Cx + Nw \tag{3}$$

As is standard, we assume $H^\mathsf{T} H > 0$ and $N N^\mathsf{T} > 0$ so that the problem is nonsingular. Our goal is to find a LTI system $\mathcal{K}$ that maps y to u, and minimizes the average infinite-horizon cost $\lim_{T \to \infty} \frac{1}{T} \mathbb{E} \int_0^T \|z(t)\|^2 \, dt$. For consistency with Section 1, define

$$\begin{bmatrix} Q & S \\ S^\mathsf{T} & R \end{bmatrix} \triangleq \begin{bmatrix} F & H \end{bmatrix}^\mathsf{T} \begin{bmatrix} F & H \end{bmatrix} = \begin{bmatrix} F^\mathsf{T} F & F^\mathsf{T} H \\ H^\mathsf{T} F & H^\mathsf{T} H \end{bmatrix}$$

$$\begin{bmatrix} W & U \\ U^\mathsf{T} & V \end{bmatrix} \triangleq \begin{bmatrix} M \\ N \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix}^\mathsf{T} = \begin{bmatrix} M M^\mathsf{T} & M N^\mathsf{T} \\ N M^\mathsf{T} & N N^\mathsf{T} \end{bmatrix}$$

By taking Laplace transforms of (1)–(3), and eliminating x, we obtain

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{21} & \mathcal{P}_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \tag{4}$$

where the $\mathcal{P}_{ij}$ are transfer functions given by

$$\mathcal{P}_{11} = \left[\begin{array}{c|c} A & M \\ \hline F & 0 \end{array} \right] \quad \mathcal{P}_{12} = \left[\begin{array}{c|c} A & B \\ \hline F & H \end{array} \right]$$
$$\mathcal{P}_{21} = \left[\begin{array}{c|c} A & M \\ \hline C & N \end{array} \right] \quad \mathcal{P}_{22} = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array} \right] \tag{5}$$

As above, we assume A is Hurwitz. Substituting $u = \mathcal{K}y$ and eliminating y and u from (4), we obtain the closed-loop map

$$z = \left(\mathcal{P}_{11} + \mathcal{P}_{12} \mathcal{K} (I - \mathcal{P}_{22} \mathcal{K})^{-1} \mathcal{P}_{21} \right) w \tag{6}$$

Since minimizing the average infinite-horizon cost is equivalent to minimizing the $\mathcal{H}_2$-norm of the closed-loop map, we seek to

$$\begin{aligned} \text{minimize} \quad & \left\| \mathcal{P}_{11} + \mathcal{P}_{12} \mathcal{K} (I - \mathcal{P}_{22} \mathcal{K})^{-1} \mathcal{P}_{21} \right\| \\ \text{subject to} \quad & \mathcal{K} \text{ is proper and rational} \\ & \mathcal{K} \text{ is stabilizing} \end{aligned} \tag{7}$$

The norm above is taken to be infinity when the linear fractional function is not strictly proper. Using the well-known Youla parameterization [19] of all stabilizing controllers, make the substitution $\mathcal{Q} = \mathcal{K}(I - \mathcal{P}_{22} \mathcal{K})^{-1}$. Since $\mathcal{P}_{22} \in \mathcal{RH}_2$, the constraint that $\mathcal{K}$ be stabilizing and proper is equivalent to the constraint that $\mathcal{Q}$ be stable. In addition, the assumptions that $H^\mathsf{T} H > 0$ and $N N^\mathsf{T} > 0$ imply that $\mathcal{Q}$ must be strictly proper to ensure finiteness of the norm. We would therefore like to solve

$$\begin{aligned} \text{minimize} \quad & \left\| \mathcal{P}_{11} + \mathcal{P}_{12} \mathcal{Q} \mathcal{P}_{21} \right\| \\ \text{subject to} \quad & \mathcal{Q} \in \mathcal{RH}_2 \end{aligned} \tag{8}$$

Note that the $\mathcal{Q}$-substitution is invertible, and its inverse is $\mathcal{K} = \mathcal{Q}(I + \mathcal{P}_{22} \mathcal{Q})^{-1}$. So solving (8) will give us a solution to the original problem (7).

