Optimal State-Feedback Control Under Sparsity and Delay Constraints

Andrew Lamperski[†] and Laurent Lessard[‡]

[†]California Institute of Technology [‡]Lund University

Abstract

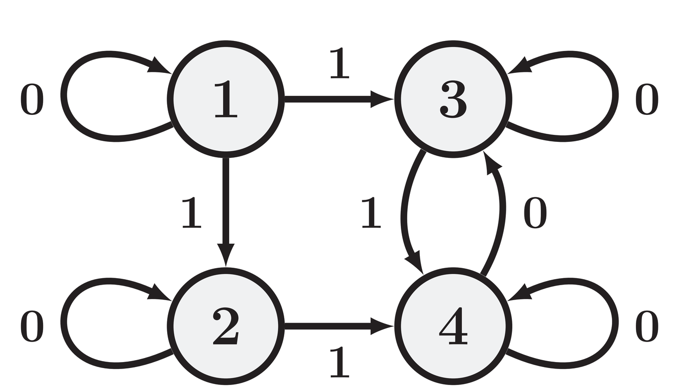
This paper presents the solution to a general decentralized state-feedback problem, in which the plant and controller must satisfy the same combination of delay constraints and sparsity constraints. The control problem is decomposed into independent subproblems, which are solved by dynamic programming. In special cases with only sparsity or only delay constraints, the controller reduces to existing solutions.

Problem

Find a policy that minimizes a quadratic cost

$$\mathbb{E}\left\{ \sum_{t=0}^{T-1} \begin{bmatrix} x_t \\ u_t \end{bmatrix}^\mathsf{T} \begin{bmatrix} Q & S \\ S^\mathsf{T} & R \end{bmatrix} \begin{bmatrix} x_t \\ u_t \end{bmatrix} + x_T^\mathsf{T} Q_f x_T \right\}$$

subject to linear dynamics and a partially nested information constraint. Plant sparsity and information constraints are encoded by a *network graph*:



$$d = \begin{bmatrix} 0 & \infty & \infty & \infty \\ 1 & 0 & \infty & \infty \\ 1 & 1 & 0 & 0 \\ 2 & 1 & 1 & 0 \end{bmatrix}$$

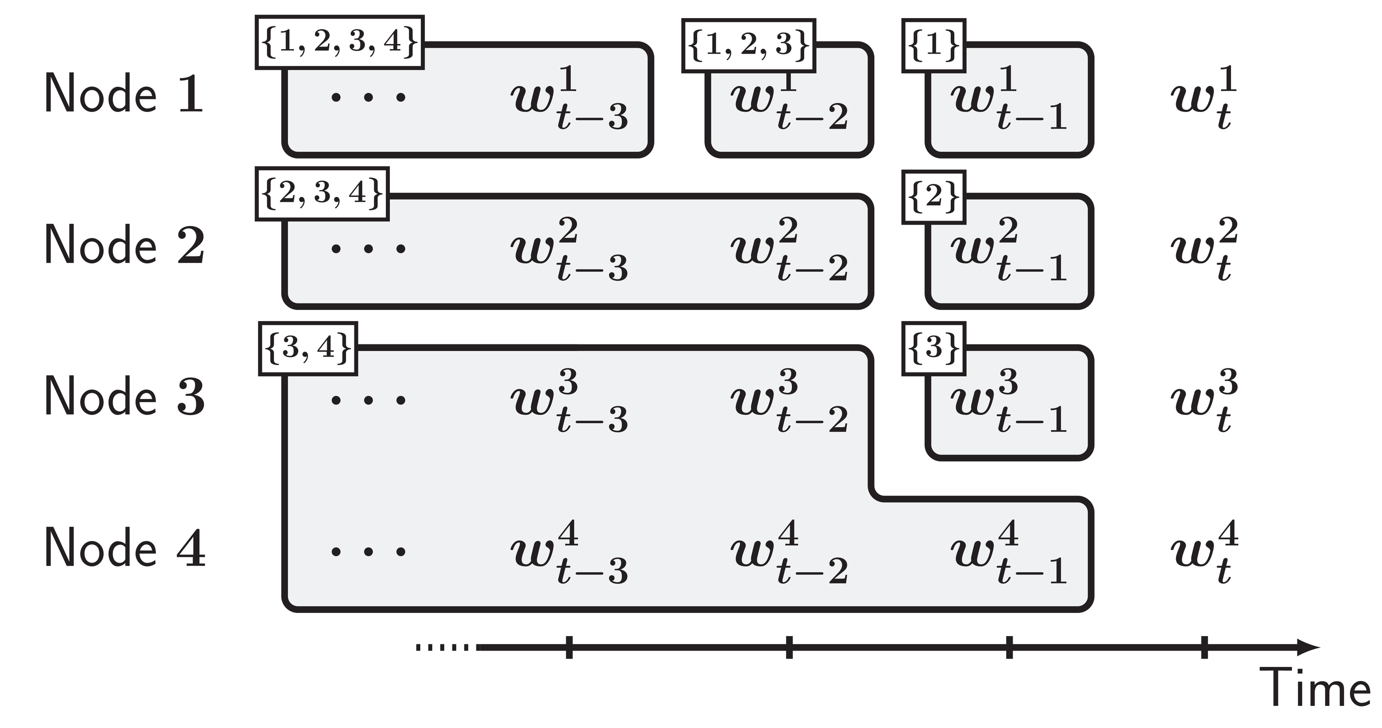
For the network graph above, dynamics and constraints are

