A Family of Smooth Strategies for Swinging up a Pendulum

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Why are Pendulums Interesting?

- Good prototypes for many control problems
  - Stabilization and manual control
  - Large transitions - swing up
  - Friction compensation
- Graduated difficulties
  - Pendulum, Pendulum on cart, Furuta pendulum, Spherical pendulum
- Well suited for interesting and instructive experiments
- Similar to many real engineering problems: power systems, phaselocked loops, Josephson junctions

Simple closed form smooth strategies
for swing up and stabilization
Idea: Shaping energy and damping

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Shaping the Potential Energy

A simple version:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \sin x_1 - u \cos x_1, \\ E &= \cos x_1 + \tfrac{1}{2} x_2^2 \end{aligned}$$

Select a potential energy which gives suitable Hamiltonian

$$H_d(x_1, x_2) = V_d(x_1) + \frac{x_2^2}{2},$$

Find a control law for the original system which matches this

$$V_d'(x_1) = -\sin x_1 + u(x_1) \cos x_1,$$

A class of *compatible* energy functions

$$V_d = \cos x_1 - a_2 \cos^2 x_1 - \cdots + constant$$

The Simplest Case

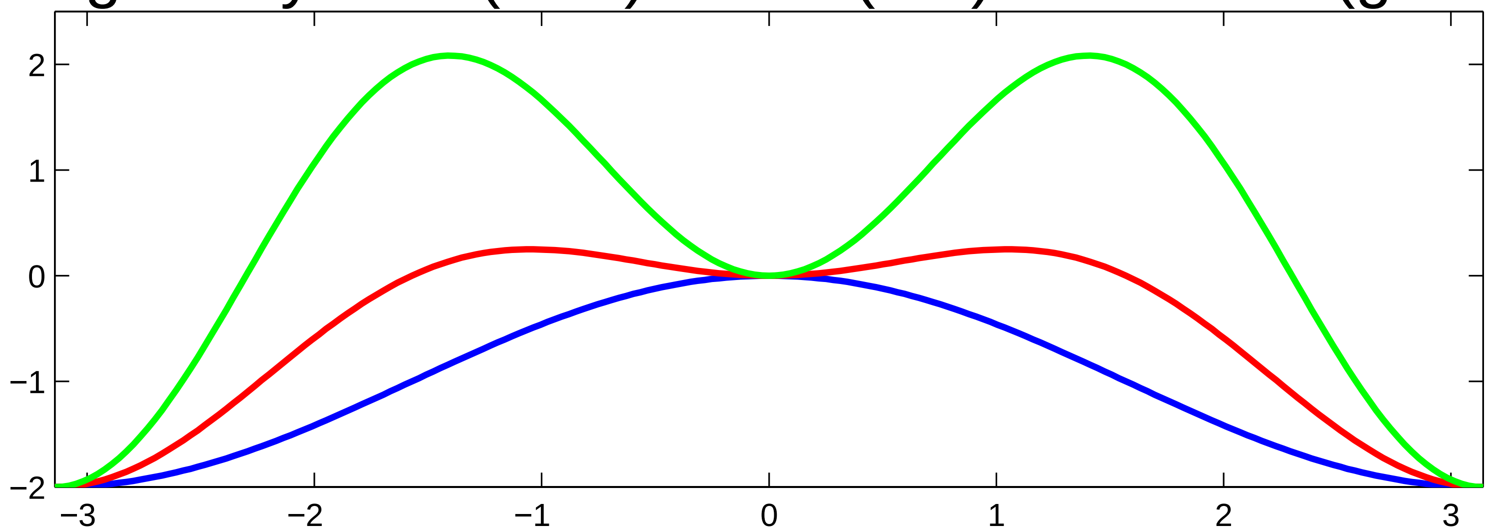
Original potential energy: $V(x_1) = \cos x_1 - 1$

The feedback $u = 2a \sin x_1$ gives the potential energy

$$V_s = \cos x_1 - a \cos^2 x_1 - 1 + a$$

Minimum at the origin if $a > 0.5$

Original system (blue) $a = 1$ (red) and $a = 3$ (green)