Lemma 2. *Suppose $\mathcal{P}_{11}$, $\mathcal{P}_{12}$, and $\mathcal{P}_{21}$ are defined by (5), and there exist stabilizing solutions to the CAREs*

$$X = \mathrm{CARE}(A, B, Q, R, S), \quad K = -R^{-1}(XB + S)^\mathsf{T}$$
$$Y = \mathrm{CARE}(A^\mathsf{T}, C^\mathsf{T}, W, V, U), \quad L = -(Y C^\mathsf{T} + U)V^{-1}$$

A solution to (8) is given by

$$\mathcal{Q}_{opt} = \left[\begin{array}{cc|c} A + BK & BK & 0 \\ 0 & A + LC & -L \\ \hline K & K & 0 \end{array} \right] \tag{9}$$

Proof. See for example [21, §14]. ∎

The solution (9) is the celebrated classical $\mathcal{H}_2$-optimal controller.

3

In Section 5, we will require the solution to the well-known $\mathcal{H}_2$ model-matching problem. This problem is more general than the above because here, $\mathcal{P}_{11}$, $\mathcal{P}_{12}$, and $\mathcal{P}_{21}$ do not share a common A-matrix in their state-space representations.

Lemma 3. *Suppose $\mathcal{P}_{11}$, $\mathcal{P}_{12}$, and $\mathcal{P}_{21}$ are matrices of stable transfer functions with state-space realizations*

$$\mathcal{P}_{11} = \left[\begin{array}{c|c} A & J \\ \hline G & 0 \end{array}\right] \quad \mathcal{P}_{12} = \left[\begin{array}{c|c} \tilde{A} & B \\ \hline F & H \end{array}\right] \quad \mathcal{P}_{21} = \left[\begin{array}{c|c} \hat{A} & M \\ \hline C & N \end{array}\right]$$

Note that A, $\tilde{A}$, and $\hat{A}$ may be different matrices. Suppose there exists stabilizing solutions to the CAREs

$$X = \mathrm{CARE}(\tilde{A}, B, Q, R, S), \qquad K = -R^{-1}(XB + S)^{\mathsf{T}}$$
$$Y = \mathrm{CARE}(\hat{A}^{\mathsf{T}}, C^{\mathsf{T}}, W, V, U), \quad L = -(YC^{\mathsf{T}} + U)V^{-1}$$

Then, there exists unique solutions to the equations

$$\begin{aligned} \tilde{Z} &= \mathrm{LYAP}\left((\tilde{A} + BK)^{\mathsf{T}}, A, (F + HK)^{\mathsf{T}}G\right) \\ \hat{Z} &= \mathrm{LYAP}\left(A, (\hat{A} + LC)^{\mathsf{T}}, J(M + LN)^{\mathsf{T}}\right) \end{aligned} \tag{10}$$

Furthermore, a solution to (8) is given by

$$\mathcal{Q}_{opt} = -\mathcal{W}_L^{-1} \left[\begin{array}{c|c} A & JN^{\mathsf{T}} + \hat{Z}C^{\mathsf{T}} \\ \hline B^{\mathsf{T}}\tilde{Z} + H^{\mathsf{T}}G & 0 \end{array}\right] \mathcal{W}_R^{-1} \tag{11}$$

where $\mathcal{W}_L$ and $\mathcal{W}_R$ are defined by

$$\mathcal{W}_L = \left[\begin{array}{c|c} \tilde{A} & B \\ \hline -R^{1/2}K & R^{1/2} \end{array}\right] \quad \mathcal{W}_R = \left[\begin{array}{c|c} \hat{A} & -LV^{1/2} \\ \hline C & V^{1/2} \end{array}\right]$$

Proof. The optimality condition [6] for (8) is

$$\mathcal{P}_{\widetilde{12}}\mathcal{P}_{11}\mathcal{P}_{\widetilde{21}} + \mathcal{P}_{\widetilde{12}}\mathcal{P}_{12}\mathcal{Q}\mathcal{P}_{21}\mathcal{P}_{\widetilde{21}} \in \mathcal{H}_2^{\perp}$$

Compute spectral factorizations, as in [21, §13]. Then $\mathcal{P}_{\widetilde{12}}\mathcal{P}_{12} = \mathcal{W}_L^{\sim}\mathcal{W}_L$ and $\mathcal{P}_{21}\mathcal{P}_{\widetilde{21}} = \mathcal{W}_R\mathcal{W}_R^{\sim}$. The optimal $\mathcal{Q}$ is thus $\mathcal{Q}_{\mathrm{opt}} = -\mathcal{W}_L^{-1}\mathbb{P}\left(\mathcal{W}_L^{-\sim}\mathcal{P}_{\widetilde{12}}\mathcal{P}_{11}\mathcal{P}_{\widetilde{21}}\mathcal{W}_R^{-\sim}\right)\mathcal{W}_R^{-1}$. Now simplify

