Scaling limits for continuous opinion dynamics systems

Giacomo Como, and Fabio Fagnani

Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, 77 Massachusetts Ave, Cambridge, MA 02139, US
e-mail: giacomo@mit.edu

Dipartimento di Matematica, Politecnico di Torino, Corso Stati Uniti 24, 10129, Torino, Italy
e-mail: fabio.fagnani@polito.it

Abstract: Scaling limits are analyzed for continuous opinion dynamics systems, also known as gossip models. In such models, agents update their vector-valued opinions to a convex combination (possibly agent- and opinion-dependent) of their current value and that of another agent. It is shown that, in the limit of large agent population size, the empirical opinion density concentrates, at an exponential probability rate, around the solution of a measure-valued ordinary differential equation describing the system's mean-field dynamics. Properties of the associated initial value problem are studied. The asymptotic behavior of the solution is analyzed for bounded-confidence opinion dynamics, and in the presence of an heterogenous influential environment.

1. Introduction

In this paper we undertake a rigorous mathematical analysis of a family of stochastic dynamical systems proposed as opinion dynamics models in the recent literature: see [10] and references therein. Such models are also known as 'gossip' models because of the nature of the propagation of information, and they have appeared in other areas, for instance, as aggregation and estimation algorithms in sensor and robotic networks: see e.g. [8, 22].

The simplest gossip model can be described as follows. Each agent a of a finite population $\mathcal{A}$ possesses an initial belief/opinion modeled as a vector $X_0^a \in \mathbb{R}^d$. Agents are activated according to independent Poisson processes in continuous time. If agent a is activated at time t, her opinion jumps from its current value $X_{t^-}^a$ to a new value $X_t^a = X_{t^-}^a + \omega(X_{t^-}^b - X_{t^-}^a)$ where b is another agent sampled from $\mathcal{A}$, and $\omega \in [0,1]$ is a parameter modeling how much agent a trusts the opinion of agent b. In general, the conditional distribution of b may depend on the activated agent a (the support of such distribution representing the out-neighborhood of a in an underlying 'social network' structure), while the parameter ω may depend on the interacting agents, a and b, as well as on their current opinions, $X_{t^-}^a$ and $X_{t^-}^b$.

Fundamental theoretical issues concern the behavior of such models for large t and $n = |\mathcal{A}|$. Rather then on the single opinions' behavior, one is interested in the emerging collective behavior of the population. Typical questions include

imsart-generic ver. 2009/08/13 file: scalingopinion5.tex date: February 24, 2010

whether a consensus is eventually achieved or rather disagreement persists, and, more in general, whether an asymptotic distribution of opinions exists, how it looks like, and how long it takes the system to approach it.

The simplest case is when the Poisson processes are all of rate 1, the conditional distribution of the observed agent is the uniform one over $\mathcal{A}$ whichever agent is activated, and the parameter ω is fixed and the same for all agents, independently of their current opinions. In this case, the model is linear and can be studied in detail: it corresponds to the asymmetric gossip model in [14]. The basic fact is that (if $\omega \in]0,1[$), almost surely, all X_t^a converge, as $t \to +\infty$ (and for any fixed n), to a consensus random value ξ which has expected value $\mathbb{E}(\xi) = n^{-1}\sum_a X_0^a$. Convergence is exponentially fast [15]:

$$\mathbb{E}\left[n^{-1}\sum_a |X_t^a - \xi|^2\right] \leq 2n^{-1}\sum_a |X_0^a|^2 \exp(-Ct)$$

where $C = -n\ln(1 - 2n^{-1}\omega\bar{\omega} - 2n^{-2}\omega^2)$ with $\bar{\omega} = 1-\omega$. The variance of ξ can be estimated as

$$\mathrm{Var}[\xi] \leq \frac{\omega}{\omega + \bar{\omega}n} n^{-1}\sum_a |X_0^a|^2 \, .$$

Moreover, using the techniques in [14] we can easily prove a concentration result of type

$$\mathbb{P}\left(|X_t - E(X_t)| \geq \varepsilon\right) \leq \exp\left(-K\varepsilon^2 n/t\right) \, .$$

Essentially, this shows that, as $n \to +\infty$, and $t = o(n)$, each agent's opinion X_t^a concentrates around a deterministic dynamics converging to $\mathbb{E}(\xi)$ as $\exp(-2\omega\bar{\omega}t)$. It is the type of results which we would like to extend to the more general models.