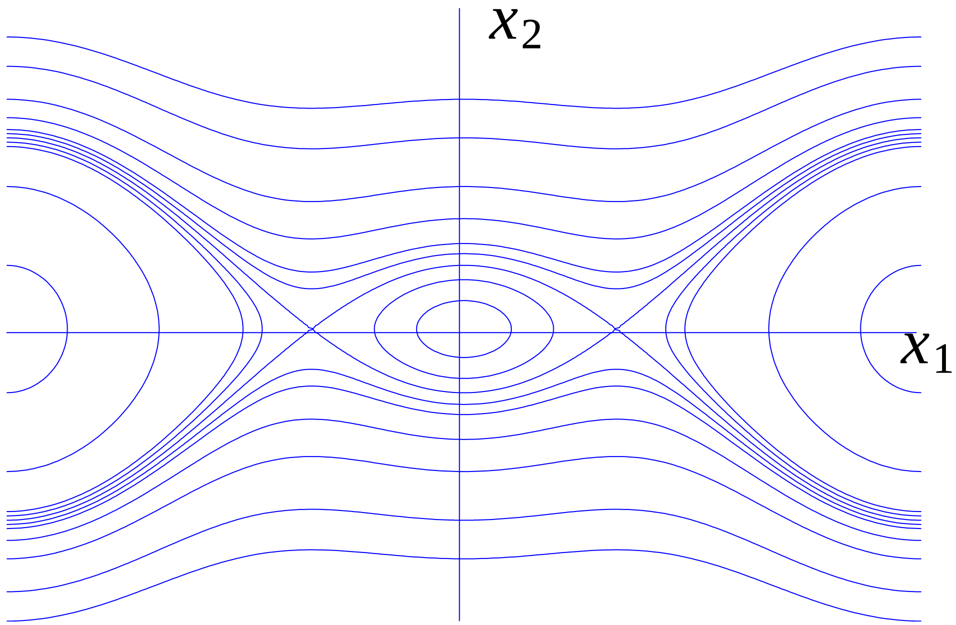


Many other choices CDC/ECC Sevilla

The Hamiltonian

The feedback $u = 2a \sin x_1$ corresponds to the Hamiltonian

$$H_d(x_1, x_2) = V_d(x_1) + \frac{x_2^2}{2},$$



Easy to influence the behavior of an Hamiltonian system

Damping and Pumping

$$H_d(x_1, x_2) = \cos x_1 - a\cos^2 x_1 + \frac{x_2^2}{2} - \frac{a}{4}$$

Introduce an additional term in the control law

$$u = 2a\sin x_1 + v(x_1, x_2)$$

If $v(x_1, 0) = 0$ it will not influence the potential energy

$$\frac{dH_d}{dt} = -x_2 v \cos x_1$$

Choose v proportional to $x_2 \cos x_1$

$$v = bx_2 F(x_1, x_2)\cos x_1$$

Damping if F is positive pumping if F negative. Control law

$$u = 2a\sin x_1 + bx_2 F(x_1, x_2)\cos x_1$$

The Control Law

Control law

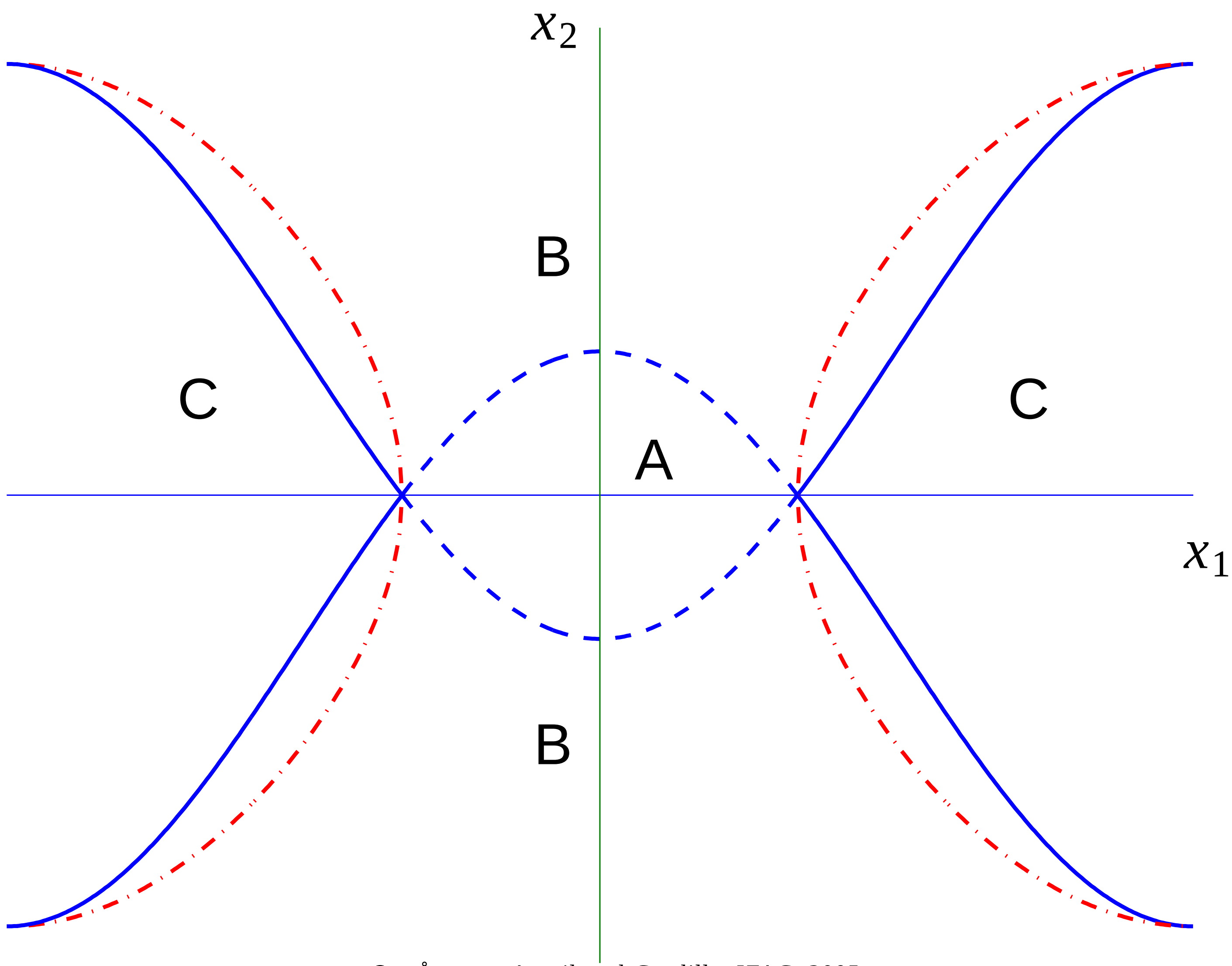
$$u = 2a\sin x_1 + bx_2F(x_1,x_2)\cos x_1$$