$$\begin{bmatrix} x_{t+1}^1 \\ x_{t+1}^2 \\ x_{t+1}^3 \\ x_{t+1}^4 \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & 0 & 0 \\ A_{21} & A_{22} & 0 & 0 \\ A_{31} & A_{32} & A_{33} & A_{34} \\ 0 & A_{42} & A_{43} & A_{44} \end{bmatrix} \begin{bmatrix} x_t^1 \\ x_t^2 \\ x_t^3 \\ x_t^4 \end{bmatrix} + \begin{bmatrix} B_{11} & 0 & 0 & 0 \\ B_{21} & B_{22} & 0 & 0 \\ B_{31} & B_{32} & B_{33} & B_{34} \\ 0 & B_{42} & B_{43} & B_{44} \end{bmatrix} \begin{bmatrix} u_t^1 \\ u_t^2 \\ u_t^3 \\ u_t^4 \end{bmatrix} + \begin{bmatrix} w_t^1 \\ w_t^2 \\ w_t^3 \\ w_t^4 \end{bmatrix}$$

$$\begin{cases} u_t^1 = \gamma_t^1(x_{0:t}^1) \\ u_t^2 = \gamma_t^2(x_{0:t-1}^1, x_{0:t}^2) \\ u_t^3 = \gamma_t^3(x_{0:t-1}^1, x_{0:t-1}^2, x_{0:t}^3, x_{0:t}^4) \\ u_t^4 = \gamma_t^4(x_{0:t-2}^1, x_{0:t-1}^2, x_{0:t-1}^3, x_{0:t}^4) \end{cases}$$

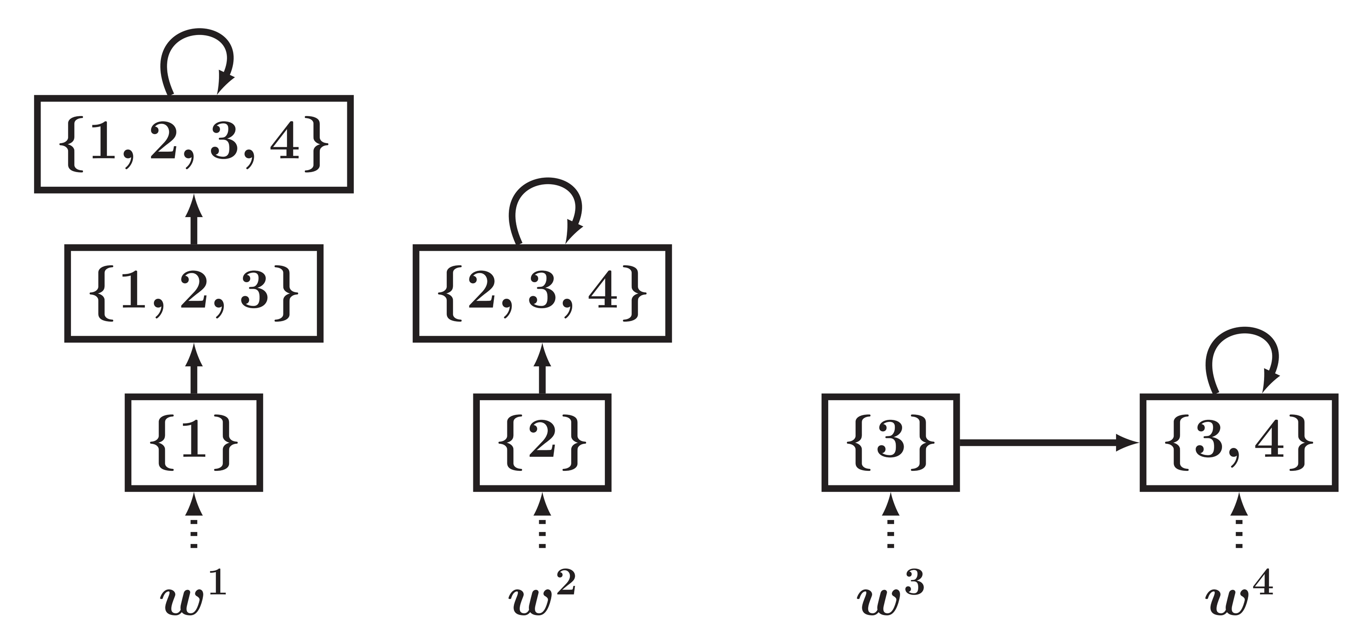
We give a general solution for *any* such problem; an arbitrary directed graph with each edge having the label '0' or '1'.