A particularly interesting setting is the homogeneous-population, state-dependent model, i.e. when the parameter ω is independent of the identity of the interacting agents, but does depend on their current opinions. The case

$$\omega = \omega_0 \mathbb{1}_{[0,R]}(|X_{t^-}^a - X_{t^-}^b|) \, , \tag{1}$$

where $R > 0$, and $\omega_0 \in]0,1[$, is known as the Deffuant-Weisbuch model [18, 13, 20] of bounded confidence opinion dynamics: Agents with opinions too far apart do not trust each other and do not interact. Another case is the so called Gaussian decay model

$$\omega = \omega_0 \exp\left(-|X_{t^-}^a - X_{t^-}^b|^2/\sigma^2\right) \, . \tag{2}$$

These models are non-linear and, to the best of the authors' knowledge, the only theoretical result [20] is that, if $\omega \in \{0\} \cup [\omega_0, 1]$ for some $\omega_0 > 0$, each $X_{t^-}^a$ converges, as $t \to +\infty$ to a limit random value ξ^a. Numerical simulations show the asymptotic emergence of opinion clusters whose number and structure depends on the initial condition but seems to be stable for large n. However, there is no theoretical result regarding concentration and scaling limits for any state-dependent model.

Finally, for the case when the parameter ω depends on the agents, as well as on their opinions, no theoretical result is available in the literature. Such models have been considered in [16] where, though, only numerical simulations have been presented. Such heterogeneous population models are going to play a very important role in opinion dynamics because they are the natural model to represent more realistic populations with agents having different attitude to change opinion, and interacting only with agents on their social neighborhood.

In this paper, we study general state-dependent gossip models for large n. We shall consider both the case of a homogeneous population, and of a heterogeneous one consisting of two classes of agents: A class of 'standard' agents, which keep on updating their opinions as a result of interactions with the whole population, and a class of 'stubborn' agents whose opinions are never updated. The latter case can be modeled as a homogeneous population model with an exogenous input describing the influence of the stubborn agents' opinions on the standard agents' ones, and interpreted as a, typically heterogenous, 'influential environment'. We believe that many more general heterogeneous models can be studied with our approach. This is however left for future research.

In our analysis, we shall adopt an Eulerian viewpoint: Instead of studying the evolution of the single agents' opinion, we shall neglect the agents' identities, and study the dynamics of the corresponding empirical opinion densities. We shall argue that the deterministic mean-field dynamics obtained in the limit of large n is governed by an ordinary differential equation (ODE) on the space of probability measures over the opinion set, presented in Sect. 2.2. As proved in Sect. 3, the initial value problem associated to the mean-field dynamics always admits a unique global solution. Moreover, at any finite time, its solution is absolutely continuous with respect to Lebesgue's measure, provided that so does the initial condition, and that some mild technical assumptions on the interaction kernel are satisfied.

The asymptotic behavior in time of the mean-field dynamics is analyzed in Sect. 4 for the state-independent heterogeneous case, and for the generally state-dependent homogeneous case. In both cases, we prove weak convergence to an equilibrium distribution, which typically does not consist of a single Dirac's delta. For the state-independent heterogeneous model, we show that the equilibrium opinion distribution is independent of the initial condition, and is uniquely characterized by its moments, which can be computed by recursively solving a lower-triangular infinite linear system. On the other hand, we prove that the equilibrium opinion distribution in the bounded-confidence model is a convex combination of Dirac's deltas. Such deltas represent opinion clusters, and their number and position depend on the initial condition. These results provide fundamental insight into two basic mechanisms which have been proposed by social scientists in order to explain persistent disagreement in the society [3], namely heterogeneity of the social environment, and homophily leading to global fragmentation.

Finally, in Sect. 5, we prove that the finite-population stochastic system concentrates around the deterministic mean-field dynamics, as the population size grows, at an exponential probability rate. We apply here a martingale argument (see e.g. [25] for the finite-dimensional case) and obtain a result in the Kantorovich-Wasserstein metric [2, 24]. The technical assumption in our results is that the, possibly stochastic, dependence of the weight ω on the opinions is Lipschitz-continuous. Hence, the case (1) is not covered by our theory. This is not a relevant drawback since one can consider suitable Lipschitz approximations of (1); on the other hand, we believe that this is just a technical question and that the result should remain valid for a larger class of functions.

