Distributed Model Predictive Control with Suboptimality and Stability Guarantees

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Abstract— **Theory for Distributed Model Predictive Control (DMPC) is developed based on dual decomposition of the convex optimization problem that is solved in each time sample. The process to be controlled is an interconnection of several subsystems, where each subsystem corresponds to a node in a graph. We present a stopping criterion for the DMPC scheme that can be locally verified by each node and that guarantees closed loop suboptimality above a pre-specified level and asymptotic stability of the interconnected system.**

One extra line

I. INTRODUCTION

Dual decomposition has been a well established concept since around 1960 when the remarkable Usawa's algorithm [1] was presented and similar ideas were exploited in large-scale optimization [7]. Very soon, the decomposition methods were also applied for computation of optimal control trajectories [22]. Over the next decades, methods for decomposition and coordination of dynamic systems were refined [15], [21], [9] and used in large-scale applications [5].

The purpose of this paper is to investigate how the same methods can be used in the context of Model Predictive Control (MPC). Our main focus is not reduction of computational complexity in a centralized computational unit, but instead coordination of many local control units, each connected to a sub-component of a large interconnected system.

The idea of MPC is to solve, in each sampling instant, a finite horizon optimization problem that predicts the future plant dynamics and minimizes a given cost function based on the predictions. When new measurements become available to the controller, another optimization takes place. Over the last decades many successful applications have been implemented [3], [16].

The most common approach for control of large-scale networked systems is to design local controllers that ignore (or minimize) the interaction between subsystems. This might, however, lead to severely deteriorated global performance. A centralized optimization-based approach could be much better, but is often impractical due to communication constraints and an overwhelming number of decision variables. Distributed MPC, where the global optimization problem is decomposed into many smaller optimization problems that can be solved locally for each subsystem, is therefore appealing. In this distributed framework, the interaction between subsystems is taken into account, while the flexibility of the decentralized approach is still there.

Some distributed MPC formulations have already been presented in the literature. In [17], proximal center decomposition was used to decompose the MPC optimization problem. It was shown that the proximal center decomposition method gives faster convergence properties than ordinary sub-gradient methods for dual decomposition. In [8] a distributed MPC-scheme for coupled nonlinear systems with decoupled constraints was presented. In [23] a cooperation based MPC-scheme was presented where global objective functions are used in each node. Closed loop stability for this scheme is guaranteed. Another distributed MPC formulation was presented in [4] based on a cooperative iteration scheme. Dual decomposition along the time axis was pursued in [2]. Further references as well as applications to power networks can be found in [18].

The distributed Model Predictive Controller in this paper is based on dual decomposition with sub-gradient updates of the lagrange multipliers. Such algorithms are know to have fairly slow convergence properties. However, in control, the performance of the closed loop system is the primary objective. A stopping criterion, which is based on relaxed dynamic programming (cite), for the distributed Model Predictive Controller is developed with significantly reduces the amount of iterations needed in the dual decomposition algorithm. The stopping criterion is designed such that closed loop performance above a certain pre-specified degree is achieved and asymptotic stability of the closed loop system is guaranteed. The set of cost functionals considered in this work are cost functionals without terminal cost or terminal constraints. In most MPC literature asymptotic stability is guaranteed for systems with certain terminal cost or terminal constraints, see [14] for a survey of such methods. Recently, stability and suboptimality results have been established for different MPC schemes without terminal cost or terminal constraints in [12], [11], [19]. This paper has been developed in parallel with [10] in which similar ideas are used in an adaptive MPC scheme for the centralized case.

The paper is organized as follows. In Section II we formulate the optimization problem that is solved in each sample of the distributed MPC-controller. In Section III we describe a dual decomposition algorithm that solves the optimization problem in a distributed fashion. MPC analysis and design tools, based on relaxed dynamic programming, are presented in Section IV. In Section V it is shown how the developed design tool can be used as a stopping criterion for distributed MPC. Numerical examples are given in Section VI in which the performance of the proposed scheme is evaluated. Finally in Section VII we give conclusions and discuss some future work directions.