$$\frac{dH_d}{dt} = -x_2 v\cos x_1$$

- $2a\sin x_1$ - *spring term*
- $bx_2F(x_1,x_2)\cos x_1$ - *damping term*
- damping if $F > 0$
- pumping if $F < 0$

How to choose the smooth function $F(x_1,x_2)$?

Match critical part of Level curve $H_d(x_1, x_2) = 0$ with $F(x_1, x_2) = 0$



A Simple Approximation

Find a simple function $W(x_1)$ that matches the potential energy function V for $x_1 \geq x_1^0 = \arccos 1/2a$. We have

$$V_d(x_1^0) = 0$$

$$V_d(\pi) = -1 - a - \frac{1}{4a} = -\frac{(2a+1)^2}{4a}$$

A simple choice is

$$W(x_1) = \frac{2a+1}{4a}(2a\cos x_1 - 1).$$

which gives

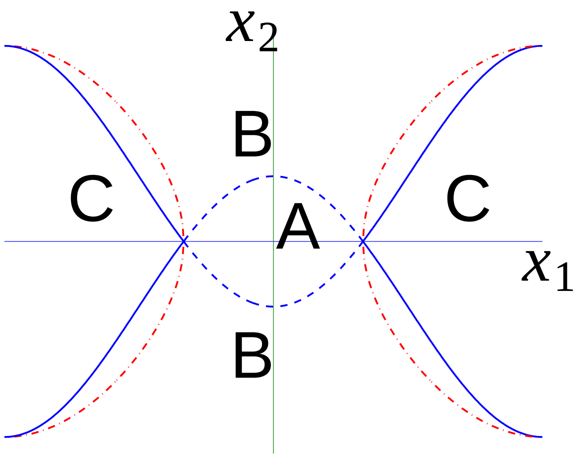
$$F(x_1, x_2) = W(x_1) + x_2^2/2$$

A Family of Control Strategies

Control law

$$u(x_1, x_2) = 2a \sin x_1 + b x_2 F(x_1, x_2) \cos x_1$$

$$F(x_1, x_2) = \frac{2a+1}{4a}(2a \cos x_1 - 1) + \frac{x_2^2}{2}.$$



- First term of u shapes the energy so that the origin is a center
- Second term introduces damping and pumping in appropriate regions
- Parameter a adjusts the width and depth of the potential well
- Parameter b adjusts the rate of damping and pumping

The Closed Loop System

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \sin x_1 - 2a \sin x_1 \cos x_1 - b x_2 F(x_1, x_2) \cos^2 x_1,$$
$$F(x_1, x_2) = \frac{2a+1}{4a}(2a \cos x_1 - 1) + \frac{x_2^2}{2}.$$

