Assessment of Achievable Performance

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Congratulations

- Seminar at MIT
- Your monumental model reduction paper
- Your tour de force on $\mathcal{H}^\infty$ theory and design methods
- Great leadership in Cambridge
  - Great research
  - International network
  - Superb students
  - $\mathcal{H}^\infty$ loop shaping (elegant theory, lots of applications)

Introduction

Goal: Capture the essence of a control design in a simple way

- Useful for an teaching and an overall assessment
- Useful for choosing the weights in $\mathcal{H}^\infty$ loopshaping

The idea

- Assume that a controller is designed with a method like $\mathcal{H}^\infty$ which guarantees robustness
- Find a way to characterize essential trade-offs qualitatively

Important Issues

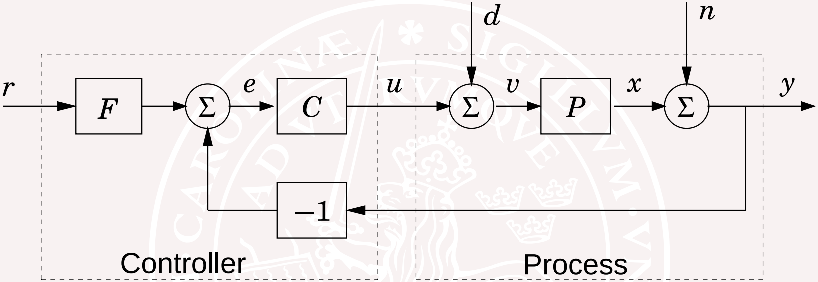
- Load disturbances
- Measurement noise
- Command signals
- Process variations
- Process dynamics, time delays, RHP poles and zeros
- Actuator resolution and saturation
- Sensor resolution and range

Results can be summarized in an assessment plot that can be generated from the process transfer function

Outline

1. Introduction
2. Preliminaries
3. Performance Trade-offs
4. Robustness Constraints for NMP systems
5. Assessment Plots
6. Summary

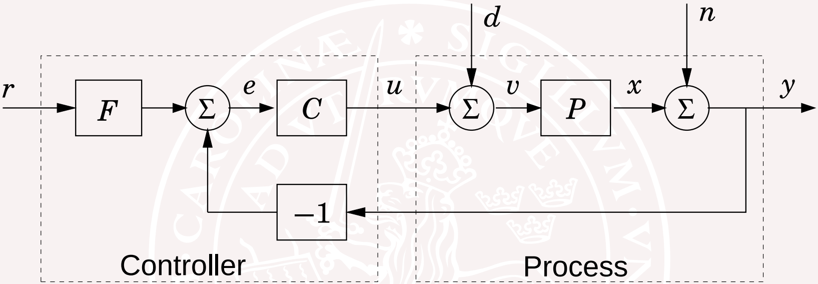
A Basic Control System



Ingredinets

- Controller: feedback C, feedforward F
- Load disturbance d : Drives the system from desired state
- Measurement noise n : Corrupts information about state x
- Command signal r : Process state x should follow r

Criteria for Control Design



Ingredients

- Attenuate effects of load disturbance d
- Do not feed in too much measurement noise n
- Make the system insensitive to process variations
- Make state x follow command r

A Separation Principle for 2DOF

Design the feedback C to achieve

- Low sensitivity to load disturbances d
- Low injection of measurement noise n
- High robustness to process variations

Then design the feedforward F to achieve the desired response to command signals r

At least six transfer functions are required to characterize the system (*the Gang of Six*)

- Many books and papers show only the set point response
- Interactive learning modules

