On Robustness Analysis of Transportation Networks

Giacomo Como Ketan Savla Daron Acemoglu Munther A. Dahleh
Emilio Frazzoli

Overview: Social planning for efficient usage of transportation networks (TNs) is attracting renewed research interest as transportation demand is approaching its infrastructure capacity. While there exists an abundant literature on socially optimal traffic assignments[1], robustness analysis of TNs has received very little attention. In this paper, we study robustness properties of a large-scale TN with respect to agents' response to rare disruptions. We consider a setup where the route choice behavior of the agents depends on their global knowledge about the TN (e.g., obtained through past experiences), as well as on real-time traffic information obtained through local (myopic) observations. We focus on the limiting case of a continuum of agents operating initially at an equilibrium condition, and analyze the dynamic response of the TN to an unexpected disruption. In such a case, the global knowledge is assumed to remain fixed, and agents act by complementing it with real-time local information. In particular, the agents beliefs on the global TNs state correspond to some equilibrium flow (which might be thought of as the outcome of the slower time-scale learning process[2]), and the post-disturbance dynamics is primarily governed by the myopic route choice behavior of the agents.

We first formulate a novel dynamical system framework for the robustness analysis of TNs, which depends on the topology, physical model of traffic and the route choice behavior of agents. We consider disturbances that reduce the maximum flow capacity of roads, and define the margin of stability of the TN to be the maximum sum of capacity losses, under which the traffic densities on all edges remain bounded over time. We then prove that, irrespective of the route choice behavior of the agents, the margin of stability of an acyclic TN is upper-bounded by the minimum of all the *node cuts* of the buffer capacities of the TN. We then consider a natural class of *soft greedy* myopic route choice behaviors and prove that this behavior yields the maximum possible margin of stability. Finally, we study the dependence of the margin of stability on the equilibrium, and formulate a simple optimization problem for finding the most robust among feasible equilibria. This is, in general, different from the classical socially optimal equilibrium, as well as from the user-optimal equilibrium. Our results provide important guidelines for social planners in terms of determining robust equilibrium operating conditions and route choice behaviors for TNs.

Formulation: For the sake of this abstract, we consider a simple scenario. The topology of TN is abstracted by an acyclic di-graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with one origin/destination (o/d) pair

[1]E.g., M. Patriksson, *The Traffic Assignment Problem: Models and Methods*, V.S.P. Intl Science, 1994.

[2]Possibly based on ficticious play, see, e.g., D. Monderer, and L.S. Shapley, "Potential Games", *Games Econ. Behav.*, 14, 124–143, 1996, or J.R. Marden, H.P. Young, G. Arslan, and J.S. Shamma, "Payoff-based dynamics for multi-player weakly acyclic games," *SIAM J. Control Optimiz.*, 48, 373–396, 2009.

and such that every $e \in \mathcal{E}$ lies at least in a path from o to d. The state of the TN is described by a vector $\rho = \{\rho_e : e \in \mathcal{E}\}$, where ρ_e denotes the density of agents on e. The flow on e is $f_e = \mu_e(\rho_e) := \overline{f}_e \min\{1, \rho_e/\overline{\rho}_e\}$, where $\overline{f}_e > 0$ and $\overline{\rho}_e > 0$ denote, the maximum flow capacity, and, respectively, the saturation density, of e. Agents enter the network from o, at a constant rate $\lambda > 0$, and leave it from d. We shall assume that the fraction of agents leaving some node v towards $e \in \mathcal{E}_v^+$ is given by $\pi_e(\rho_{\mathcal{E}_v^+})$, where $\rho_{\mathcal{E}_v^+} := \{\rho_e : e \in \mathcal{E}_v^+\}$ is the vector of current densities on the outgoing edges from v, giving the local real-time information, and the functional dependence $\pi_e(\,\cdot\,)$encodes the agent's response to deviations from some equilibrium flow f^*, reflecting the static agents' beliefs on the global status of the TN. Hence, in particular, $\pi_e(\rho^*_{\mathcal{E}_v^+}) = f_e^*/\sum_{j\in\mathcal{E}_v^+} f_j^*$ for some equilibrium density ρ^*. As an example, we may consider, for some $\beta > 0$, the logit response $\pi_e(\rho_{\mathcal{E}_v^+}) = f_e^* e^{-\beta(\rho_e-\rho_e^*)}/\sum_{j\in\mathcal{E}_v^+} f_j^* e^{-\beta(\rho_j-\rho_j^*)}$.