II. Problem Setup

Consider a dynamical system with the state vector $x = [x_1; x_2 \dots x_{\mathscr{J}}]$ and the dynamics

$$x_i(t+1) = \sum_{j=1}^{\mathscr{J}} A_{ij} x_j(t) + B_i u_i(t) \qquad x_i(0) = \bar{x}_i \qquad (1)$$

for all $i = 1, \dots, \mathscr{J}$, where $x_i \in \mathbb{R}^{n_i}$ and $u_i \in \mathbb{R}^{m_i}$. The system has an associated graph, with one node for every i and an edge connecting j and i unless A_{ij} and A_{ji} are both zero. The dynamics of the full system can be written as

$$x(t+1) = Ax(t) + Bu(t) \qquad x(0) = \bar{x} \qquad (2)$$

where $x \in \mathbb{R}^n$, $n = \sum_i n_i$ and $u \in \mathbb{R}^m$, $m = \sum_i m_i$.

The control objective is to minimize the following infinite horizon cost:

$$V^\infty(\bar{x}) := \min_u \sum_{t=0}^{\infty} \underbrace{\sum_{i=1}^{\mathscr{J}} \ell_i(x_i(t), u_i(t))}_{\ell(x(t),u(t))} \qquad (3)$$

subject to convex constraints

$$x_i(t) \in X_i \text{ and } u_i(t) \in U_i \text{ for all } i, t \qquad (4)$$

Under general assumptions (essentially convexity of ℓ), we will see that the problem can be solved to arbitrary accuracy with a distributed Model Predictive Control (MPC) scheme, where the only communication that is allowed is between neighboring nodes. Hence node i may exchange information with all nodes j that are connected to i by an edge of the graph.

For the centralized case, the MPC-controller is based on iterative solutions of the following finite horizon approximation of (3):

$$V^N(x(t)) := \min_u \sum_{\tau=0}^{N} \ell(x(t,\tau), u(t,\tau)) \qquad (5)$$

subject to

$$x_i(t,\tau) \in X_i \text{ and } u_i(t,\tau) \in U_i \text{ for all } i, \tau \qquad (6)$$

and the plant predictions:

$$x(t,\tau+1) = Ax(t,\tau) + Bu(t,\tau) \qquad x(t,0) = x(t).$$

The objective function in the minimization is a straight forward truncation of the infinite horizon objective. This means that no terminal cost or terminal constraints are present, as in most MPC literature. From the optimization (5) a control sequence $u(t,\tau)$ is obtained. The first of those control actions, $u(t,0)$, is applied to the process giving the following closed loop dynamics

$$x(t+1) = Ax(t) + Bu(t,0) \qquad x(0) = \bar{x} \qquad (7)$$

Note that the predicted plant evolution in the controller at time t is denoted $x(t,\tau)$ where τ is the internal time, while the actual closed loop state at time t is denoted $x(t)$.

The optimal performance, from initial state to the origin, is defined in (3), while the actual performance of the MPC controller is defined as

$$V^\infty_{MPC}(\bar{x}) := \sum_{t=0}^{\infty} \ell(x(t), u(t,0)) \qquad (8)$$

where the state evolution is defined by (7). The ultimate objective of this paper is to create a distributed MPC scheme such that $V^\infty_{MPC}(\bar{x})$ is within a certain pre-specified factor of the optimal performance $V^\infty(\bar{x})$.

To distribute the optimization problem (5) over all the nodes in the graph associated with the dynamical system, dual decomposition is used.

III. Dual Decomposition

The problem (5) can be decomposed using so called dual decomposition. For this purpose, we follow the notation of [6] and introduce the decoupled state equations

$$x_i(\tau+1) = A_{ii} x_i(\tau) + B_i u_i(\tau) + v_i(\tau) \qquad x_i(0) = \bar{x}_i \qquad (9)$$

with the additional constraints that

$$v_i(\tau) = \sum_{j \neq i} A_{ij} x_j(\tau) \qquad \text{for all } \tau \qquad (10)$$

For notational convenience we have dropped the t-parameter in $x(t,\tau)$ and $u(t,\tau)$ in this section. The variable v_i can be interpreted as the expected influence of other agents in the update of x_i. The constraints (10) are then relaxed by introduction of corresponding Lagrange multipliers in the cost function. This gives

$$\begin{aligned} &\max_p \min_{u,v,x} \sum_{\tau=0}^{N} \sum_{i=1}^{\mathscr{J}} \left[\ell_i(x_i,u_i) + p_i^T \left(v_i - \textstyle\sum_{j \neq i} A_{ij} x_j \right) \right] = \\ &= \max_p \sum_i \min_{u_i,x_i,v_i} \sum_{\tau=0}^{N} \underbrace{\left[\ell_i(x_i,u_i) + p_i^T v_i - x_i^T \left(\textstyle\sum_{j \neq i} A_{ji}^T p_j \right) \right]}_{\ell_i^p(x_i,u_i,v_i)} \end{aligned}$$

subject to (6), (9) and the restriction that $p(N) = 0$, since we have only N equality constraints.