Process Control

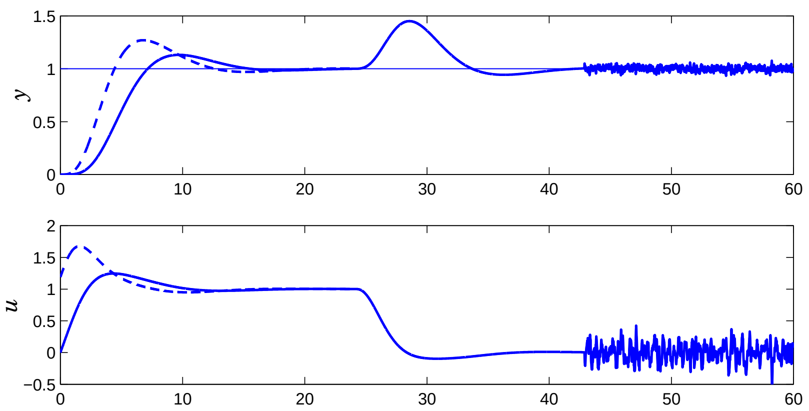
The tuning debate: Should controllers be tuned for set-point response or for load disturbance response?

- Different tuning rules for PID controllers
- Shinskey: Set-point disturbances are less common than load changes.
- Resolved by set-point weighting (poor mans 2DOF)

$$u(t) = k\big(\beta r(t) - y(t)\big) + k_i \int_0^t \big(r(\tau) - y(\tau)\big) d\tau + k_d \Big(\gamma \frac{dr}{dt} - \frac{dy_f}{dt}\Big)$$

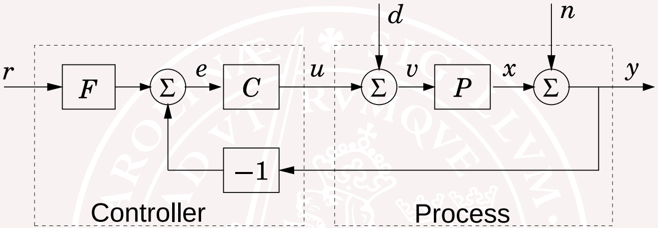
- Tune k, k_i, and k_d for load disturbances, filtering for measurement noise and β, and γ for set-points

PID Control with Set-Point Weighting



Outline

Disturbance Modeling gives Weights



$$Y = PSD - SN, \qquad U = -PCSD - CSN$$

Stochastic modeling of d (drifting) and n (white noise) and solution to the LQG problem

$$J = E \int_0^\infty \big(y^2(\tau) + \rho u^2(\tau)\big) d\tau$$

gives a controller with integral action and high frequency roll-off and weights for an $\mathcal{H}^\infty$ problem (mixed $\mathcal{H}^2$ - $\mathcal{H}^\infty$)

Performance Assessment

Disturbance reduction by feedback

$$\frac{Y_{cl}(s)}{Y_{ol}(s)} = \frac{1}{1+PC}$$

Load disturbance attenuation (typically low frequencies)

$$\frac{X}{D} = \frac{Y}{D} = \frac{P}{1+PC}, \qquad \frac{U}{D} = -\frac{PC}{1+PC}$$

Measurement noise injection (typically high frequencies)

$$\frac{X}{N} = \frac{PC}{1+PC}, \qquad \frac{U}{N} = -\frac{C}{1+PC}$$

Command signal following

$$\frac{X}{R} = \frac{Y}{R} = \frac{PCF}{1+PC}, \qquad \frac{U}{R} = \frac{CF}{1+PC}$$

Robustness

Robustness to process variations (large, additive, stable ΔP)

$$\left|\frac{\Delta P}{P}\right| < \frac{|1+PC|}{|PC|} = \frac{1}{|T|}$$

Sensitivity of command signal response (small variations)

$$\frac{dG_{xr}}{G_{xr}} = \frac{1}{1+PC}\frac{dP}{P}$$

Sensible design methods like $\mathcal{H}^\infty$ loop shaping guarantees good sensitivities.