In business as usual, i.e. in the absence of disruptions, the dynamics is then governed by the ODE system $\frac{\mathrm{d}}{\mathrm{d}t}\rho_e(t) = (\lambda \mathbb{1}_{\{o\}}(v) + \sum_{j\in\mathcal{E}_v^-} \mu_j(\rho_j))\pi_e(\rho) - \mu_e(\rho_e)$, for $v \neq d \in \mathcal{V}$, $e \in \mathcal{E}_v^+$. We then consider perturbations of the system above in which $\mu_e(\rho_e)$ is replaced by $\mu_e^{\delta_e}(\rho_e) := \min\{\mu_e(\rho_e), \overline{f}_e - \delta_e\}$ for some $0 \leq \delta_e \leq \overline{f}_e$. Let $\rho^\delta(t)$ be the solution of the perturbed system, with initial condition $\rho^\delta(0) = \rho^*$ coinciding with the equilibrium of the unperturbed system. Let $\Delta_U(\pi) := \{\delta : \limsup_{t\to\infty} ||\rho^\delta|| = +\infty\}$ be the set of destabilizing perturbations. We measure robustness of the TN in terms of its margin of stability $\gamma\,(\pi) := \inf_{\delta\in\Delta_U(\pi)} \|\delta\|_1$.

Results: Our primary objective is to study the dependence of $\gamma(\pi)$, and, specifically, to identify the behavior π maximizing $\gamma(\pi)$ for a given equilibrium f^*. To that effect:

(a) We first establish that, irrespective of π, $\gamma(\pi) \leq \overline{\gamma}(f^*) := \min_{v\in\mathcal{V}} \sum_{e\in\mathcal{E}_v^+} \left(\overline{f}_e - f_e^*\right)$. The proof of this result is based on the observation that the incoming flow at any node is unaffected by the changes on links *downstream* from it, and by applying mass conservation laws. It is instructive to note that $\overline{\gamma}(f^*)$ is typically strictly smaller than the upper bound on $\gamma(\pi)$ which would follow by a straightforward application of the max-flow min-cut theorem. In contrast to the latter, our result captures the loss in performance due to the limited real-time information available to the agents, and to their greedy behavior.

(b) For the logit-type route choice with any $\beta > 0$, it holds that $\gamma(\pi) = \overline{\gamma}(f^*)$. The proof of this result is based on properties of local stability and diffusivity of the associated dynamical system. In other words, the logit-type route choice function gives the maximum possible margin of stability for a given network equilibrium flow.

The preceding discussion studied the dependence of margin of stability on route choice functions for a given network equilibrium state. It is desirable from a social planner's perspective to find the most robust equilibrium state, i.e., to find a feasible f^* maximizing $\overline{\gamma}(f^*)$. In fact, $\overline{\gamma}(f^*)$ is concave in f^* and the set of feasible f^* forms a polytope. Hence, the maximization problem is tractable. Notice that this optimization problem is independent of congestion properties and hence, in general, the robust equilibrium is different than the classical socially optimal equilibrium or the user-optimal (e.g., Wardrop) equilibrium.

Conclusion: In this paper, we studied robustness properties of myopic route choice functions and equilibrium operating conditions for a TN. In future, we plan to extend our analysis to TNs with more general topologies, more realistic physical models of traffic, and different kinds of disturbances. We also plan to take into consideration the constraints imposed by the incentive mechanisms on the feasibility of route choices and equilibria.