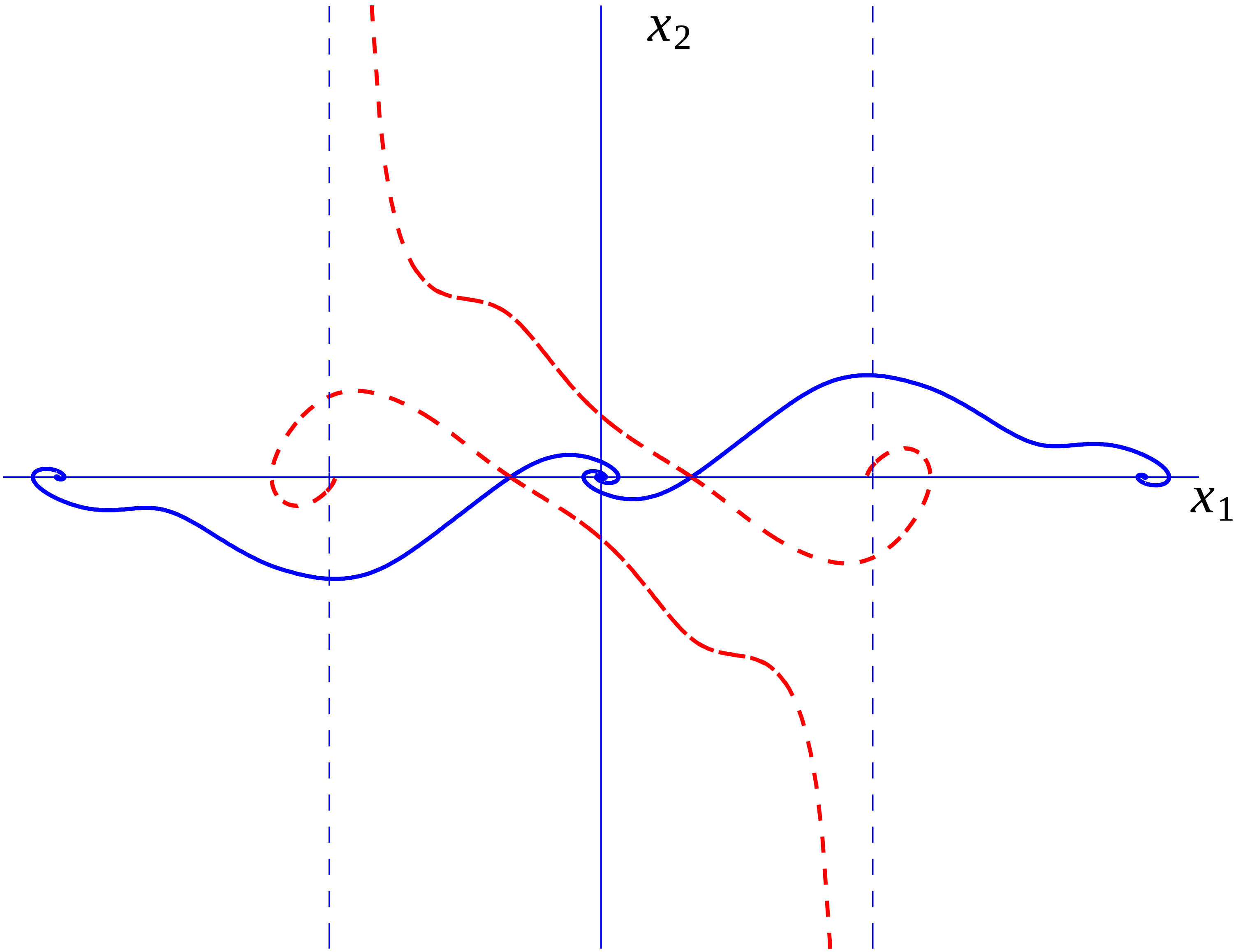
Equilibria

- $x_1 = 0,\ x_2 = 0$
- $x_1 = \pm \arccos 1/2a,\ x_2 = 0$

Large x_2

$$\frac{dx_1}{dt} = x_2$$
$$\frac{dx_2}{dt} \approx -\frac{b}{2} x_2^3 \cos^2 x_1$$

Separatrices

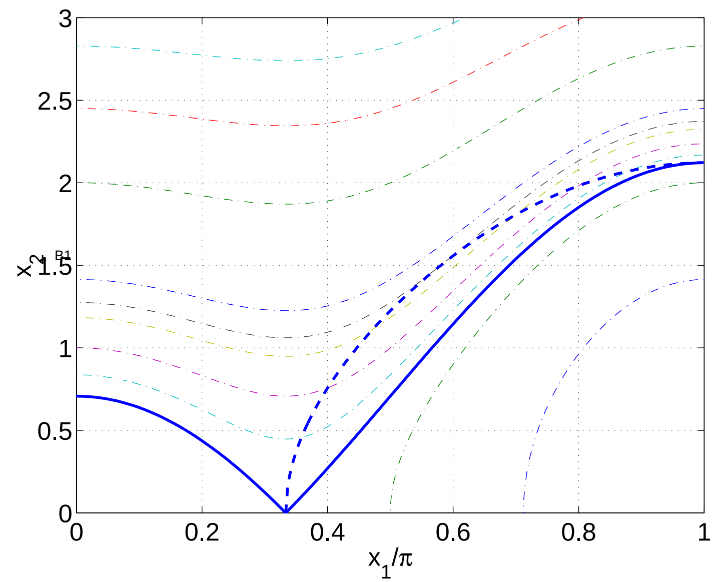


Main Result

Let $x_0 = \arccos(1/(2a))$ and introduce

$$\varphi_H(x) = \sqrt{\frac{1}{2a} + 2a\cos^2 x - 2\cos x}$$

$$\varphi_F(x) = \sqrt{\frac{1+2a}{2a}(1 - 2a\cos x)}, \quad x \geq x_0$$



$$\Phi(a) = \int_0^{x_0} \varphi_H(x)\cos^2(x)F(x,\varphi_H(x))dx + \int_{x_0}^{\pi} \varphi_F(x)\cos^2(x)F(x,\varphi_H(x))dx$$