Simple Performance Assessment

$$\frac{Y_{cl}(s)}{Y_{ol}(s)} = \frac{1}{1+PC}$$

$$X = \frac{P}{1+PC}D - \frac{PC}{1+PC}N$$

$$U = -\frac{PC}{1+PC}D - \frac{C}{1+PC}N$$

Load disturbances typically have low frequencies, and measurement noise typically has high frequencies $\rightarrow$ integral action and high frequency roll-off

Minimum Phase Systems

Any transfer function can be realized. No limitations because of system dynamics. High bandwidth attenuates disturbances effectively but measurement noise is also amplified. Gain crossover frequency ω_{gc} captures

- Disturbance attenuation

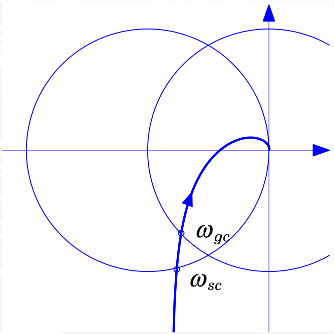
$$Y_{cl} = SY_{ol}$$

- Noise injection to state

$$X = -TN$$

- How about noise injection to u?

$$U = -CSN$$



Effect of Noise on Control Signal

Loop shaping design

- Determine desired crossover frequency ω_{gc}
- Required phase lead at crossover frequency

$$\varphi_l = -\arg P(i\omega_{gc}) - \pi + \varphi_m$$

- Add phase lead to give desired phase margin
- Adjust gain to make loop gain 1 at ω_{gc}

Phase lead is requires gain.

Gain of a Simple Lead Networks

$$G_n(s) = \left(\frac{s+a}{s/\sqrt[n]{K}+a}\right)^n.$$

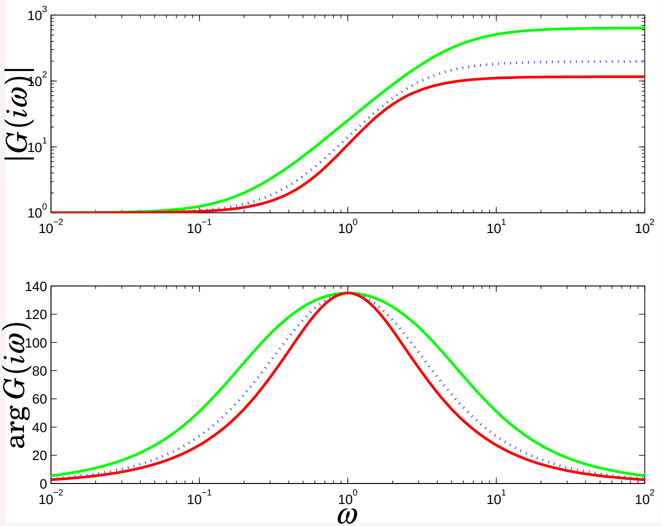
Phase lead $\varphi = n \arctan \dfrac{\sqrt[n]{K}-1}{2\sqrt[2n]{K}}.$

Gain $K_n = \left(1 + 2\tan^2\frac{\varphi}{n} + 2\tan\frac{\varphi}{n}\sqrt{1+\tan^2\frac{\varphi}{n}}\ \right)^n$

Phase lead	*n=2*	*n=4*	*n=6*	*n=8*	*n=∞*
90°	34	25	24	24	23
180°	-	1150	730	630	540
225°	-	14000	4800	3300	2600

As n goes to infinity $K_n \to K_\infty = e^{2\varphi}$, exponential increase

Lead Networks of 2nd 3rd and 10th Order



Bode's Phase Area Formula

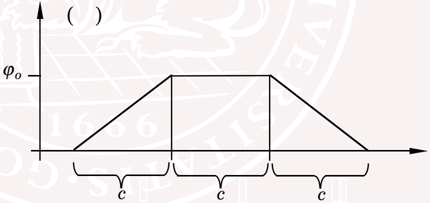
Let $G(s)$ be a transfer function with no poles and zeros in the right half plane. Assume that $\lim_{s\to\infty} G(s) = G_\infty$. Then

$$\log \frac{G(\infty)}{G(0)} = \frac{2}{\pi}\int_0^\infty \arg G(i\omega)\frac{d\omega}{\omega} = \frac{2}{\pi}\int_{-\infty}^\infty \arg \bar{G}(iu)du$$