After introduction of dual variables the problem can be interpreted as a game with two players for each graph node. Given the prices, the objective of the first player in node i is to select the inputs $u_i(0), \dots, u_i(N)$ to minimize the local cost $\sum_{\tau=0}^{N} \ell_i^p(x_i,u_i,v_i)$, which can be decomposed as

$$\underbrace{\sum_{\tau=0}^{N} \ell_i(x_i,u_i)}_{\text{local cost}} + \overbrace{\sum_{\tau=0}^{N} p_i^T v_i}^{\text{what he expects others to charge him}} - \underbrace{\sum_{\tau=0}^{N} x_i^T \left(\textstyle\sum_{j \neq i} A_{ji}^T p_j \right)}_{\text{what he is payed by others}}$$

The other player in node i chooses $p_i(0),\dots,p_i(N)$ with the objective to minimize $\sum_{\tau=0}^{N} p_i^T\left(\sum_{j\neq i} A_{ij}x_j - v_i\right)$.

To summarize, a decomposition of the objective as well as distributed optimality conditions are given by the following proposition.

Proposition 1: Suppose that $\ell_1,\dots,\ell_{\mathscr{J}}$ are convex and that the minimum in (5) is attained. Then

$$V^N(\bar{x}) = \max_p \sum_{i=1}^{\mathscr{J}} \min_{x_i,u_i,v_i} \left(\sum_{\tau=0}^{N} \ell_i^p(x_i(\tau),u_i(\tau),v_i(\tau)) \right) \quad (11)$$

where maximization is subject to $p(N) = 0$, (6) and (9). Moreover, the maximum in (11) is attained if and only if the constraints (1) are satisfied.

Proof. The equality (11) is an instance of standard Lagrangian duality. The maximum in (11) is attained if and only if the gradient with respect to p is zero. The gradient with respect to $p_i(\tau)$ is $v_i(\tau) - \sum_{j\neq i} A_{ij}x_j(\tau)$, so all the constraints (1) must be satisfied at optimum. □

Proposition 1 shows that the computation of x_i, u_i and v_i for given prices p_j is completely decentralized. However, finding the optimal prices requires coordination. The expressions on the right hand side of (11) are concave functions of p. Hence optimal prices can be found as the limits of a gradient iteration: Given some price prediction sequence $\{p_i^k(\tau)\}_{\tau=0}^{N}$, corresponding state predictions $\{x_i^k(\tau)\}_{\tau=1}^{N}$ and input predictions $\{u_i^k(\tau)\}_{\tau=0}^{N}$ are computed locally by minimization of $\sum_{\tau=0}^{N} \ell_i^p\left(x_i(\tau),u_i(\tau),v_i(\tau)\right)$ subject to (6) and (9). Then prices can then be updated distributively by a gradient step

$$p_i^{k+1}(\tau) = p_i^k(\tau) + \gamma_i^k \left[v_i^k(\tau) - \sum_{j\neq i} A_{ij}x_j^k(\tau) \right] \quad (12)$$

for $\tau = 0,\dots,N$. Convergence of such gradient algorithms has been proved under different types of assumptions on the step size sequence γ_i^k, see [20]. In the continuation we assume that the γ_i^k are such that the dual decomposition iterations converge towards the optimum. However, the convergence rate of such algorithm may be fairly slow. This undesirable property is addressed in the following sections where a stopping criterion is developed which guarantees certain closed loop performance and stability. This criterion shows to significantly decrease the number of iterations needed compared to if the optimum was to be found.

Before we continue with the development of the stopping criterions, we need the following definition:

$$V_i^{N,k}(\bar{x}_i) := \sum_{\tau=0}^{N} \left[\ell_i(x_i^k,u_i^k) + (p_i^k)^T \left(v_i^k - \sum_{j\neq i} A_{ij}x_j^k \right) \right]$$

where k denotes the iteration number and all variables are optimized according to (11). Also note that by standard duality we have that

$$V^{N,k}(\bar{x}) := \sum_{i=1}^{\mathscr{J}} V_i^{N,k}(\bar{x}_i) \leq V^N(\bar{x})$$

for any k. If the conditions in the stopping criterions are satisfied for $k = K \in \mathbb{N}_1$, the control action to be applied to the process is

$$u(t,0) = [u_1^K(0); u_2^K(0) \dots u_{\mathscr{J}}^K(0)] \quad (13)$$

which together with (7) defines the closed loop solution.