We conclude this section with a brief overview of some related work. A special instance of the measure-valued ODE analyzed in the present paper has already been proposed in [4] for probability densities (in this case it becomes an integro-differential equation), but with no proof of either well-posedness or concentration of the stochastic finite system. In [5, 9, 6], deterministic, bounded-confidence, opinion dynamics models with possibly a continuum of agents have been studied both in discrete and continuous time. In particular, the continuous time dynamics studied in [9, 6] is governed by a partial differential equation in the space of probability measures, whose solution behaves similarly to the scaling limit of the stochastic model considered here. Finally, it is worth mentioning the work [17], where mean-field limits have been analyzed for the Cucker-Smale flocking dynamics [11, 12].

2. Problem setting and main results

In this section, we formally state the model and present our main results.

Before proceeding, let us establish some notation to be followed throughout the paper. For $x, y \in \mathbb{R}^d$, for some $d \in \mathbb{N}$, $|x-y|$ and $x \cdot y$ will denote their Euclidean distance, and scalar product, respectively. The indicator function of a set A will be denoted by $\mathbb{1}_A$. Given an open subset $\mathcal{X} \subseteq \mathbb{R}^d$, we denote by $\mathcal{B}(\mathcal{X})$ its Borel σ-algebra, and by $\mathcal{M}(\mathcal{X})$ the space of finite signed Borel measures on $\mathcal{X}$, while $\mathcal{M}^+(\mathcal{X}) \subseteq \mathcal{M}(\mathcal{X})$ denotes the closed convex cone of Borel non-negative measures and $\mathcal{P}(\mathcal{X}) \subseteq \mathcal{M}^+(\mathcal{X})$ the convex set of probability measures. The space of real-valued continuous bounded (resp. compact-supported, vanishing at infinity) functions on $\mathcal{X}$, equipped with the supremum norm $||\varphi||_\infty := \sup\{|\varphi(x)| : x \in \mathcal{X}\}$, will be denoted by $\mathcal{C}_b(\mathcal{X})$ (resp. $\mathcal{C}_c(\mathcal{X})$, $\mathcal{C}_0(\mathcal{X})$). The Dirac delta measure centered in $x \in \mathcal{X}$ will be denoted by δ_x. For $\mu \in \mathcal{M}(\mathcal{X})$, and $\varphi \in \mathcal{C}_b(\mathcal{X})$, we shall write $\langle \mu, \varphi \rangle$ for the integral $\int \varphi(x) \mathrm{d}\mu(x)$, with the convention that, whenever not explicitly indicated, the domain of integration is assumed to be the entire space $\mathcal{X}$. The total variation of $\mu \in \mathcal{M}(\mathcal{X})$ will be denoted by $||\mu||$. The symbol λ will denote Lebesgue's measure on $\mathcal{X}$, $\mu << \lambda$ will stand for absolute continuity, and $\mathrm{d}\mu/\mathrm{d}\lambda$ for the Radon-Nikodym derivative, of μ with respect to λ. Finally, we shall denote by $\mathcal{P}_1(\mathcal{X}) := \{\mu \in \mathcal{P}(X) : \int |x| \mathrm{d}\mu(x) < +\infty\}$ the metric space of probability measures with finite first moment, equipped with the order-1 Kantorovich-Wasserstein distance. The latter is defined by [2, 24] $W_1(\mu, \nu) := \inf\left\{\iint |x-y| \mathrm{d}\xi(x,y)\right\}$, where the infimization runs over all couplings of μ and ν, i.e. joint probability measures $\xi \in \mathcal{P}(\mathcal{X} \times \mathcal{X})$ having marginals given by μ, and ν, respectively.

2.1. Stochastic models of continuous opinion dynamics

The present paper is concerned with continuous opinion dynamics systems. Agents belong to a finite population $\mathcal{A}$ of cardinality $|\mathcal{A}| = n$. At time $t \in \mathbb{R}^+$ each agent $a \in \mathcal{A}$ maintains an opinion $X_t^a \in \mathcal{X}$, where $\mathcal{X} \subseteq \mathbb{R}^d$ is an open set. The vector of the opinions will be denoted by $X_t := \{X_t^a\} \subseteq \mathcal{X}^{\mathcal{A}}$.

