Course Outline

Tuesday 08:45
- Optimal control
- Dynamic programming
- Bellman's equation
- Value iteration
- Approximate DynP

Wednesday 08:45
- Exercise summary
- MPC with terminal cost
- MPC without terminal cost
- Adaptive horizon
- Receding horizon estimation

Tuesday 09:45
- Model Predicitve Control
- Stability
- Terminal constraints
- Introduction to exercise

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- Dual Decomposition
- Distributed MPC
- Gradient methods
- Large-scale systems

Tuesday afternoon: Computer exercise

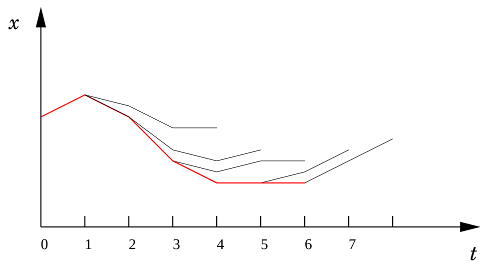
Conclusions of Lecture 1

- Dynamic programming — Off-line controller optimization
- Main limitation: Complexity of optimal value function
- Relaxed Dynamic Programming (with error bounds)
- Next: On-line optimization — Model Predictive Control

Lecture 2

- **Model Predicitve Control**
- Stability
- Stability from terminal constraints
- Introduction to exercise

Model Predicitive Control (Receding Horizon Control)



At time t:

1. Measure the state $x(t)$
2. Use model to optimize trajectory for $t+1, \dots, t+N$
3. Apply the optimization result $u(t)$ to the system
4. After one sample, go to 1 to repeat the procedure

The History of MPC

- **A.I. Propoi**, *Use of Linear Programming methods for synthesizing sampled-data automatic systems*, 1963 Automation and Remote Control
- Used industrially since 1970s, see for example **J. Richalet**, *Model predictive heuristic control — application to industrial processes*, Automatica, 1978.
- Many industrial products: DMC (Aspen Tech), IDCOM (Adersa), RMPCT (Honeywell), SMCA (Setpoint Inc), SMOC (Shell Global), 3dMPC (ABB), ...
- Strong theory development since about 1980 (linear) and 1990 (nonlinear)

Why is MPC popular?

- Models support understanding
- Systematic multi-input-multi-output design
- MPC controllers can handle constraints
- Systematic treatment of nonlinearities

The General Problem

Consider a nonlinear discrete time system

$$x(t+1) = f(x(t), u(t)), \quad x(0) = x_0$$

with $x(t) \in X$, $u(t) \in U$. Find control law $u = \mu(x)$ minimizing

$$J_\infty(x_0, u) = \sum_{t=0}^{\infty} \ell(x(t), u(t))$$

The minimal value is denoted $V_\infty(x_0)$.

For simplicity, we will assume that $0 \in X$, $0 \in U$, $f(0,0) = 0$ and $\ell(x,u) \geq 0$ with equality for $x = 0$, $u = 0$.

The MPC Control Law

Consider

$$V_N(x_0) = \inf_{u,x} \sum_{t=0}^{N-1} \ell(x(t), u(t))$$

where infimum is taken over $x(t) \in X$, $u(t) \in U$ satisfying $x(t+1) = f(x(t), u(t))$ and $x(0) = x_0$.

The MPC control law

$$\mu_N(x) := \arg\min_u \{V_{N-1}(f(x,u)) + \ell(x,u)\}$$

gives the cost

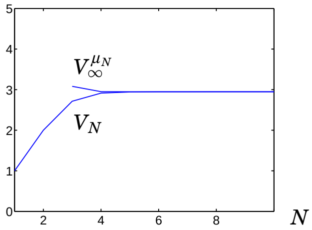
$$V_\infty^{\mu_N}(x_0) = \sum_{t=0}^{\infty} \ell(x_{\mu_N}(t), \mu_N(x_{\mu_N}(t)))$$