IV. MPC Tools

In this section two tools for DMPC based on dual decomposition are developed. The first tool is an analysis tools based on the relaxed dynamic programming inequality. If the conditions of this analysis tool are satisfied, asymptotic stability and closed loop suboptimality to a certain degree are guaranteed. This analysis tool is then developed to a design tool that can be used as a stopping criterion for the number of iterations needed in the DMPC scheme to ensure asymptotic stability and closed loop suboptimality to a pre-specified degree. For both tools in this section it is assumed that data from all nodes are available when checking the conditions. In the next section, it is shown how the conditions of the design tool can be verified in a distributed manner suitable for implementation of the DMPC scheme.

A. MPC Analysis Tool

The analysis tool presented here is based on the work about relaxed dynamic programming, see [13]. In [19] asymptotic stability and a certain degree of suboptimality is proved if the relaxed dynamic programming inequality

$$V^N(x(t)) \geq V^N(Ax(t) + Bu(t,0)) + \alpha\ell(x(t),u(t,0)) \quad (14)$$

holds for some $\alpha \in (0,1)$ and for all time steps in the closed loop trajectory. Further in [11] it is shown that using some controllability assumptions on the running cost, ℓ, a minimal control horizon N such that (14) is satisfied for all $x \in X$ can be calculated for the class of systems satisfying the controllability assumptions. Thus, in the continuation of this paper we use the following assumption:

Assumption 1: Assume that for a pre-specified value of $\alpha \in (0,1)$ a control horizon N is known such that

$$V^N(x(t)) \geq V^N(Ax(t) + Bu(t,0)) + \alpha\ell(x(t),u(t,0))$$

holds for all $x \in X$.
In the distributed MPC-scheme the control horizon is chosen such that Assumption 1 holds.

The work in [11] considers MPC in which the optimum of the each optimization problem is attained. Next we will state two theorems for unfinished optimizations based on the relaxed dynamic programming inequality (14) that ensures a certain degree of suboptimality and asymptotic stability respectively. The first theorem is about suboptimality and is an variation of [10, Theorem 1] to include unfinished optimizations.

Theorem 1: Consider the closed loop solution $x(\cdot)$ according to (7) with control signal (13) applied after $K(t)$ iterations. Assume that there is an $\alpha \in (0,1)$ such that

$$V^{N,K(t)}(x(t)) \geq V^{N,K(t+1)}(x(t+1)) + \alpha\ell(x(t),u(t,0)) + s(t) \quad (15)$$

where

$$V^{N,K(t)}(x(t)) \geq 0 \quad (16)$$

and

$$s(t) = s(t-1) + \alpha \ell(x(t-1), u(t-1,0)) + \\ + V^{N,K(t)}(x(t)) - V^{N,K(t-1)}(x(t-1)) \quad (17)$$

and $s(0) = 0$ hold for all $t \in \mathbb{N}_0$. Then

$$\alpha V^{\infty}_{MPC}(x(0)) \leq V^{\infty}(x(0))$$

Proof. Induction of (17) gives

$$\begin{aligned} s(T) &= s(T-1) + \alpha \ell(x(T-1), u(T-1,0)) + \\ &\quad + V^{N,K(T)}(x(T)) - V^{N,K(T-1)}(x(T-1) \\ &= s(T-2) + \alpha \sum_{t=T-2}^{T-1} \ell(x(t-1), u(t-1,0)) + \\ &\quad + V^{N,K(T)}(x(T)) - V^{N,K(T-2)}(x(T-2) \\ &= \ldots = \alpha \sum_{t=0}^{T-1} \ell(x(t-1), u(t-1,0)) + \\ &\quad + V^{N,K(T)}(x(T)) - V^{N,K(0)}(x(0)) \end{aligned}$$

Insertion of this into (15) gives for any $T \in \mathbb{N}_0$

$$\begin{aligned} &\alpha \sum_{t=0}^{T} \ell(x(t), u(t,0)) \leq \\ &\leq V^{N,K(0)}(x(0)) - V^{N,K(T+1)}(x(T+1)) \\ &\leq V^{N,K(0)}(x(0)) \leq V^{N}(x(0)) \leq V^{\infty}(x(0)) \end{aligned}$$

where the second inequality comes from (16). The third inequality is a direct consequence of duality theory which says that a dual feasible point is less than or equal to the primal optimal point. The last inequality is due to the observation that longer control horizon gives larger cost since no terminal constraint or terminal cost is present. The result follows from the definition of $V^{\infty}_{MPC}(x(0))$ as $T \to \infty$. □

Our next objective is to prove asymptotic stability of the system if the conditions of Theorem 1 are satisfied. Before we do that, the following assumptions on the running cost is needed:

Assumption 2: Assume that there exist a $\beta > 0$ such that $\min_u \ell(x,u) \geq \beta \|x\|_2^2$.