We shall assume the initial opinions X_0 to be a collection of independent and identically distributed random variables, the law of each X_0^a given by some $\mu_0 \in \mathcal{P}(\mathbb{R}^d)$. The trajectories of opinion profile vector $\{X_t : t \in \mathbb{R}^+\}$ are right-continuous and evolve according to the following jump Markov process: Agents have clocks which tick at the times of independent rate-1 Poisson processes. If her clock ticks at time t, agent a updates her opinion X_{t-}^a to a new value X_t^a which depends on the observation of the current opinion of some other agent and of her own one. In particular, she observes the opinion of some b sampled uniformly from $\mathcal{A}$, and then updates her opinion to a random value X_t^a, which has conditional probability law $\kappa(\,\cdot\,|X_{t-}^a, X_{t-}^b)$. Here $\kappa(\,\cdot\,|\,\cdot\,,\,\cdot\,)$ is a stochastic kernel, i.e., for all $x, y \in \mathcal{X}$, $\kappa(\,\cdot\,|x, y)$ is a probability measure on $\mathcal{X}$, and $(x, y) \mapsto \kappa(B|x, y)$ is a measurable map from $\mathcal{X} \times \mathcal{X}$ to $[0, 1]$, for all measurable sets $B \subseteq \mathcal{X}$. We shall refer to κ as the interaction kernel of the model. We shall assume that the above stochastic process is defined on some filtrated probability space $(\Omega, \{\mathcal{F}_t\}_{t\in\mathbb{R}^+}, \mathbb{P})$, and denote by $0 = T_0 \le T_1 \le \ldots$, the times at which some opinion update occurs. Observe that $\{T_{k+1} - T_k : k \in \mathbb{Z}^+\}$ is a family of independent rate-n Poisson random variables.

Interaction kernels of interest in opinion dynamics are typically 'locally aggregating'. Specifically, in most of the models considered in the literature, $\mathcal{X} \subseteq \mathbb{R}^d$ is an convex open set containing the support of the initial condition, and the interaction kernel has the following form:

$$\langle \kappa(\,\cdot\,|x, y), \varphi\rangle = \alpha \int \varphi\left(\overline{\omega}x + \omega y\right) \mathrm{d}\theta^i(\omega|x, y) + \overline{\alpha} \int\!\!\int \varphi(\overline{\upsilon}x + \upsilon z)\mathrm{d}\theta^e(\upsilon|x, z)\mathrm{d}\psi(z) \tag{3}$$

where $\alpha = 1 - \overline{\alpha} \in [0, 1]$, $\theta^i(\,\cdot\,|\,\cdot\,,\,\cdot\,)$ and $\theta^e(\,\cdot\,|\,\cdot\,,\,\cdot\,)$ are stochastic kernels from $\mathcal{X} \times \mathcal{X}$ to $[0, 1]$, $\overline{\omega} = 1 - \omega$, $\overline{\upsilon} = 1 - \upsilon$, and $\psi \in \mathcal{P}(\mathcal{X})$. This models a situation in which, with probability α, the activated agent updates her opinion towards a convex combination of her current opinion x and the opinion y of an observed agent. The weight ω in such a convex combination measures the confidence that the activated agent has on the observed opinion of another agent, and is assumed to depend, through the stochastic kernel $\theta^i(\,\cdot\,|\,\cdot\,,\,\cdot\,)$, on both the activated and the observed agent's opinions, x and y. On the other hand, with probability $\overline{\alpha}$, the activated agent observes an external signal z, sampled from a probability distribution ψ, playing the role of an exogenous source of influence, or influential environment, and she updates her opinion toward a convex combination of her current opinion x and the observed signal z. The dependence of the weight υ of such convex combination is captured by the stochastic kernel $\theta^e(\,\cdot\,|\,\cdot\,,\,\cdot\,)$.

While for most of the results of our paper we shall not need the interaction kernel κ to have the specific form (3), we shall focus on kernels of this form in

Sect. 4 when proving properties of the solution of the corresponding measure-valued ODE. Observe that, if μ_0 and ψ are compact-supported, then the system dynamics becomes naturally restricted to the convex hull of the supports of μ_0 and ψ.