$$\mathbb{P}\left(\mathcal{W}_L^{-\sim}\mathcal{P}_{\widetilde{12}}\mathcal{P}_{11}\mathcal{P}_{\widetilde{21}}\mathcal{W}_R^{-\sim}\right) =$$
$$\mathbb{P}\left(\left[\begin{array}{c|c} \tilde{A} + BK & BR^{-1/2} \\ \hline F + HK & HR^{-1/2} \end{array}\right]^{\sim} \left[\begin{array}{c|c} A & J \\ \hline G & 0 \end{array}\right] \right.$$
$$\left. \times \left[\begin{array}{c|c} \hat{A} + LC & M + LN \\ \hline V^{-1/2}C & V^{-1/2}N \end{array}\right]^{\sim}\right)$$

We may compute this projection by applying Lemma 1 twice. This results in the two Sylvester equations (10). Upon simplification, we obtain the final solution (11). ∎

Remark 4. *The general problem considered in Lemma 3 simplifies to the classical problem in Lemma 2 if we set $\tilde{A} \triangleq A$, $\hat{A} \triangleq A$, $G \triangleq M$, and $J \triangleq F$. Under these assumptions, the Riccati and Sylvester equations have the same solutions. Indeed, we find $\tilde{Z} = X$ and $\hat{Z} = Y$. This is why (9) is so much simpler than (11).*

5 The Two-Player Problem

Many of the equations for the centralized problem covered in Section 4 still hold for the two-player problem. In particular, (1)–(6) are the same, but we have some additional structure:

$$A \triangleq \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix} \quad B \triangleq \begin{bmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{bmatrix} \quad C \triangleq \begin{bmatrix} C_{11} & 0 \\ C_{21} & C_{22} \end{bmatrix}$$

We also impose a similar structure on our controller $\mathcal{K}$. We denote the set of block lower-triangular operators as $\mathcal{S}$, and omit the specific class of operators from this notation for convenience. We therefore write the constraint as $\mathcal{K} \in \mathcal{S}$. To ease notation, define

$$E_1 \triangleq \begin{bmatrix} I \\ 0 \end{bmatrix} \quad \text{and} \quad E_2 \triangleq \begin{bmatrix} 0 \\ I \end{bmatrix}$$

where sizes of the identity matrices involved are determined by context. We also partition B by its block-columns and C by its block-rows. Thus, $B_1 \triangleq BE_1$, $B_2 \triangleq BE_2$, $C_1 \triangleq E_1^{\mathsf{T}}C$, and $C_2 \triangleq E_2^{\mathsf{T}}C$. The optimization problem (7) becomes

$$\begin{aligned} & \text{minimize} && \left\|\mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}\mathcal{P}_{21}\right\| \\ & \text{subject to} && \mathcal{K} \text{ is proper and rational} \\ & && \mathcal{K} \text{ is stabilizing} \\ & && \mathcal{K} \in \mathcal{S} \end{aligned} \tag{12}$$

We make the same substitution $\mathcal{Q} = \mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}$. Note that from (5), $\mathcal{P}_{22} \in \mathcal{S}$. It follows that $\mathcal{K} \in \mathcal{S}$ if and only if $\mathcal{Q} \in \mathcal{S}$. This property allows us to write a convex optimization problem in $\mathcal{Q}$:

$$\begin{aligned} & \text{minimize} && \left\|\mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{Q}\mathcal{P}_{21}\right\| \\ & \text{subject to} && \mathcal{Q} \in \mathcal{R}\mathcal{H}_2 \cap \mathcal{S} \end{aligned} \tag{13}$$

The optimality condition for (13) is

$$\mathcal{P}_{\widetilde{12}}\mathcal{P}_{11}\mathcal{P}_{\widetilde{21}} + \mathcal{P}_{\widetilde{12}}\mathcal{P}_{12} \begin{bmatrix} \mathcal{Q}_{11} & 0 \\ \mathcal{Q}_{21} & \mathcal{Q}_{22} \end{bmatrix} \mathcal{P}_{21}\mathcal{P}_{\widetilde{21}} \in \begin{bmatrix} \mathcal{H}_2^{\perp} & \mathcal{L}_2 \\ \mathcal{H}_2^{\perp} & \mathcal{H}_2^{\perp} \end{bmatrix}$$

At this point, our solution diverges from that of the centralized case. Indeed, the spectral factorization approach of Lemma 3 fails because in general, structured spectral factors may not exist.