Theorem 2: Consider the closed loop trajectory (7) with control action (13). Assume that

$$V_{\infty}^{MPC}(x(0)) \leq M \quad (18)$$

where M is a finite positive real number. Then then $\|x(t)\|_2^2 \to 0$ as $t \to \infty$.

Proof. We show this by a contradiction argument. We have that

$$V_{\infty}^{MPC}(x(0)) = \sum_{t=0}^{\infty} \ell(x(t), u(t,0)) \leq M \quad (19)$$

where M is a finite positive real number. Assume that $\|x(t)\|_2^2 \not\to 0$ as $t \to \infty$, then there is an $\varepsilon > 0$ such that $\|x(t)\|_2^2 \geq \varepsilon$ for all $t \in \mathbb{N}_0$. Further

$$\sum_{t=0}^{\infty} \ell(x(t), u(t,0)) \geq \sum_{t=0}^{\infty} \beta \|x(t)\|_2^2 \geq \beta \varepsilon \sum_{t=0}^{\infty} 1 \quad (20)$$

which is unbounded. Thus by contradiction the assertion holds. □

The two theorems presented here are analysis tools that can be verified in run-time. The objective of the next section is to use these analysis tools as stopping criterion for the distributed MPC scheme. There are two main considerations when doing this. The first is that the conditions of the theorems, that hold for a centralized MPC-scheme, must be guaranteed to hold using local stopping criterions in each node where local optimizations take place. The second is that the value function of the optimization in the following time step is not known, but an upper bound can be calculated as a primal solution with control horizon N.

B. MPC Design Tool

The objective of this section is to develop a design tool that utilizes the analysis tool developed in Section IV-A. The analysis tool cannot be used directly as a design tool, since at time t information about the dual value function, $V^{N,K(t+1)}(x(t+1))$, at time $t+1$ is needed. However, if an upper bound, denoted $\bar{V}^{N,K(t+1)}(x(t+1))$, to $V^{N,K(t+1)}(x(t+1))$ is known at time t the conditions of Theorem 1 can be changed to

$$V^{N,K(t)}(x(t)) \geq \bar{V}^{N,K(t+1)}(x(t+1)) + \alpha \ell(x(t), u(t,0)) + s(t)$$

where

$$V^{N,K(t)}(x(t)) \geq 0$$

and

$$s(t) = s(t-1) + \alpha \ell(x(t-1), u(t-1,0)) + \\ + \bar{V}^{N,K(t)}(x(t)) - V^{N,K(t-1)}(x(t-1))$$

and the results from Theorem 1 clearly holds. Due to the upper bound used, the conditions get more conservative. Most of this conservatism can be eliminated by changing the update of the slack variable $s(t)$ as in the following theorem.

Theorem 3: Consider a closed loop trajectory (7) with control action (13). Assume that for a pre-specified $\alpha \in (0,1)$ that

$$V^{N,K(t)}(x(t)) \geq \bar{V}^{N,K(t+1)}(x(t+1)) + \alpha \ell(x(t), u(t,0)) + s(t) \quad (21)$$

where

$$V^{N,K(t)}(x(t)) \geq 0 \quad (22)$$

and

$$s(t) = s(t-1) + \alpha \ell(x(t-1), u(t-1,0)) \\ + \bar{V}^{N,K(t)}(x(t)) - \bar{V}^{N,K(t-1)}(x(t-1)) \quad (23)$$

for $t \geq 2$ and

$$s(1) = \alpha \ell(x(0), u(0,0)) + \bar{V}^{N,K(1)}(x(1)) - V^{N,K(0)}(x(0)) \quad (24)$$

and $s(0)=0$. Then

$$\alpha V^{\infty}_{MPC}(x(0)) \leq V^{\infty}(x(0))$$

and $\|x(t)\|_2^2 \to 0$ as $t \to \infty$.