Remark 1. The models considered in the cited literature usually assume the interaction to be symmetric: when agent a wakes up and connects to agent b, both agents update their opinions. This symmetric model may be more suitable in certain applicative contexts, the asymmetric one in some others. However, while for finite population sizes some of the properties of the two models differ (for example, in the symmetric model the average of the opinions is preserved, while this is not necessarily the case for the asymmetric model [14]), all the results and proofs of this paper hold, with minor changes, for the symmetric model too.

2.2. The Eulerian approach and main results

As the main interest is in the global behavior of the opinion dynamics system, rather than on that of the single agents' opinions, it proves convenient to undertake an Eulerian approach, studying the evolution of the empirical densities of the agents' opinions. Formally, this is accomplished by considering the random flow of probability measures

$$\mu_t^n := \frac{1}{n}\sum_{a\in\mathcal{A}}\delta_{X_t^a} \in \mathcal{P}(\mathcal{X}), \qquad t \in \mathbb{R}^+ .$$

This is a $\mathcal{P}(\mathcal{X})$-valued process whose trajectories are piecewise-constant and right-continuous. In particular, one has

$$\mu_t^n = M_k , \qquad \forall t \in [T_k, T_{k+1}[, \quad k \in \mathbb{Z}^+ ,$$

where $\{M_k : k \in \mathbb{Z}^+\}$ is a $\mathcal{P}(\mathcal{X})$-valued Markov chain.

Furthermore, consider the operator $F : \mathcal{M}^+(\mathcal{X}) \to \mathcal{M}^+(\mathcal{X})$, defined by

$$\langle F(\mu), \varphi\rangle := \iiint \varphi(z)\, \mathrm{d}\kappa(z|x, y)\mathrm{d}\mu(x)\mathrm{d}\mu(y) , \tag{4}$$

for all $\varphi \in \mathcal{C}_0(\mathcal{X})$. It is immediate to verify that

$$\mathbb{E}\left[\langle M_k, \varphi\rangle | \, M_k\right] = \left(1 - n^{-1}\right)\langle M_k, \varphi\rangle + n^{-1}\langle F(M_k), \varphi\rangle ,$$

for all $\varphi \in \mathcal{C}_0(\mathcal{X})$, and $k \in \mathbb{Z}_+$. One may rewrite this in the form

$$\langle M_{k+1}, \varphi\rangle - \langle M_k, \varphi\rangle = n^{-1}\langle F(M_k) - M_k, \varphi\rangle + n^{-1}\langle \Lambda_{k+1}, \varphi\rangle , \tag{5}$$

where the random signed measure Λ_{k+1} satisfies

$$\mathbb{E}\left[\Lambda_{k+1}|\mathcal{F}_{T_k}\right] = 0 , \qquad ||\Lambda_{k+1}|| \le n||M_{k+1} - M_k|| + ||F(M_k) - M_k|| \le 4 . \tag{6}$$

Equation (6) implies that $\{\langle \Lambda_k, \varphi \rangle : k \in \mathbb{N}\}$ is a sequence of bounded martingale differences, which can be thought as 'noise'. This suggests to think of the stochastic process $\{M_k : k \in \mathbb{Z}^+\}$ as of a noisy discretization, or Euler approximation in the numerical analysis language, of the probability-measure-valued ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu_t = F(\mu_t) - \mu_t\,. \tag{7}$$

with stepsize $1/n$. We shall refer to a solution of (7) as the mean-field dynamics of the system.

More precisely, we shall define a solution of (7) to be a family $\{\mu_t : t \in [0, +\infty)\} \subseteq \mathcal{P}(\mathcal{X})$ such that, for every function $\varphi \in \mathcal{C}_0(\mathcal{X})$, the real-valued map $t \mapsto \langle \mu_t, \varphi \rangle$ is differentiable on $\mathbb{R}^+$, and satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle \mu_t, \varphi \rangle = \langle F(\mu_t), \varphi \rangle - \langle \mu_t, \varphi \rangle\,, \tag{8}$$

for every $t > 0$. The main result of this paper, stated below, guarantees that (7) admits a unique solution $\{\mu_t\}$, and that the stochastic process $\{\mu_t^n\}$ concentrates around $\{\mu_t\}$ exponentially fast in n.