Notice that $V_1 \leq V_2 \leq \ldots \leq V_N \leq \ldots \leq V_\infty \leq V_\infty^{\mu_N}$

Example 1 — Double Integrator

$$V_N(x_0) = \inf_{u,x} \sum_{t=0}^{N-1} (|x(t)|^2 + u(t)^2)$$

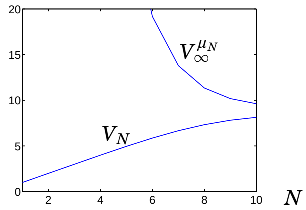
$$x(t+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \qquad x(0) = x_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



Example 1 — Double Integrator

$$V_N(x_0) = \inf_{u,x} \sum_{t=0}^{N-1} (|x(t)|^2 + 1000u(t)^2)$$

$$x(t+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \qquad x(0) = x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



Longer horizon required. Why?

Major Issues of MPC Theory

- Can we guarantee stability?
- Can we guarantee performance?
- What prediction horizon is needed?

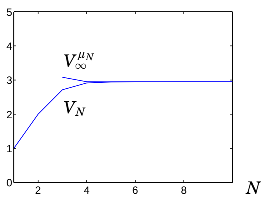
Lecture 2

- Model Predicitve Control
- **Stability**
- Stability from terminal constraints
- Introduction to exercise

Example 2 — Things can go bad

$$\inf_{u,x} \sum_{t=0}^{N-1} (|x(t)|^2 + u(t)^2)$$

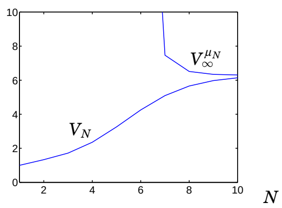
$$x(t+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$



Marginally unstable for $N \leq 2$.

$$\inf_{u,x} \sum_{t=0}^{N-1} (|x(t)|^2 + u(t)^2)$$

$$x(t+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -2 \\ 1 \end{bmatrix} u(t)$$



Exponentially unstable for $3 \leq N \leq 5$!

Long horizon need not help!

For the system

$$\begin{cases} x_1(t+1) = u(t) \\ x_2(t+1) = -2x_1(t) + u(t) \end{cases}$$

the cost function

$$\sum_{t=0}^{N-1} x_2(t)^2$$

is minimized by the control law $u(t) = 2x_1(t)$, which gives the unstable dynamcs

$$x_1(t+1) = 2x_1(t)$$

The transfer function from u to x_2 has an unstable zero at $z = 2$!

The effect of unstable zeros

Consider and input-output map $Y(z) = G(z)U(z)$ where $G(z)$ has a real unstable zero at $z = a$. If U is a step, the step response $y(0), y(1), y(2), \ldots$, with Zeta-transform $Y(z)$, satisfies

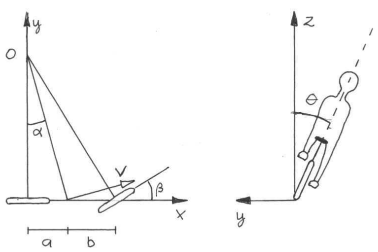
$$0 = Y(a) = \sum_{t=0}^{\infty} y(t)a^{-t}$$

so $y(t)$ must take both positive and negative values. Moreover, this must happen before the exponential decaying a^{-t} becomes dominating.

Hence an unstable zero implies that the response to control action initially goes in the "wrong" direction. The time constant of the unstable zero puts a bound on how fast the feedback loop can become.