Proof. Induction over (23) gives

$$\begin{aligned}
s(T) &= s(T-1) + \alpha \ell(x(T-1), u(T-1,0)) + \\
&\quad + \bar{V}^{N,K(T)}(x(T)) - \bar{V}^{N,K(T-1)}(x(T-1)) \\
&= s(T-2) + \alpha \sum_{t=T-2}^{T-1} \ell(x(t), u(t,0)) + \\
&\quad + \bar{V}^{N,K(T)}(x(T)) - \bar{V}^{N,K(T-2)}(x(T-2)) \\
&= \ldots = s(1) + \alpha \sum_{t=1}^{T-1} \ell(x(t), u(t,0)) + \\
&\qquad + \bar{V}^{N,K(T)}(x(T)) - \bar{V}^{N,K(1)}(x(1)) \\
&= \alpha \sum_{t=0}^{T-1} \ell(x(t), u(t,0)) + \\
&\quad + \bar{V}^{N,K(T)}(x(T)) - V^{N,K(0)}(x(0))
\end{aligned}$$

where the last inequality comes from (24). Insertion of this into (21) gives for any $T \in \mathbb{N}_0$

$$\begin{aligned}
&\alpha \sum_{t=0}^{T} \ell(x(t), u(t,0)) \leq \\
&\quad \leq V^{N,K(0)}(x(0)) - \bar{V}^{N,K(T+1)}(x(T+1)) + \\
&\quad\quad + V^{N,K(T)}(x(T)) - \bar{V}^{N,K(T)}(x(T)) \\
&\quad \leq V^{N,K(0)}(x(0)) - \bar{V}^{N,K(T+1)}(x(T+1)) \\
&\quad \leq V^{N,K(0)}(x(0)) \leq V^{N}(x(0)) \leq V^{\infty}(x(0))
\end{aligned}$$

where the second and third inequalities come from that $\bar{V}^{N,K(T)}(x(T))$ is an upper bound to $V^{N,K(T)}(x(T))$ which is positive. The fourth inequality is an application of duality theory which says that the value of a dual feasible point is less than the primal optimal point. The last inequality is due to the observation that a longer control horizon gives larger cost in absence of terminal cost and terminal constraints. The assertion about suboptimality follows from the definition of $V^{\infty}_{MPC}(x(0))$ as $T \to \infty$.

Since $V^{\infty}_{MPC}(x(0))$ is finite, Theorem 2 gives that $\|x(t)\|_2^2 \to 0$ as $t \to \infty$.

This completes the proof. □

Remark 1: The slack variable $s(t)$ in this theorem actually consists of two parts. The first part is the unused slack from the inequality of the previous time step

$$\begin{aligned}
s(t) = s(t-1) + \alpha \ell(x(t-1), u(t-1,0)) + \\
+ \bar{V}^{N,K(t)}(x(t)) - V^{N,K(t-1)}(x(t-1)).
\end{aligned}$$

The second part is added to reduce conservatism due to the use of upper bounds from two time steps ago

$$V^{N,K(t-1)}(x(t-1)) - \bar{V}^{N,K(t-1)}(x(t-1)).$$

Also note that $s(t) \leq 0$ for all $t \in \mathbb{N}_0$.

Remark 2: When the system is operating far from the origin the value $V^{\infty}_{MPC}(x(0))$ is much more affected than when operating close to the origin. Thus, the control performance might deteriorate close to the origin since the overall objective is not affected. If desired, extra conditions might be added to avoid such behaviour without compromising the results. For instance one might want the system to satisfy

$$V^{N,K(t)}(x(t)) \geq \bar{V}^{N,K(t+1)}(x(t+1)) + \alpha_{lb} \ell(x(t), u(t,0))$$

in every time step where $\alpha_{lb} < \alpha$.

V. Distributed Model Predictive Control

The objective of this section is to use the developed design tool as a stopping criterion for the number of iterations needed in the DMPC scheme based on dual decomposition. If the conditions hold in every sample, we can guarantee suboptimality to a certain degree for the closed loop system and asymptotic stability. Since all communication in the dual decomposition scheme is between neighbouring nodes, the conditions of Theorem 3 should also be guaranteed by conditions that can be locally verified with communication between closest neighbours. Throughout this section we assume that the control horizon, N, in the DMPC scheme is such that Assumption 1 holds for a pre-specified value $\alpha \in (0,1)$.