Theorem 1. *Let $\mu \in \mathcal{P}(\mathcal{X})$ be arbitrary. Then:*

(a) *There exists a unique solution $\{\mu_t : t \in \mathbb{R}^+\}$ of (7) with initial condition $\mu_0 = \mu$;*

(b) *If $\mathcal{X} \subseteq \mathbb{R}^d$ is bounded, and the stochastic kernel κ is globally Lispchitz continuous as a map from $\mathcal{X} \times \mathcal{X}$ to $\mathcal{P}_1(\mathcal{X})$, then, for every $\tau \in (0, +\infty)$, for sufficiently small $\varepsilon > 0$ and sufficiently large $n \in \mathbb{N}$, it holds*

$$\mathbb{P}\left(\sup\{W_1(\mu_t^n, \mu_t) : t \in [0, \tau]\} \geq \varepsilon\right) \leq \exp(-K\varepsilon^3 n)\,,$$

where K is a positive constant depending on $\mathcal{X}$, κ, and τ only.

Points (a) of Theorem 1 will be proved in Sect. 3.1, while point (b) will be proved in Sect. 5.

3. Well-posedness of the measure-valued ODE

In this section, we shall first prove point (a) of Theorem 1, i.e. that the initial value problem associated to the ODE (7) admits a unique solution. Then, under further technical assumptions, we shall show that, if the initial measure μ_0 admits a density, so does the solution μ_t at any finite time t.

3.1. Weak solutions

To start with, we extend the ODE to the space of signed measures $\mathcal{M}^+(\mathcal{X})$. In order to do this we need to extend the operator F and introduce another operator G in the following way. For $\mu \in M(\mathcal{X})$ put

$$F(\mu) := F(\mu^+)\,, \qquad G(\mu) := \mu^+(\mathcal{X})\mu\,, \tag{9}$$

where $\mu = \mu^+ - \mu^-$ denotes the Hahn-Jordan decomposition of $\mu \in \mathcal{M}(\mathcal{X})$. It is not hard to check that both F and G are locally Lipschitz continuous with respect to the total variation norm, i.e., for every bounded set $\Theta \subseteq \mathcal{M}(\mathcal{X})$, there exist nonnegative constants K_F, K_G such that

$$||F(\mu_1) - F(\mu_2)|| \leq K_F ||\mu_1 - \mu_2|| \,, \qquad ||G(\mu_1) - G(\mu_2)|| \leq K_G ||\mu_1 - \mu_2|| \,, \quad (10)$$

for all $\mu_1, \mu_2 \in \Theta$. Moreover,

$$F(\mu)(\mathcal{X}) = G(\mu)(\mathcal{X}) = \mu(\mathcal{X})^2 \,, \qquad \forall \mu \in \mathcal{M}^+(\mathcal{X}) \,. \quad (11)$$

In the following we want to study the well-posedness of initial value problems associated to the measure-valued ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu_t = F(\mu_t) - G(\mu_t) \,, \quad (12)$$

where (12) means that, for every $\varphi \in \mathcal{C}_0(\mathcal{X})$, the real-valued map $t \mapsto \langle \mu_t, \varphi \rangle$ is differentiable on $\mathbb{R}^+$, and satisfies $\frac{\mathrm{d}}{\mathrm{d}t}\langle \mu_t, \varphi \rangle = \langle F(\mu_t), \varphi \rangle - \langle G(\mu_t), \varphi \rangle$, for every $t > 0$.

Proposition 1. *Suppose that $F, G : \mathcal{M}(\mathcal{X}) \to \mathcal{M}^+(\mathcal{X})$ satisfy properties (10), and (11). Then, for every $\mu \in \mathcal{M}^+(\mathcal{X})$, there exists a unique solution $\{\mu_t\}_{t \in \mathbb{R}^+} \subseteq \mathcal{M}^+(\mathcal{X})$ to (12) such that $\mu_0 = \mu$. Moreover, $\mu_t(\mathcal{X}) = \mu(\mathcal{X})$ for every $t \geq 0$.*

