Optimal Controller Synthesis for the Decentralized Two-Player Problem with Output Feedback

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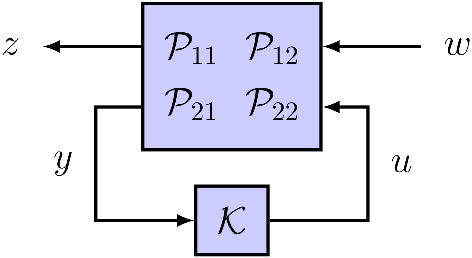
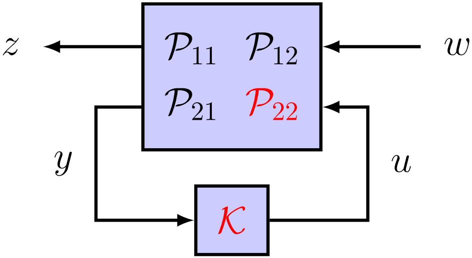
The classical (centralized) LQG problem



- $\mathcal{P}_{ij}$ and $\mathcal{K}$ are matrices of proper rational transfer functions.
- Find a stabilizing $\mathcal{K}$ that minimizes

$$\left\| \mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{K}\left(I - \mathcal{P}_{22}\mathcal{K}\right)^{-1}\mathcal{P}_{21} \right\|_{\mathcal{H}_2}$$

The decentralized LQG problem



- $\mathcal{K}$ has sparsity pattern $\mathcal{S}$. e.g.

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} \mathcal{K}_{11} & 0 \\ \mathcal{K}_{21} & \mathcal{K}_{22} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

- **Assumption:** $\mathcal{P}_{22}$ also has sparsity pattern $\mathcal{S}$.

Previous Work

Bad news:

- Optimal controller is nonlinear even for simple LQG examples
  [Witsenhausen 1968]
- Decentralized controller synthesis is hard in general
  [Blondel, Tsitsiklis 2000]

Good news:

- Large classes of decentralized problems can be convexified
  [Ho, Chu 1972; Qi, Salapaka, et al 2004; Rotkowitz, Lall 2006]
- In many such cases, the optimal controller is linear
  [Ho, Chu 1972; Rotkowitz 2008]

Great news:

- Explicit state-space solution for state feedback with sparsity
  [Swigart, Lall 2010; Shah, Parrilo 2010]
- Characterization of separable problems
  [Kim, Lall 2012]

Centralized solution

State-space formula for the plant:

$$\begin{bmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{21} & \mathcal{P}_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B \\ \hline C_1 & 0 & D_{12} \\ C & D_{21} & 0 \end{array}\right]$$

Optimal controller:

$$\mathcal{K}_{\mathsf{opt}} = \left[\begin{array}{c|c} A + BK + LC & -L \\ \hline K & 0 \end{array}\right] \qquad \begin{aligned} K &= \mathbf{are}(A, B, C_1, D_{12}) \\ L &= \mathbf{are}(A^\mathsf{T}, C^\mathsf{T}, B_1^\mathsf{T}, D_{21}^\mathsf{T})^\mathsf{T} \end{aligned}$$

Interpretation:

$$\begin{aligned} \dot{\xi} &= A\xi + Bu - L(y - C\xi) \\ u &= K\xi \end{aligned}$$

$$\xi = \mathbf{E}\left(x \mid \mathcal{Y}\right)$$

Two-player solution

State-space formula for the plant:

$$\begin{bmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} \\ \mathcal{P}_{21} & \mathcal{P}_{22} \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B \\ \hline C_1 & 0 & D_{12} \\ C & D_{21} & 0 \end{array}\right] \qquad A = \begin{bmatrix} A_{11} & 0 \\ A_{21} & A_{22} \end{bmatrix}, \text{ etc.}$$

Optimal controller:

$$\mathcal{K}_{\text{opt}} = \left[\begin{array}{cc|c} A + BK + \hat{L}C & 0 & -\hat{L} \\ B(K - \hat{K}) & A + B\hat{K} + LC & -L \\ \hline K - \hat{K} & \hat{K} & 0 \end{array}\right]$$

- K and L same as in centralized case
- $\hat{K} \sim \begin{bmatrix} 0 & 0 \\ * & * \end{bmatrix}$ and $\hat{L} \sim \begin{bmatrix} * & 0 \\ * & 0 \end{bmatrix}$, so $\mathcal{K}_{\text{opt}} \in \mathcal{S}$

Two-player solution

Optimal controller:

$$\mathcal{K}_{\mathsf{opt}} = \left[\begin{array}{cc|c} A + BK + \hat{L}C & 0 & -\hat{L} \\ B(K - \hat{K}) & A + B\hat{K} + LC & -L \\ \hline K - \hat{K} & \hat{K} & 0 \end{array}\right]$$