To ensure the conditions in the design tool, i.e. in Theorem 3, an upper bound is needed. Any primal feasible solution over N time steps, defined as

$$P^N(x(t), u(t,\cdot)) = \sum_{\tau=0}^{N} \ell(x(t,\tau), u(t,\tau))$$

where

$$x(t,\tau+1) = Ax(t,\tau) + Bu(t,\tau)\ , \ \ x(t,0) = x(t)$$

with initial state, $x(t+1)$, is an upper bound to $V^{N,K(t+1)}(x(t+1))$ since

$$P^N(x(t+1), u(t,\cdot)) \geq V^N(x(t+1)) \geq V^{N,K(t+1)}(x(t+1)).$$

To account for the fact that the primal solution might be infeasible, we define $P^N(x(t), u(t,\cdot)) = \infty$ if the solution is infeasible. The local part of the primal cost is

$$P_i^N(x_i(t), u_i(t,\cdot)) = \sum_{\tau=0}^{N} \ell_i(x_i(t,\tau), u_i(t,\tau))$$

which gives

$$P^N(x(t), u(t,\cdot)) = \sum_{i=1}^{\mathscr{J}} P_i^N(x_i(t), u_i(t,\cdot)).$$

To calculate the primal cost, the following control sequence, based on the control sequence in the current iteration of the dual decomposition scheme, $u^k(t,\cdot)$, is used

$$u_P^k(t,\tau) = \begin{cases} u^k(t,\tau+1), & \tau = 0,\ldots,N-1 \\ 0, & \tau = N \end{cases}$$

This gives the following upper bound to be used in the DMPC scheme

$$P^N(x(t+1), u_P^k(t,\cdot))$$

where $x(t+1)$ is the predicted next state if the current control action, $u^k(t,0)$, is applied. The upper bound can be computed locally with neighbouring communication by forward simulation of the system.

To locally verify the conditions of Theorem 3 the following conditions are used in each node

$$V_i^{N,k}(x_i(t)) - P_i^N(x_i(t+1), u_{i,P}^k(t,\cdot)) \geq \\ \geq \alpha \ell_i(x_i(t), u_i^k(t,0)) + s_i(t) \quad (25)$$

where

$$V_i^{N,k}(x_i(t)) \geq 0 \quad (26)$$

and

$$s_i(t) = s_i(t-1) + \alpha \ell_i(x_i(t-1), u_i(t-1,0)) + \\ + P_i^N(x_i(t), u_{i,P}^{K(t)}(t,\cdot)) - \\ - P_i^N(x_i(t-1), u_{i,P}^{K(t-1)}(t-1,\cdot)) \quad (27)$$

and

$$s_i(1) = \alpha \ell_i(x_i(0), u_i(0,0)) + \\ + P_i^N(x_i(1), u_{i,P}^{K(1)}(1,\cdot)) - V_i^{N,K(0)}(x_i(0)) \quad (28)$$

and $s_i(0) = 0$.

Under Assumption 1 the conditions hold for the global system after sufficiently many iterations. However, it is not certain the these distributed tests will pass even at optimum. The conditions must be complemented by the following optimality condition

$$V_i^{N,k}(x_i(t)) = P_i^N(x_i(t), u^k(t,\cdot)). \quad (29)$$

A distributed MPC scheme that guarantee the conditions of Theorem 3 is summarized in the following theorem

Theorem 4: Consider a closed loop trajectory (7) with control action (13) which is applied after $K(t)$ iterations where the $K(t) = k$ such that

$$(25), (26), (27), (28), s_i(0) = 0 \text{ or } (29)$$

holds for all $t \in \mathbb{N}_0$. Further suppose that Assumption 1 holds. Then the conditions of Theorem 3 hold which guarantee

$$\alpha V_{MPC}^{\infty}(x(0)) \leq V^{\infty}(x(0))$$

and $\|x(t)\|_2^2 \to 0$ as $t \to \infty$ for the global system.

Proof. Summation over i of (25), (26), (27), (28), $s_i(0) = 0$ directly gives the conditions of Theorem 3.

Condition (29) is satisfied at the optimum. Assumption 1 gives that the conditions of Theorem 3 holds when the optimum is reached since $s(t) \leq 0$ for all $t \in \mathbb{N}_0$. □

Practical examples have shown that if the neighbouring interaction is not too large, the optimality conditions never need to be used as stopping criterion.