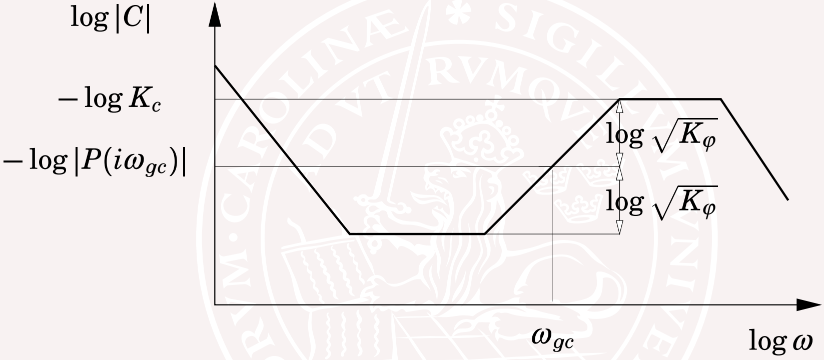
The gain K required to obtain a given phase lead φ is an exponential function of the area under the phase curve

$$K = e^{4c\varphi_0/\pi} = e^{2\gamma\varphi_0}$$

$$\gamma = \frac{2c}{\pi}$$



Estimate of Controller Gain



$$K_c = \max_{\omega \geq \omega_{gc}} |C(i\omega)| = \frac{\sqrt{K_\varphi}}{|P(i\omega_{gc})|} = \frac{e^{\gamma\varphi_l}}{|P(i\omega_{gc})|} = \frac{e^{\gamma(-\pi+\varphi_m-\arg P(i\omega_{gc}))}}{|P(i\omega_{gc})|}.$$

Right hand side only depends on the process!

Estimating Controller Gain

This largest high frequency gain of the controller is approximately given by ($\gamma \approx 1$)

$$K_c = \max_{\omega \geq \omega_{gc}} |C(i\omega)| = \frac{e^{\gamma\varphi_l}}{|P(i\omega_{gc})|} = \frac{e^{\gamma(-\pi+\varphi_m-\arg P(i\omega_{gc}))}}{|P(i\omega_{gc})|}$$

Notice that K_c only depends on the process

- Compensation for process gain $1/|P(i\omega_{gc})|$
- Gain required for phase lead: $e^{\gamma(-\pi+\varphi_m-\arg P(i\omega_{gc}))}$

The largest allowable gain is determined by sensor noise and resolution and saturation levels of the actuator. Results also hold for NMP systems but there are other limitations for such systems.

A Classic Problem

- For linear systems it follows Bode's phase area formula that phase advance requires gain
- An observation: higher order compensator gives lower gain
- A key question: Can we get a given phase advance with less gain by using a nonlinear systems?
- The Clegg integrator
- A problem worth revisiting?

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Robustness and Gain Crossover Frequency

Factor process transfer function as $P(s) = P_{mp}(s)P_{nmp}(s)$ such that $|P_{nmp}(i\omega)| = 1$ and P_{nmp} has negative phase. Requiring a phase margin φ_m we get

$$\begin{aligned}\arg L(i\omega_{gc}) &= \arg P_{nmp}(i\omega_{gc}) + \arg P_{mp}(i\omega_{gc}) + \arg C(i\omega_{gc})\\ &\geq -\pi + \varphi_m\end{aligned}$$

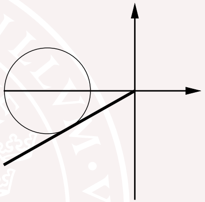
But $\arg P_{mp}C \approx n\pi/2$, where n is the slope at the crossover frequency. (Exact for Bodes ideal loop transfer function $P_{mp}(s)C(s) = (s/\omega_{gc})^n$). Hence

$$\arg P_{nmp}(i\omega_{gc}) \geq -\pi + \varphi_m - n\frac{\pi}{2}$$

The phase crossover inequality implies that robustness constraints for NMP systems can be expressed in terms of ω_{gc}.