A key observation is that if we assume $\mathcal{Q}_{11}$ is known, and possibly suboptimal, then the problem of finding the optimal $\begin{bmatrix} \mathcal{Q}_{21} & \mathcal{Q}_{22} \end{bmatrix}$ is centralized:

$$\begin{aligned} & \min && \left\|\left(\mathcal{P}_{11} + \mathcal{P}_{12}E_1\mathcal{Q}_{11}E_1^{\mathsf{T}}\mathcal{P}_{21}\right) + \mathcal{P}_{12}E_2\begin{bmatrix} \mathcal{Q}_{21} & \mathcal{Q}_{22} \end{bmatrix}\mathcal{P}_{21}\right\| \\ & \text{s.t.} && \begin{bmatrix} \mathcal{Q}_{21} & \mathcal{Q}_{22} \end{bmatrix} \in \mathcal{R}\mathcal{H}_2 \end{aligned} \tag{14}$$

and its solution is given in the following lemma.

Lemma 5. *Suppose $\mathcal{Q}_{11} \in \mathcal{R}\mathcal{H}_2$ and has a realization*

$$\mathcal{Q}_{11} = \left[\begin{array}{c|c} A_P & B_P \\ \hline C_P & 0 \end{array}\right]$$

Suppose that stabilizing solutions exist to the CAREs

$$Y = \mathrm{CARE}(A^\mathsf{T}, C^\mathsf{T}, W, V, U), \quad L = -(YC^\mathsf{T} + U)V^{-1}$$

$$\tilde{X} = \mathrm{CARE}(A, B_2, Q, R_{22}, SE_2),$$

$$\tilde{K} = -R_{22}^{-1}(\tilde{X}B_2 + SE_2)^\mathsf{T} = \begin{bmatrix} \tilde{K}_1 & \tilde{K}_2 \end{bmatrix}$$

Then there exists a unique solution to the equation

$$\begin{bmatrix} \Phi & \tilde{Z}_3 \end{bmatrix} = \mathrm{LYAP}\left((A_{22} + B_{22}\tilde{K}_2)^\mathsf{T}, \begin{bmatrix} A_{11} & 0 \\ B_P C_{11} & A_P \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} 0 & (E_2^\mathsf{T} \tilde{X} B_1 + S_{21} + \tilde{K}_2^\mathsf{T} R_{21})C_p \end{bmatrix} \right) \quad (15)$$

Furthermore, a solution to (14) is given by

$$\begin{bmatrix} \mathcal{Q}_{21} & \mathcal{Q}_{22} \end{bmatrix}_{opt} = \left[\begin{array}{c|c} A_{22} + B_{22}\tilde{K}_2 & B_{22} \\ \hline \tilde{K}_2 & I \end{array} \right]$$

$$\times \left[\begin{array}{ccc|c} A & 0 & B_1 C_P & 0 \\ 0 & A + LC & 0 & -L \\ 0 & 0 & A_P & B_P E_1^\mathsf{T} \\ \hline \tilde{K} & \hat{K} & -R_{22}^{-1}(B_{22}^\mathsf{T}\tilde{Z}_3 + R_{21}C_P) & 0 \end{array} \right]$$

$$(16)$$

where we have defined $\hat{K} = \tilde{K} - R_{22}^{-1}B_{22}^\mathsf{T}\Phi E_1^\mathsf{T}$.

Proof. The components of (14) may be simplified. Routine algebraic manipulations yield

$$\mathcal{P}_{11} + \mathcal{P}_{12}E_1\mathcal{Q}_{11}E_1^\mathsf{T}\mathcal{P}_{21} =$$

$$\left[\begin{array}{ccc|c} A & 0 & B_1 C_P & 0 \\ 0 & A & 0 & M \\ 0 & B_p C_1 & A_P & B_P E_1^\mathsf{T} N \\ \hline F & F & H E_1 C_P & 0 \end{array} \right]$$

and

$$\mathcal{P}_{12}E_2 = \left[\begin{array}{c|c} A_{22} & B_{22} \\ \hline F E_2 & H E_2 \end{array} \right]$$