VI. Numerical Example

The performance of the developed distributed MPC scheme is evaluated by applying it to an artificial example with equally sized water containers. The water containers are connected in series and the flow between neighbouring containers are proportional to the relative difference in water level. Between every second container there are pumps that can control the water flow between the two containers they are connected to. In this example we consider the case of ten water containers and five pumps. The system is decomposed to consist of five subsystems, each with two containers and one pump. The local subsystems have the following dynamics:

$$x_i(t+1) = A_{i,i}x_i(t) + A_{i,i-1}x_{i-1}(t) + \\ + A_{i,i+1}x_{i+1}(t) + B_i u_i(t)$$

where

$$A_{1,1} = \begin{pmatrix} 0.9 & 0.1 \\ 0.1 & 0.8 \end{pmatrix} \qquad A_{5,5} = \begin{pmatrix} 0.8 & 0.1 \\ 0.1 & 0.9 \end{pmatrix}$$

$$A_{i,i} = \begin{pmatrix} 0.8 & 0.1 \\ 0.1 & 0.8 \end{pmatrix} \qquad \text{for } i = 2,3,4$$

and

$$A_{i,i-1} = \begin{pmatrix} 0 & 0.1 \\ 0 & 0 \end{pmatrix} = A_{i,i+1}^T \qquad B_i = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

for $i = 1, \dots, 5$ where $A_{1,0} = A_{5,6} = 0$

The mean water level of the system is actually uncontrollable since the total amount of water in the containers is constant. By requiring that $\mathbf{1}^T x(0) = 0$ the mean water level is defined to be zero. The objective is to control the individual water levels to the mean value of the water levels, i.e. to zero, while minimizing the following local running cost:

$$\ell_i(x_i, u_i) = x_i^T x_i + u_i^T u_i.$$

The control horizon is chosen to $N = 10$ which satisfies Assumption 1. The following table presents the results obtained for different schemes when suboptimality specified by $\alpha = 0.8$ is desired.

MPC scheme comparisons				
Scheme	cond	mean # iters	α_{calc}	p_{upd}
DMPC	all	1.95	0.841	prev
DMPC	all	7.05	0.891	0
DMPC	(25),(29)	150.0	0.889	prev
DMPC	(29)	161.6	0.893	prev
DC	-	-	0.720	-
C	-	-	0.893	-

TABLE I
Results from experiments with different MPC schemes

The first column describes what conditions are used as stopping criterion in the Distributed MPC scheme. The second column presents the mean number if iterations required for the conditions to hold. The third column specifies the resulting performance where

$$\alpha_{calc} = \frac{V^{N,K(0)}(x(0))}{\sum_{t=0}^{T} \ell(x(t), u(t,0))}.$$

The last column p_{upd} specifies how the price-updates are performed between optimizations. A zero means that the prices initially are chosen to 0 in each optimization. If the entry says 'prev' the previously calculated prices are shifted one time step and used as initial prices for the new optimization.

The first four rows present result when using the DMPC scheme presented in this article, with different conditions as stopping criterion. When the full scheme is used, presented in the first row, only 1.95 iterations are needed on average while still guaranteeing the suboptimality requirements. This can be compared to the second row, where the prices are set to zero between every new optimization. Then 7.05 iterations must be performed on average to guarantee the conditions. The scheme behind the results in row three has $s_i(t) = 0$ in (25). The number of iterations get very large using this scheme. This shows that the introduced slacks $s_i(t)$ has a large effect on the number of iterations needed to ensure a certain suboptimality bound. The condition in row four is that the optimum in the optimization should be found. This requires a large number of iterations on average.

Row five, labeled DC, corresponds to decentralized control in which the local optimizations are performed ignoring the coupling between systems. Using the method, no guarantees about performance or stability can be made, and the system in this case do not reach the desired performance. Row six, labeled C, presents results when applying centralized control. This is actually results in the same control strategy as the one in row four where the optimum is found in each sample.

VII. Conclusions and Future Work

We have presented theory for distributed Model Predictive Control based on dual decomposition, where the process to be controlled is an interconnection of several linear subsystems. We have developed stopping criterions which can be verified locally in each node, that guarantees closed loop asymptotic stability and suboptimality to a pre-specified degree for the global system. The provided numerical examples show that the number of iterations needed to guarantee the conditions is significantly smaller than if the optimum was to be reach in all optimizations.

Future work might be to extend this suboptimality framework to fit other decomposition methods for distributed MPC that show better convergence properties than the sub-gradient method proposed here.

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