Bike example

A (linearized) torque balance for a bicycle can be approximated as



$$J\frac{d^2\theta}{dt^2} = mg\ell\theta + \frac{mV_0\ell}{b}\left(V_0\beta + a\frac{d\beta}{dt}\right)$$

Klein's Bicycle with Rear Wheel Steering

Richard Klein at UIUC has built several UnRidable Bicycles (URBs). We have versions in Lund

Transfer function

$$P(s) = \frac{am\ell V_0}{bJ} \frac{-s + \dfrac{V_0}{a}}{s^2 - \dfrac{mg\ell}{J}}$$

Pole at $p = \sqrt{\dfrac{mg\ell}{J}} \approx 3 \;\; \mathrm{rad/s}$

RHP zero at $z = \dfrac{V_0}{a}$

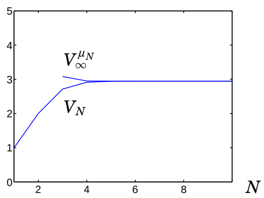
Pole independent of velocity but zero proportional to velocity. There is a velocity such that $z = p$ and the system is uncontrollable. The system is difficult to control robustly if z/p is in the range of 0.25 to 4.

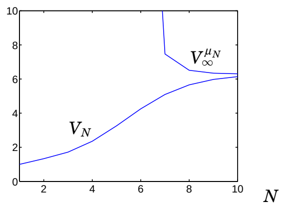
UCSB Version

Example 2 — Things can go bad

$$\inf_{u,x} \sum_{t=0}^{N-1} (|x(t)|^2 + u(t)^2) \qquad \inf_{u,x} \sum_{t=0}^{N-1} (|x(t)|^2 + u(t)^2)$$

$$x(t+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \qquad x(t+1) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -2 \\ 1 \end{bmatrix} u(t)$$





No finite unstable zero. Unstable zero at $z = 1.5$.

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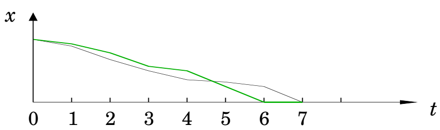
MPC with Equilibrium Terminal Constraint

$$J_N(x_0, u) = \sum_{t=0}^{N-1} \ell(x(t), u(t)) \qquad V_N(x_0) = \inf_{u,x} J_N(x_0, u)$$

subject to $x(t) \in X$, $u(t) \in U$, $x(t+1) = f(x(t), u(t))$, $x(0) = x_0$ and the *terminal constraint* $x(N) = 0$.

$\mu_N(x) := \arg\min_u \{V_{N-1}(f(x,u)) + \ell(x,u)\}$

The terminal constraint gives $V_N(x) \le V_{N-1}(x)$.



MPC with Equilibrium Terminal Constraint

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$\mu_N(x) := \arg\min_u \{V_{N-1}(f(x,u)) + \ell(x,u)\}$

The terminal constraint gives $V_N(x) \le V_{N-1}(x)$. Hence

$$\begin{aligned} V_N(x) &= \ell(x, \mu_N(x)) + V_{N-1}(f(x, \mu_N(x))) \\ &\ge \ell(x, \mu_N(x)) + V_N(f(x, \mu_N(x))) \end{aligned}$$

so V_N is a Lyapunov function that proves stability! Moreover $V_\infty \le V_\infty^{\mu_N} \le V_N$.

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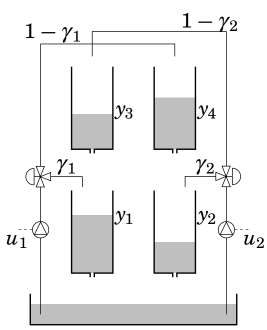
Control of a Quadruple Tank



The transfer matrix from (u_1, u_2) to (y_1, y_2)

$$\begin{pmatrix} \dfrac{\gamma_1 c_1}{1+sT_1} & \dfrac{(1-\gamma_2)c_1}{(1+sT_1)(1+sT_3)} \\ \dfrac{(1-\gamma_1)c_2}{(1+sT_2)(1+sT_4)} & \dfrac{\gamma_2 c_2}{1+sT_2} \end{pmatrix}$$

has unstable zero if and only if

$$0 \le \gamma_1 + \gamma_2 < 1$$

Is there a step response with wrong direction?

Lab material

can be found on `www.control.lth.se/disc-summer-school/`

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