Noise Partition



Partition the noise history based on which inputs at time t are influenced.

Information Graph



Redraw the noise partition as a graph; each edge is a time increment.

State and Input Projection

$$x_t = \begin{bmatrix} I \\ 0 \\ 0 \\ 0 \end{bmatrix} \zeta_t^{\{1\}} + \begin{bmatrix} 0 \\ I \\ 0 \\ 0 \end{bmatrix} \zeta_t^{\{2\}} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ I & 0 \\ 0 & I \end{bmatrix} \zeta_t^{\{3,4\}} + \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix} \zeta_t^{\{1,2,3\}} + \dots$$

Each ζ_t^r corresponds to a node in the information graph. The $\{\zeta_t^r\}_r$ are independent because they are functions of different noise terms.

Main Result

Define matrices X_t^r recursively by $X_T^r = Q_f^{rr}$ and

$$X_t^r = Q^{rr} + A^{sr\mathsf{T}} X_{t+1}^s A^{sr} - \left(S^{rr} + A^{sr\mathsf{T}} X_{t+1}^s B^{sr}\right) \times \left(R^{rr} + B^{sr\mathsf{T}} X_{t+1}^s B^{sr}\right)^{-1} \left(S^{rr} + A^{sr\mathsf{T}} X_{t+1}^s B^{sr}\right)^\mathsf{T}$$

where s is the unique node with $r \to s$ in the information graph. Define gains K_t^r by

$$K_t^r = -\left(R^{rr} + B^{sr\mathsf{T}} X_{t+1}^s B^{sr}\right)^{-1} \left(S^{rr} + A^{sr\mathsf{T}} X_{t+1}^s B^{sr}\right)^\mathsf{T}$$

The optimal control policy is given by $u_t = \sum_{s \in U} I^{V,s} K_t^s \zeta_t^s$.

The states ζ_t^s evolve according to $\zeta_0^s = \sum_{w^i \to s} I^{s,\{i\}} x_0^i$ and

$$\zeta_{t+1}^s = \sum_{r \to s} \left(A^{sr} + B^{sr} K_t^r\right) \zeta_t^r + \sum_{w^i \to s} I^{s,\{i\}} w_t^i$$

Notation: M^{sr} is the block submatrix $(M_{ij})_{i \in s, j \in r}$

Special Cases

Sparsity only [1, 2]: Acyclic graph with delay 0 on all edges.

Delay only [3]: Strongly connected graph with delay 1 on all edges.

References

[1] J. Swigart, *Optimal Controller Synthesis for Decentralized Systems.* PhD thesis, Stanford University, 2010.

[2] P. Shah and P. A. Parrilo, "$\mathcal{H}_2$-optimal decentralized control over posets: A state space solution for state-feedback," in *IEEE CDC*, 2010.

[3] A. Lamperski and J. C. Doyle, "Dynamic programming solutions for decentralized state-feedback LQG problems with communication delays," in *ACC*, 2012.

Download

Further information is available on our websites (see below).

Our paper is available in PDF format here:

This poster is available in PDF format here:

Andrew Lamperski (andyl@caltech.edu): http://www.cds.caltech.edu/~andyl/professional/ **Laurent Lessard** (laurent.lessard@control.lth.se): http://www.control.lth.se/lessard/