Since (14) is centralized, we may apply Lemma 3, and the optimal $\begin{bmatrix} \mathcal{Q}_{21} & \mathcal{Q}_{22} \end{bmatrix}$ is given by (11). This formula can be simplified considerably if we take a closer look at the Sylvester equations (10). The estimation equation,

$$\hat{Z} = \mathrm{LYAP}\left(\begin{bmatrix} A & 0 & B_1 C_P \\ 0 & A & 0 \\ 0 & B_p C_1 & A_P \end{bmatrix}, (A + LC)^\mathsf{T}, \right.$$

$$\left. \begin{bmatrix} 0 \\ W + UL^\mathsf{T} \\ B_P E_1^\mathsf{T}(U^\mathsf{T} + VL^\mathsf{T}) \end{bmatrix} \right)$$

is satisfied by $\hat{Z} = \begin{bmatrix} 0 & Y & 0 \end{bmatrix}^\mathsf{T}$, which does not depend on A_P, B_P, or C_P. The control equation,

$$\tilde{Z} = \mathrm{LYAP}\left((A_{22} + B_{22}\tilde{K}_2)^\mathsf{T}, \begin{bmatrix} A & 0 & B_1 C_P \\ 0 & A & 0 \\ 0 & B_P C_1 & A_P \end{bmatrix}, \right.$$

$$\left. E_2^\mathsf{T} \begin{bmatrix} Q & Q & SE_1 C_P \end{bmatrix} + \tilde{K}_2^\mathsf{T} E_2^\mathsf{T} \begin{bmatrix} S^\mathsf{T} & S^\mathsf{T} & R E_1 C_P \end{bmatrix} \right)$$

has a first subequation that decouples from the rest, and whose solution is $E_2^\mathsf{T}\tilde{X}$. Indeed, $\tilde{Z}$ must be of the form:

$$\tilde{Z} = \begin{bmatrix} E_2^\mathsf{T}\tilde{X} & E_2^\mathsf{T}\tilde{X} + \Phi E_1^\mathsf{T} & \tilde{Z}_3 \end{bmatrix}$$

where Φ and $\tilde{Z}_3$ satisfy (15). Substituting into (11) and simplifying, we obtain (16). ∎

A similar result holds if we fix $\mathcal{Q}_{22}$. Our centralized optimization problem is then:

$$\min \quad \left\| (\mathcal{P}_{11} + \mathcal{P}_{12}E_2\mathcal{Q}_{22}E_2^\mathsf{T}\mathcal{P}_{21}) + \mathcal{P}_{12}\begin{bmatrix} \mathcal{Q}_{11} \\ \mathcal{Q}_{21} \end{bmatrix} E_1^\mathsf{T}\mathcal{P}_{21} \right\|$$

$$\text{s.t.} \quad \begin{bmatrix} \mathcal{Q}_{11} \\ \mathcal{Q}_{21} \end{bmatrix} \in \mathcal{RH}_2$$

$$(17)$$

and its solution is given in the following lemma.

Lemma 6. *Suppose* $\mathcal{Q}_{22} \in \mathcal{RH}_2$ *and has a realization*

$$\mathcal{Q}_{22} = \left[\begin{array}{c|c} A_Q & B_Q \\ \hline C_Q & 0 \end{array} \right]$$

Suppose that stabilizing solutions exist to the CAREs

$$X = \mathrm{CARE}(A, B, Q, R, S), \quad K = -R^{-1}(XB + S)^\mathsf{T}$$

$$\tilde{Y} = \mathrm{CARE}(A^\mathsf{T}, C_1^\mathsf{T}, W, V_{11}, UE_1),$$

$$\tilde{L} = -(\tilde{Y}C_1^\mathsf{T} + UE_1)V_{11}^{-1} = \begin{bmatrix} \tilde{L}_1 \\ \tilde{L}_2 \end{bmatrix}$$

Then there exists a unique solution to the equation

$$\begin{bmatrix} \Psi \\ \hat{Z}_3 \end{bmatrix} = \mathrm{LYAP}\left(\begin{bmatrix} A_{22} & B_{22}C_Q \\ 0 & A_Q \end{bmatrix}, (A_{11} + \tilde{L}_1 C_{11})^\mathsf{T}, \right.$$

$$\left. \begin{bmatrix} 0 \\ B_Q(C_2\tilde{Y}E_1 + U_{12}^\mathsf{T} + V_{21}\tilde{L}_1^\mathsf{T}) \end{bmatrix} \right) \quad (18)$$

Furthermore, a solution to (17) is given by

$$\begin{bmatrix} \mathcal{Q}_{11} \\ \mathcal{Q}_{21} \end{bmatrix}_{opt} =$$

$$\left[\begin{array}{ccc|c} A + BK & 0 & 0 & -\hat{L} \\ 0 & A & 0 & -\tilde{L} \\ 0 & B_Q C_2 & A_Q & (B_Q V_{21} + \hat{Z}_3 C_{11}^\mathsf{T})V_{11}^{-1} \\ \hline K & 0 & -E_2 C_Q & 0 \end{array} \right]$$

$$\times \left[\begin{array}{c|c} A_{11} + \tilde{L}_1 C_{11} & \tilde{L}_1 \\ \hline C_{11} & I \end{array} \right] \quad (19)$$

where we have defined $\hat{L} = \tilde{L} - E_2\Psi C_{11}^\mathsf{T}V_{11}^{-1}$.