Bode's Ideal Cut-off Characteristics

The repeater problem. Large gain variations in vacuum tube amplifiers. What should a loop transfer function look like to make the properties independent of open-loop gain?



$$L(s) = \left(\frac{s}{\omega_{gc}}\right)^n$$

Phase margin invariant with loop gain. For this transfer function we have $\arg L(i\omega) = n\pi/2$.

The slope $n = -1.5$ gives the phase margin $\varphi_m = 45°$.

Horowitz extended Bodes ideas to deal with arbitrary plant variations not just gain variations in the QFT method.

The Crossover Frequency Inequality

The inequality

$$\arg P_{nmp}(i\omega_{gc}) \geq -\pi + \varphi_m - n_{gc}\frac{\pi}{2}$$

implies that robustness requires that the phase lag of the non-minimum phase component P_{nmp} at the crossover frequency is not too large!

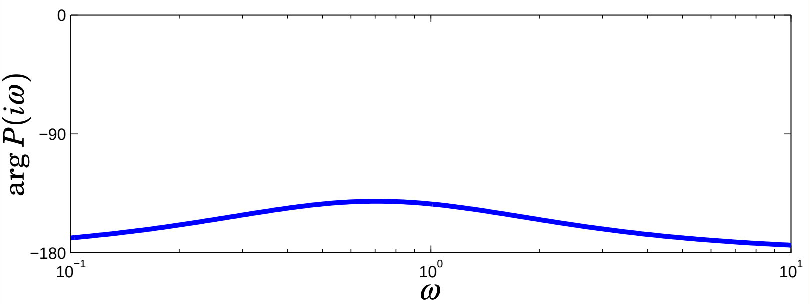
Simple rule of thumb:

- $\varphi_m = 45°,\ n_{gc} = -1/2 \Rightarrow \arg P_{nmp}(i\omega_{gc}) \geq -\frac{\pi}{2}$
- $\varphi_m = 60°,\ n_{gc} = -2/3 \Rightarrow \arg P_{nmp}(i\omega_{gc}) \geq -\frac{\pi}{3}$
- $\varphi_m = 45°,\ n_{gc} = -1 \Rightarrow \arg P_{nmp}(i\omega_{gc}) \geq -\frac{\pi}{4}$

Useful to Plot the Phase of P_{nmp}

Example from Doyle, Francis and Tannenbaum 1992 and the Bhattacharya fragility debate.

$$P(s) = \frac{s-1}{s^2+0.5s-0.5}, \qquad P_{nmp} = \frac{(1-s)(s+0.5)}{(1+s)(s-0.5)}$$



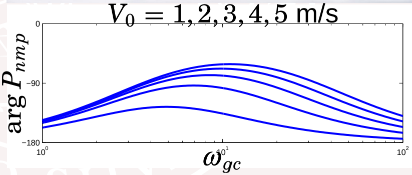
Bicycle with Rear Wheel Steering

Transfer function

$$P(s) = \frac{am\ell V_0}{bJ} \frac{-s + \frac{V_0}{a}}{s^2 - \frac{mg\ell}{J}}$$