Interpretation:

$$\dot{\zeta} = A\zeta + B\hat{u} - \hat{L}(y - C\zeta)$$
$$\hat{u} = K\zeta$$

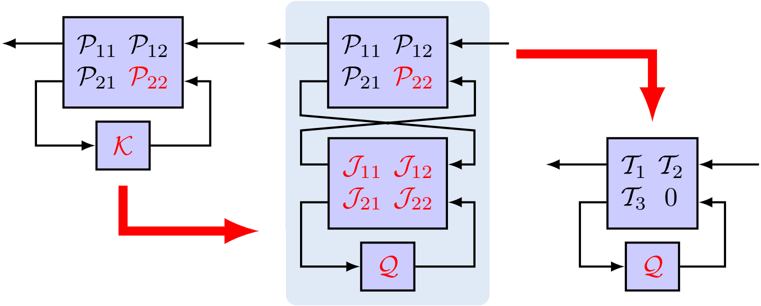
$$\zeta = \mathbf{E}\left(x \mid \mathcal{Y}_1\right)$$

$$\dot{\xi} = A\xi + Bu - L(y - C\xi)$$
$$u = K\zeta + \hat{K}(\xi - \zeta)$$

$$\xi = \mathbf{E}\left(x \mid \mathcal{Y}_{1,2}\right)$$

Outline of solution method

1. Modified Youla parameterization:



2. Model-matching problem

$$\underset{\substack{\mathcal{K}\text{ stabilizing}\\ \mathcal{K}\in\mathcal{S}}}{\text{minimize}} \left\| \mathcal{P}_{11} + \mathcal{P}_{12}\mathcal{K}(I - \mathcal{P}_{22}\mathcal{K})^{-1}\mathcal{P}_{21} \right\|$$

becomes: $$\underset{\substack{\mathcal{Q}\text{ stable}\\ \mathcal{Q}\in\mathcal{S}}}{\text{minimize}} \left\| \mathcal{T}_1 + \mathcal{T}_2\mathcal{Q}\mathcal{T}_3 \right\|$$

Outline of solution method

3. Person-by-person optimality

$$\underset{Q_{ij} \text{ stable}}{\text{minimize}} \left\| \mathcal{T}_1 + \mathcal{T}_2 \begin{bmatrix} \mathcal{Q}_{11} & 0 \\ \mathcal{Q}_{21} & \mathcal{Q}_{22} \end{bmatrix} \mathcal{T}_3 \right\|$$

- fix $\mathcal{Q}_{11}$ and solve for $\mathcal{Q}_{21}$, $\mathcal{Q}_{22}$.
- fix $\mathcal{Q}_{22}$ and solve for $\mathcal{Q}_{11}$, $\mathcal{Q}_{21}$.

4. Impose matching conditions **(key step!)**

5. Transform back $\mathcal{Q} \to \mathcal{K}$.

Solution details

Standard AREs:

$$(X, K) = \mathbf{are}(A, B, C_1, D_{12}) \qquad (\tilde{X}, J) = \mathbf{are}(A_{22}, B_{22}, C_1E_2, D_{12}E_2)$$

$$(Y, L) = \mathbf{are}(A^\mathsf{T}, C^\mathsf{T}, B_1^\mathsf{T}, D_{21}^\mathsf{T})^\mathsf{T} \quad (\tilde{Y}, M) = \mathbf{are}(A_{11}^\mathsf{T}, C_{11}^\mathsf{T}, E_1^\mathsf{T}B_1^\mathsf{T}, E_1^\mathsf{T}D_{21}^\mathsf{T})^\mathsf{T}$$

Coupled linear equations:

$$(A_{22} + B_{22}J)^\mathsf{T}\Phi + \Phi(A_{11} + MC_{11}) - (\tilde{X} - X_{22})(\Psi C_{11}^\mathsf{T} + U_{21})V_{11}^{-1}C_{11} + \\ (Q_{21} + J^\mathsf{T}S_{12}^\mathsf{T} + \tilde{X}A_{21} - X_{21}MC_{11}) = 0$$

$$(A_{22} + B_{22}J)\Psi + \Psi(A_{11} + MC_{11})^\mathsf{T} - B_{22}R_{22}^{-1}(B_{22}^\mathsf{T}\Phi + S_{12}^\mathsf{T})(\tilde{Y} - Y_{11}) + \\ (W_{21} + U_{21}M^\mathsf{T} + A_{21}\tilde{Y} - B_{22}JY_{21}) = 0$$

New control and estimation gains:

$$\hat{K} = \begin{bmatrix} 0 & 0 \\ -R_{22}^{-1}(B_{22}^\mathsf{T}\Phi + S_{12}^\mathsf{T}) & J \end{bmatrix} \qquad \hat{L} = \begin{bmatrix} M & 0 \\ -(\Psi C_{11}^\mathsf{T} + U_{21})V_{11}^{-1} & 0 \end{bmatrix}$$

Optimal decentralized cost

$$\mathcal{J}_{\text{opt}} = \left\| \left[\begin{array}{c|c} A + BK & B_1 \\ \hline C_1 + D_{12}K & 0 \end{array} \right] \right\|^2$$

centralized cost

$$+ \left\| \left[\begin{array}{c|c} A + LC & B_1 + LD_{21} \\ \hline D_{12}K & 0 \end{array} \right] \right\|^2$$

cost of decentralization

$$+ \left\| \left[\begin{array}{c|c} A + B\hat{K} + \hat{L}C & (\hat{L} - L)D_{21} \\ \hline D_{12}(\hat{K} - K) & 0 \end{array} \right] \right\|^2$$

Summary

For two-player output feedback:

- Each player estimates the global state
- Some gains are decoupled, found by solving separate AREs
- Others are coupled, found by solving **linear** equations
- Can quantify the cost due to decentralization

Rough structure:

$$u_{\mathsf{opt}} = K\hat{x}_{|1} + \hat{K}\left(\hat{x}_{|2} - \hat{x}_{|1}\right)$$

Further Information

- Complete proof of the two-player problem (submitted TAC)
- Extension to the broadcast case (submitted CDC'12)

`http://www.control.lth.se/lessard/`

Thank you!