Proof. The proof is omitted, as it is analogous to that of Lemma 5. ∎

Remark 7. *If we isolate the optimal $\mathcal{Q}_{22}$ from Lemma 5, it simplifies greatly. Indeed, if we multiply (16) on the right by E_2, we obtain*

$$\mathcal{Q}_{22} = \left[\begin{array}{cc|c} A_{22} + B_{22}\tilde{K}_2 & B_{22}\hat{K} & 0 \\ 0 & A + LC & -LE_2 \\ \hline \tilde{K}_2 & \hat{K} & 0 \end{array}\right] \quad (20)$$

Similarly, the optimal $\mathcal{Q}_{11}$ from Lemma 6 simplifies to

$$\mathcal{Q}_{11} = \left[\begin{array}{cc|c} A + BK & \hat{L}C_{11} & -\hat{L} \\ 0 & A_{11} + \tilde{L}_1 C_{11} & -\tilde{L}_1 \\ \hline E_1^\mathsf{T} K & 0 & 0 \end{array}\right] \quad (21)$$

Remark 7 is the key observation that allows us to find a relatively simple analytic formula for the optimal controller. By substituting the result of Lemma 6 into Lemma 5, or vice-versa, we can obtain a simple set of equations that characterize the optimal controller.

6   Main Results

In this section, we present our main result: an explicit solution to (13). We begin by presenting some assumptions that will be needed to guarantee a solution.

A1. (A, B_2) is stabilizable.

A2. $R = H^\mathsf{T} H > 0$

A3. $\begin{bmatrix} A - j\omega I & B \\ F & H \end{bmatrix}$ has full column rank for all $\omega \in \mathbb{R}$.

A4. (C_1, A) is detectable.

A5. $V = NN^\mathsf{T} > 0$

A6. $\begin{bmatrix} A - j\omega I & M \\ C & N \end{bmatrix}$ has full row rank for all $\omega \in \mathbb{R}$.

Note that because we assumed A is stable, assumptions A1 and A4 are redundant. Next, we present the equations we will need to solve in order to construct the optimal controller. First, we have two control CAREs and their associated gains

$$X = \text{CARE}(A, B, Q, R, S)$$
$$K = -R^{-1}(XB + S)^\mathsf{T} \quad (22)$$

$$\tilde{X} = \text{CARE}(A, B_2, Q, R_{22}, SE_2)$$
$$\tilde{K} = -R_{22}^{-1}(\tilde{X}B_2 + SE_2)^\mathsf{T} = \begin{bmatrix} \tilde{K}_1 & \tilde{K}_2 \end{bmatrix} \quad (23)$$

Next, we have the analogous set of estimation equations.

$$Y = \text{CARE}(A^\mathsf{T}, C^\mathsf{T}, W, V, U)$$
$$L = -(YC^\mathsf{T} + U)V^{-1} \quad (24)$$

$$\tilde{Y} = \text{CARE}(A^\mathsf{T}, C_1^\mathsf{T}, W, V_{11}, UE_1)$$
$$\tilde{L} = -(\tilde{Y}C_1^\mathsf{T} + UE_1)V_{11}^{-1} = \begin{bmatrix} \tilde{L}_1 \\ \tilde{L}_2 \end{bmatrix} \quad (25)$$

Finally, we define a pair of coupled linear equations that must also be solved for Φ and Ψ.

$$\begin{aligned} (A_{22} &+ B_{22}\tilde{K}_2)^\mathsf{T}\Phi + \Phi(A_{11} + \tilde{L}_1 C_{11}) \\ &+ E_2^\mathsf{T}(\tilde{X} - X)(\tilde{L} - E_2\Psi C_{11}^\mathsf{T} V_{11}^{-1})C_{11} = 0 \\ (A_{22} &+ B_{22}\tilde{K}_2)\Psi + \Psi(A_{11} + \tilde{L}_1 C_{11})^\mathsf{T} \\ &+ B_{22}(\tilde{K} - R_{22}^{-1}B_{22}^\mathsf{T}\Phi E_1^\mathsf{T})(\tilde{Y} - Y)E_1 = 0 \end{aligned} \quad (26)$$

Note that these equations are linear in Φ and Ψ and can be solved easily; for example, they may be written in standard $Ax = b$ form using the Kronecker product. For convenience, we reiterate here the gains $\hat{K}$ and $\hat{L}$ that were defined in Lemmas 5 and 6.

$$\hat{K} = \tilde{K} - R_{22}^{-1}B_{22}^\mathsf{T}\Phi E_1^\mathsf{T}$$
$$\hat{L} = \tilde{L} - E_2\Psi C_{11}^\mathsf{T} V_{11}^{-1} \quad (27)$$

The following lemma guarantees the existence of solutions to the CAREs.