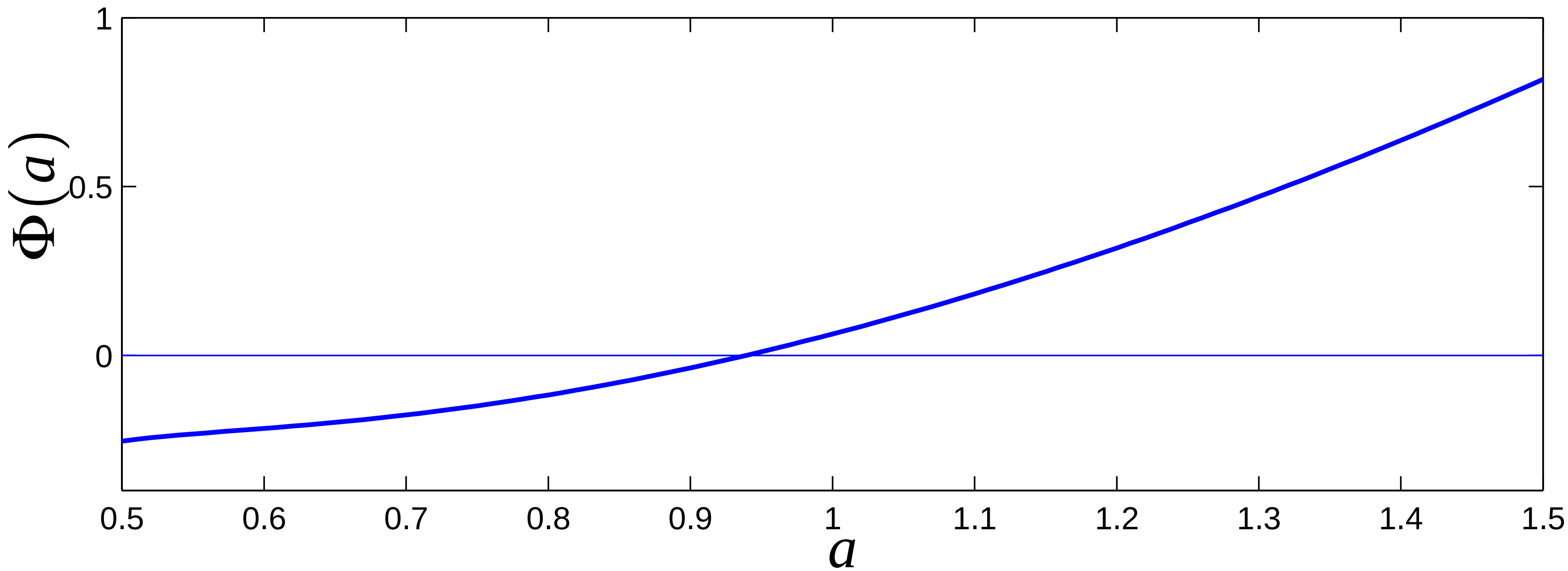
Theorem Sufficiency condition

Let a be such that $\Phi(a) > 0$ and let $b > 0$ then all solutions except those starting at $x_1 = \pm\pi$, $x_1 = 0$ and on the separatrices converge to $x_1 = 0$ and $x_2 = 0$.