RHP pole at $\sqrt{mg\ell/J}$

RHP zero at V_0/a



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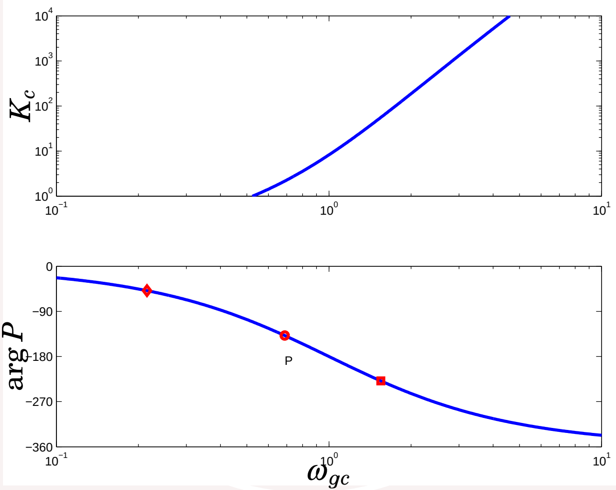
The Assessment Plot

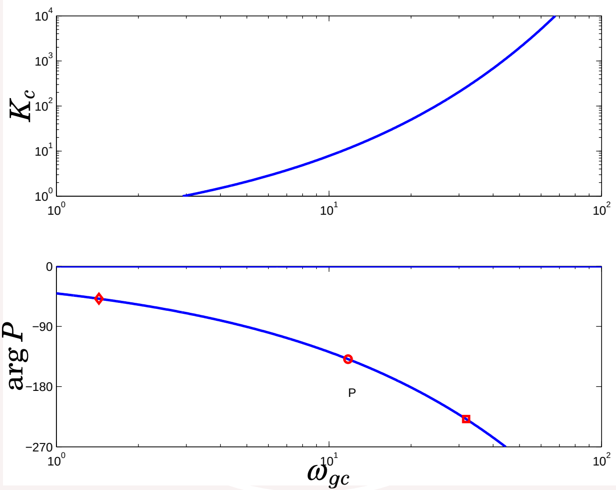
The *assessment plot* has a gain curve $K_c(\omega_{gc})$ and two phase curves $\arg P(i\omega)$ and $\arg P_{nmp}(i\omega)$

- Attenuation of disturbance captured by ω_{gc}
- Injection of measurement noise captured by the high frequency gain of the controller $K_c(\omega_{gc})$
- Robustness limitations due to time delays and RHP poles and zeros captured by $\arg P_{nmp}(\omega_{gc})$
- Controller complexity is captured by $\arg P(i\omega_{gc})$

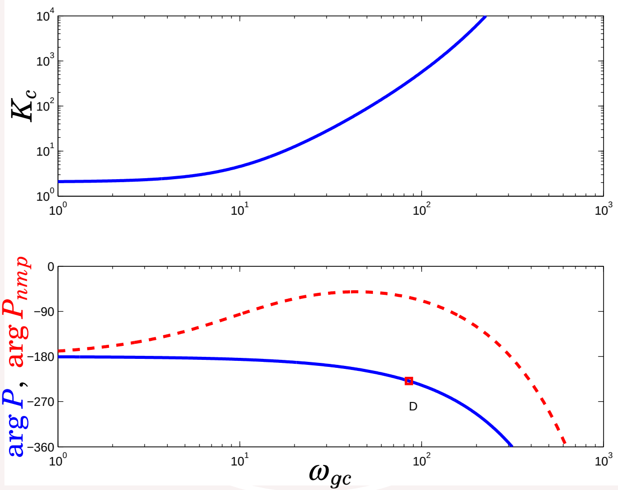
Assessment Plot for $P(s) = 1/(s+1)^4$



Assessment Plot for $P(s) = e^{-\sqrt{s}}$



Assessment Plot for $P(s) = e^{-0.01s}/(s^2 - 100)$



Outline

Summary

Issues in control system design

- Load disturbances, measurement noise, command signals
- Robustness to rocess variations
- Process dynamics, time delays, RHP poles and zeros
- Resolution and range of actuators and sensors

The assessment plot captures many issues qualitatively.

- Trading off attenuation of load disturbance against injection of measurement noise
- Robustness for NMP systems
- Assessment of controller complexity
- Weights can be chosen based on gain crossover frequency