Lemma 8. *The assumptions A1–A6 are necessary and sufficient for the existence of stabilizing solutions to the CAREs (22)–(25).*

Proof. This is a standard result regarding CAREs. See for example [21, §13]. ∎

Remark 9. *A simpler sufficient (but not necessary) set of conditions that guarantees the existence of stabilizing solutions to (22)–(25) is given by:*

B1. $R > 0$ and $V > 0$

B2. (A, B_2) and (A, W) are controllable

B3. (C_1, A) and (Q, A) are observable

What follows is the main result of the paper.

Theorem 10. *Suppose assumptions A1–A6 or B1–B3 hold. Then (26) has a unique solution, and an optimal solution to (13) is given by*

$$\mathcal{Q}_{opt} =$$
$$\left[\begin{array}{ccc|c} A + BK & -\hat{L}C_1 & 0 & -\hat{L}E_1^\mathsf{T} \\ 0 & A + B_2\hat{K} + \hat{L}C_1 & -B_2\hat{K} & \hat{L}E_1^\mathsf{T} \\ 0 & 0 & A + LC & L \\ \hline K & E_2\hat{K} & -E_2\hat{K} & 0 \end{array}\right] \quad (28)$$

Proof. Solving (13) is equivalent to simultaneously solving (14) and (17). To see why, write the optimality conditions for each one

$$E_2^\mathsf{T} P_{12}^\sim \left(\mathcal{P}_{11} + \mathcal{P}_{12} \begin{bmatrix} \mathcal{Q}_{11} & 0 \\ \mathcal{Q}_{21} & \mathcal{Q}_{22} \end{bmatrix} \mathcal{P}_{21} \right) \mathcal{P}_{21}^\sim \in \begin{bmatrix} \mathcal{H}_2^\perp & \mathcal{H}_2^\perp \end{bmatrix}$$

$$P_{12}^\sim \left(\mathcal{P}_{11} + \mathcal{P}_{12} \begin{bmatrix} \mathcal{Q}_{11} & 0 \\ \mathcal{Q}_{21} & \mathcal{Q}_{22} \end{bmatrix} \mathcal{P}_{21} \right) \mathcal{P}_{21}^\sim E_1 \in \begin{bmatrix} \mathcal{H}_2^\perp \\ \mathcal{H}_2^\perp \end{bmatrix}$$

and note that they are equivalent to

$$P_{12}^{\sim}\left(\mathcal{P}_{11} + \mathcal{P}_{12}\begin{bmatrix} \mathcal{Q}_{11} & 0 \\ \mathcal{Q}_{21} & \mathcal{Q}_{22} \end{bmatrix}\mathcal{P}_{21}\right)\mathcal{P}_{21}^{\sim} \in \begin{bmatrix} \mathcal{H}_2^{\perp} & \mathcal{L}_2 \\ \mathcal{H}_2^{\perp} & \mathcal{H}_2^{\perp} \end{bmatrix}$$
$$(29)$$

which is the optimality condition for (13). There always exists an optimal rational controller [10]. Therefore, a solution to (29) exists, and hence there must also exist a simultaneous solution to (14) and (17).

By Lemma 8, we have stabilizing solutions to (22)–(25), so we may apply Lemmas 5 and 6. Thus, there must exist Φ, Ψ, $\tilde{Z}_3$, and $\hat{Z}_3$ that simultaneously satisfy (15) and (18). Substituting (21) as (A_P, B_P, C_P) in (15) and similarly (20) as (A_Q, B_Q, C_Q) in (18), we obtain two augmented Sylvester equations. Algebraic manipulation shows that we must have

$$\tilde{Z}_3 = \begin{bmatrix} E_2^{\mathsf{T}}(\tilde{X} - X) & \Phi \end{bmatrix} \quad \text{and} \quad \hat{Z}_3 = \begin{bmatrix} \Psi \\ (\tilde{Y} - Y)E_1 \end{bmatrix}$$

where Φ and Ψ satisfy (26). This establishes existence and uniqueness of a solution to (26). Upon substituting these values back into (16) or (19), we obtain an explicit formula for the blocks of $\mathcal{Q}$. Upon simplification, we obtain (28). ∎