The Function $\Phi(a)$

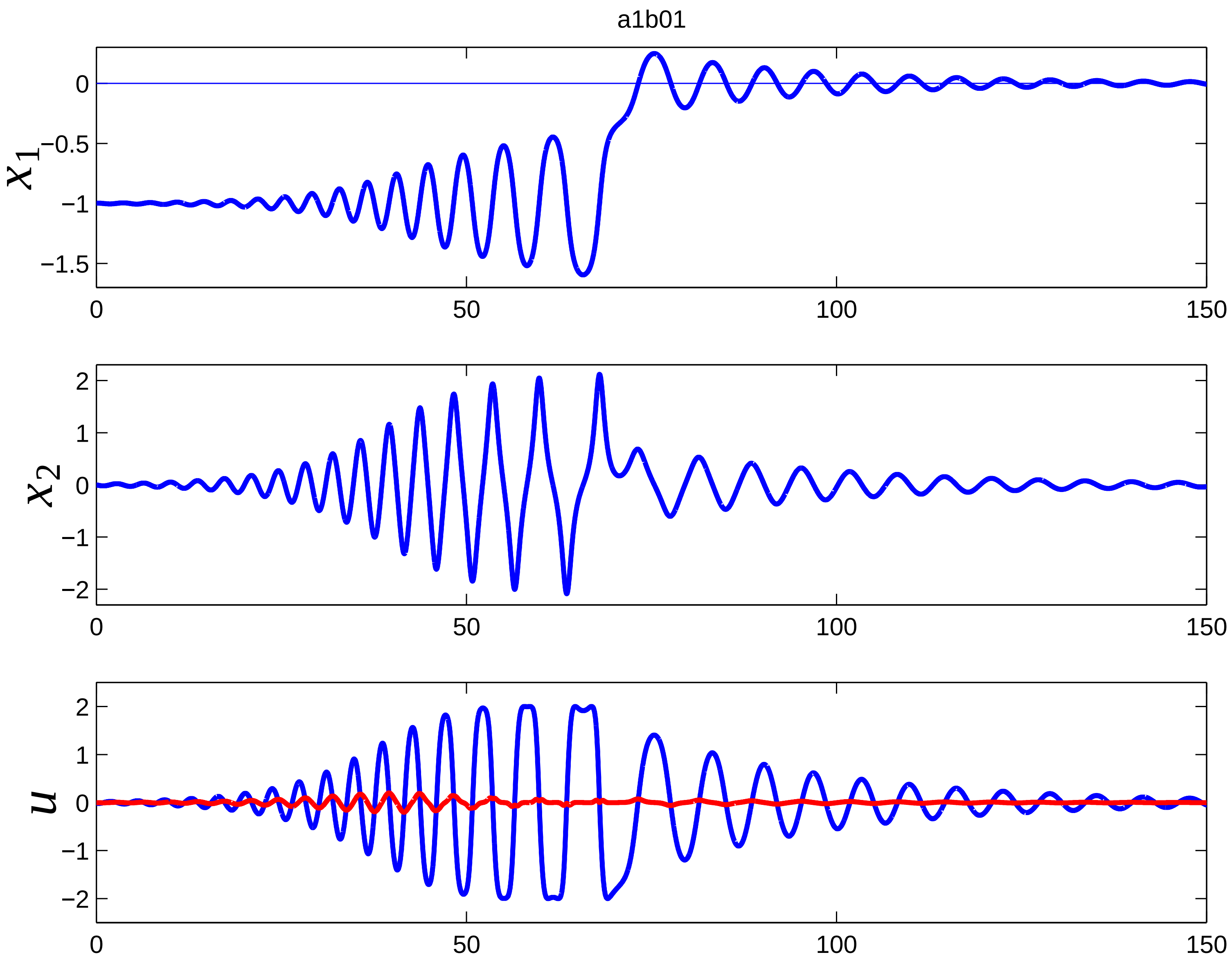
$$\Phi(a) = \int_0^{x_0} \varphi_H(x)\cos^2(x)F(x,\varphi_H(x))\,dx + \int_{x_0}^{\pi} \varphi_F(x)\cos^2(x)F(x,\varphi_H(x))\,dx$$

$$\varphi_H(x) = \sqrt{\frac{1}{2a} + 2a\cos^2 x - 2\cos x}, \qquad \varphi_F(x) = \sqrt{\frac{1+2a}{2a}(1-2a\cos x)}$$

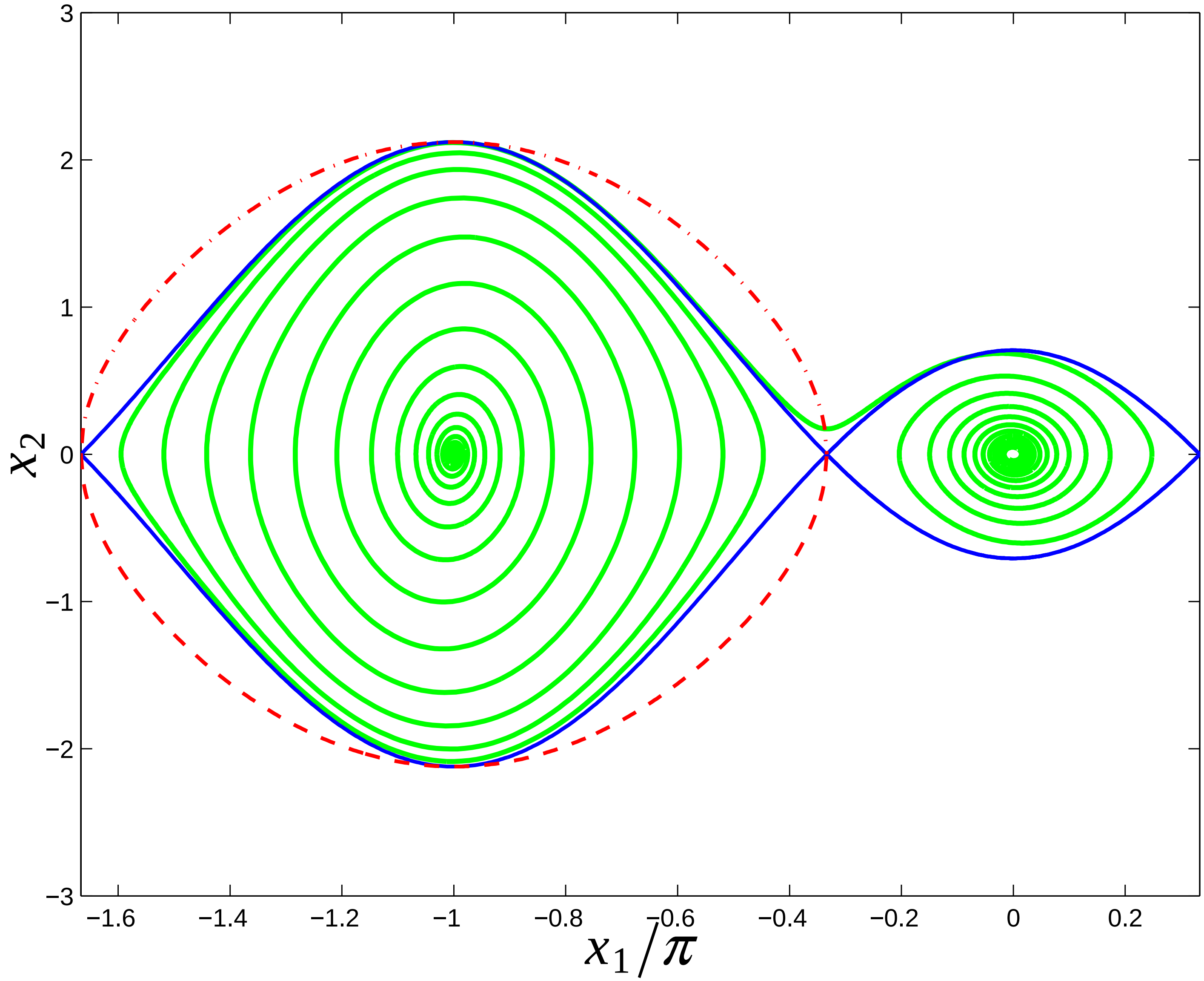


$a > 0.94$ (0.84) suffices! Potential well requires $a > 0.5$

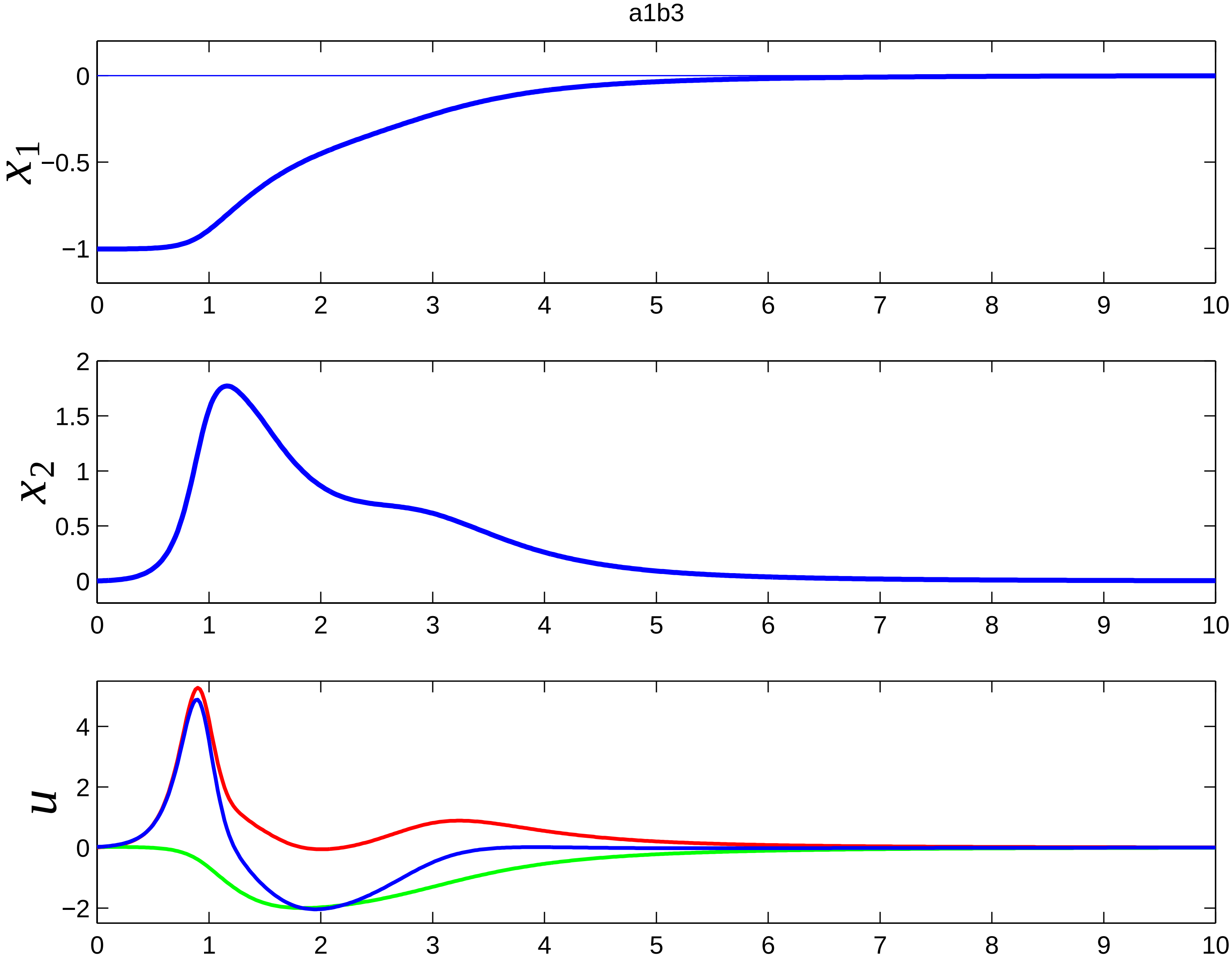
Simulated Swingup $a = 1$ and $b = 0.1$



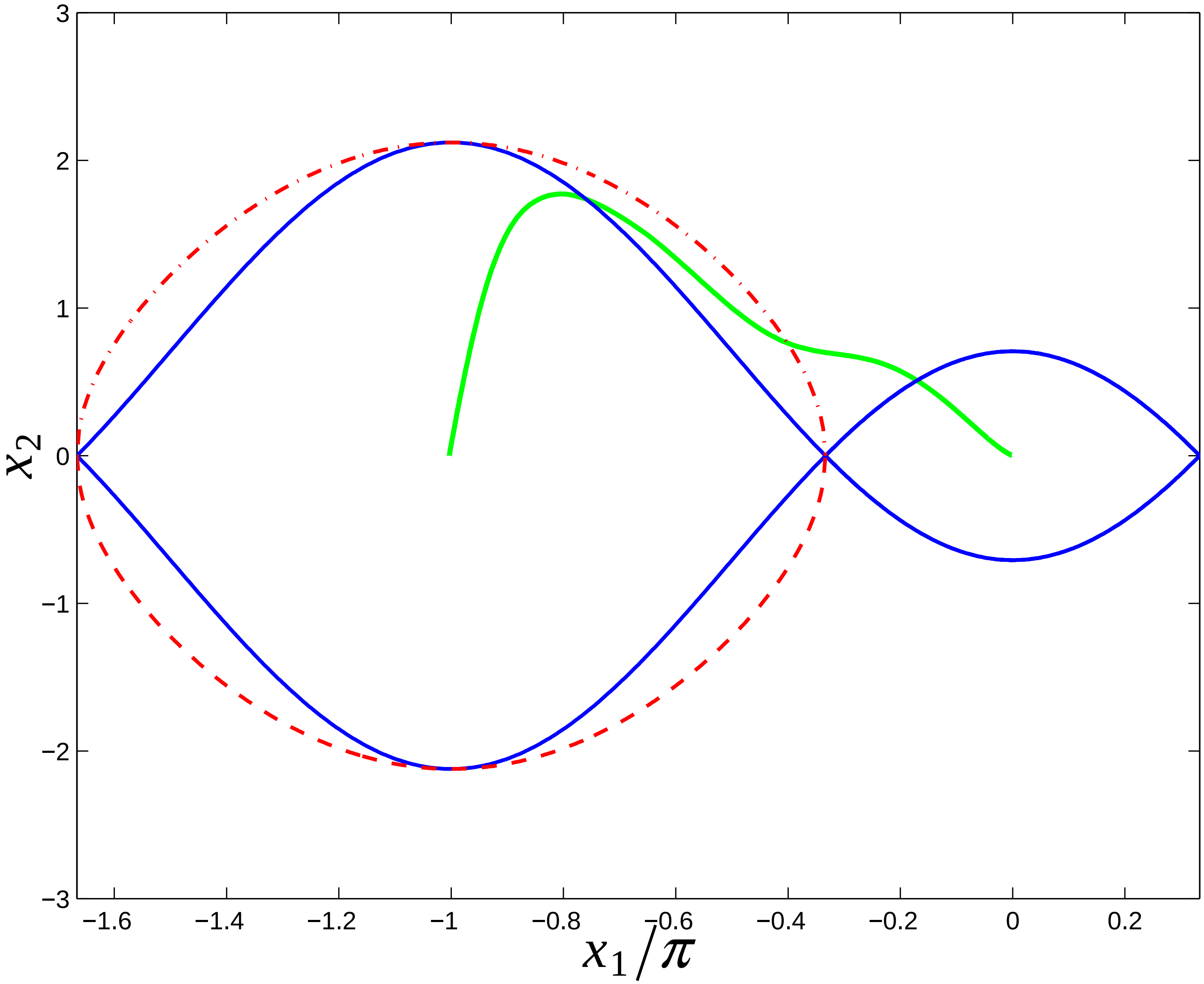
A Phase Plane



Simulated Swingup $a = 1$ and $b = 3$



A Phase Plane



Summary

- Two simple ideas

  Shape potential energy

  Shape the damping

- The control law

$$u(x_1, x_2) = 2a \sin x_1 + bx_2 F(x_1, x_2) \cos x_1$$
$$F(x_1, x_2) = \frac{2a+1}{4a}(2a \cos x_1 - 1) + \frac{x_2^2}{2}.$$

- Parameters a and b have good physical interpretations
- Many other versions ECC/CDC Sevilla
- Magnitude of control signal
- Pendulum and cart
- Furuta and spherical pendulums