Theorem 11. *Suppose assumptions A1–A6 or B1–B3 hold. An optimal solution to* (12) *is given by*

$$\mathcal{K}_{opt} = \left[\begin{array}{cc|c} A + BK + \hat{L}C_1 & 0 & -\hat{L}E_1^{\mathsf{T}} \\ BK - B_2\hat{K} & A + LC + B_2\hat{K} & -L \\ \hline K - E_2\hat{K} & E_2\hat{K} & 0 \end{array}\right]$$
$$(30)$$

Proof. Obtain $\mathcal{Q}_{opt}$ from Theorem 10, and transform using $\mathcal{K}_{opt} = \mathcal{Q}_{opt}(I + \mathcal{P}_{22}\mathcal{Q}_{opt})^{-1}$. After some algebraic manipulations and reductions, we arrive at (30). ∎

7   State Dimension

First, note that $\mathcal{Q}_{opt}$ and $\mathcal{K}_{opt}$ have the correct sparsity pattern. Indeed, all the blocks in the state-space representation are block-lower-triangular. We can also verify that $\mathcal{Q}_{opt}$ is stable; the eigenvalues of its A-matrix are the eigenvalues of $A + BK$, $A_{11} + \tilde{L}_1C_{11}$, $A_{22} + B_{22}\tilde{K}_2$, and $A + LC$, which are stable by construction.

This is the first time a state-space formula has been found for the two-player output-feedback problem. In particular, we now know the state dimension of the optimal controller. If $A_{11} \in \mathbb{R}^{n_1 \times n_1}$ and $A_{22} \in \mathbb{R}^{n_2 \times n_2}$, then $\mathcal{K}_{opt}$ has at most $2n_1 + 2n_2$ states. However, notice that the numbers of states above may not represent the number of states required for a decentralized implementation. In particular, if the two controllers cannot communicate, then the first controller needs a realization of $\mathcal{K}_{11}$ and the second controller needs a realization of $\begin{bmatrix} \mathcal{K}_{21} & \mathcal{K}_{22} \end{bmatrix}$.

In this case, the first controller will have $n_1 + n_2$ states, and the second controller will have $2n_1 + 2n_2$ states.

In the table below, we compare the number of states required for each player's optimal controller in a variety of special cases appearing previously in the literature.

Special Cases	Player 1	Player 2
State-feedback [13, 15]	n_2	n_2
Partial output-feedback [16]	n_2	$2n_2$
Dynamically decoupled [3]	$n_1 + n_2$	$n_1 + 2n_2$
General output-feedback	$n_1 + n_2$	$2n_1 + 2n_2$

As expected, the general output-feedback solution presented herein requires more states than any of special cases. In every case above, Player 2's state includes Player 1's state. Thus, the number of states for the whole controller $\mathcal{K}_{opt}$ is the same as the number of states for Player 2. Note as well that if we make the problem centralized by removing the structural constraint on the controller, the optimal controller requires $n_1 + n_2$ states.

8   Estimation Structure

The estimation structure is also revealed in (30). If we label the states of $\mathcal{K}_{opt}$ as ζ and ξ, then the state-space equations are:

$$\dot{\zeta} = A\zeta + BK\zeta - \hat{L}(y_1 - C_1\zeta)$$
$$\dot{\xi} = A\xi + Bu - L(y - C\xi)$$
$$u = K\zeta + E_2\hat{K}(\xi - \zeta)$$

The second equation is the optimal centralized state estimator, given by the Kalman filter. Thus, $\xi = \mathbb{E}(x \,|\, y)$. It can also be shown that $\zeta = \mathbb{E}(x \,|\, y_1)$, but we omit the proof due to space constraints. This fact is not obvious, because the equation for ζ depends on $\hat{L}$, which depends on Ψ, and in turn depends on all the parameters of the problem. The controller output u is the centralized LQR controller plus a correction term which depends on the discrepancy between both state estimates.

9   Conclusion

We have shown how to construct the $\mathcal{H}_2$-optimal controller for a two-player output-feedback architecture. The optimal controller, which was not previously known, has twice as many states as the original system. Computing it requires solving four standard AREs and one pair of linearly coupled Sylvester equations.

10   Acknowledgements

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