Proof For $\tau \in (0, +\infty)$, let $\mathcal{C}([0,\tau], \mathcal{M}(\mathcal{X}))$ be the space of continuous curves in $\mathcal{M}(\mathcal{X})$ equipped with the sup norm $||\{\mu_t\}||_\tau := \sup\{||\mu_t|| : \, t \in [0,\tau]\}$. Given a curve $\{\mu_s\} \in \mathcal{C}([0,\tau], \mathcal{M}(\mathcal{X}))$, and a bounded measurable function $\varphi \in \mathcal{C}_0(\mathcal{X})$, define

$$\langle \Phi(\{\mu_s\})_t, \varphi \rangle := \langle \mu, \varphi \rangle + \int_0^t \langle F(\mu_s), \varphi \rangle \mathrm{d}s - \int_0^t \langle G(\mu_s), \varphi \rangle \mathrm{d}s \,, \quad \forall t \in [0,\tau] \,. \quad (13)$$

Observe that (12) with the initial condition $\mu_0 = \mu$ is equivalent to

$$\langle \mu_t, \varphi \rangle = \langle \Phi\left(\{\mu_s\}\right)_t, \varphi \rangle \,, \qquad \forall \varphi \in \mathcal{C}_0(\mathcal{X}), \; t \geq 0 \,. \quad (14)$$

Notice that, for every $t \in [0,\tau]$, $\Phi(\{\mu_s\})_t$ can be seen as the difference of two bounded linear positive functionals on $\mathcal{C}_0(\mathcal{X})$, so that $\Phi(\{\mu_s\})_t \in \mathcal{M}(\mathcal{X})$. Moreover, the map $t \mapsto \Phi(\{\mu_s\})_t$ is continuous over $[0,\tau]$, since

$$\begin{aligned} ||\Phi(\{\mu_s\})_{t+\varepsilon} - \Phi(\{\mu_s\})_t|| &= \int_t^{t+\varepsilon} ||G(\mu_s)|| \mathrm{d}s + \int_t^{t+\varepsilon} ||F(\mu_s)|| \mathrm{d}s \\ &\leq \varepsilon \left[||\{G(\mu_s)\}||_\tau + ||\{F(\mu_s)\}||_\tau \right] . \end{aligned} \quad (15)$$

Therefore, the operator Φ takes values in $\mathcal{C}([0,\tau], \mathcal{M}(\mathcal{X}))$. Now, let us consider $\Theta := \{\nu \in \mathcal{M}(\mathcal{X}) : \, ||\nu|| \leq 2||\mu||\}$, let K_F, K_G be the Lipschitz constants relative to Θ of F, and G, respectively. For every $\nu \in \Theta$, (11) and (10) imply that

$$||F(\nu)|| \leq ||F(\nu) - F(\mu)|| + ||F(\mu)|| \leq 4K_F||\mu|| \quad (16)$$

Similarly,

$$||G(\nu)|| \leq 4K_G||\mu|| \tag{17}$$

Define now the set $\mathcal{S} := \{\{\mu_t\} \in \mathcal{C}([0,\tau], \mathcal{M}(\mathcal{X})) : \mu_0 = \mu,\, \mu_t \in \Theta,\, \forall t \in [0,\tau]\}$. For all $\{\mu_t\} \in S$, using (16) and (17), and arguing like in (15), we obtain

$$||\Phi(\{\mu_t\})||_\tau \leq (1 + 4\tau K)||\mu|| \tag{18}$$

where $K := K_F + K_G$. Moreover, if both $\{\mu_t\}$ and $\{\nu_t\}$ belong to $\mathcal{S}$, then,

$$\begin{array}{rcl} ||\Phi(\{\mu_t\}) - \Phi(\{\nu_t\})||_\tau & = & \displaystyle\sup_{0\leq t\leq\tau} \int_0^t (||F(\mu_s) - F(\nu_s)|| + ||G(\nu_s) - G(\mu_s)||)\, \mathrm{d}s \\ & \leq & \tau K||\{\mu_t\} - \{\nu_t\}||_\tau\,. \end{array} \tag{19}$$

We now assume to have chosen $\tau \in]0, \frac{1}{4K}]$. Then, by (18), $\Phi(\mathcal{S}) \subseteq \mathcal{S}$ and, by (19), Φ is a contraction of $\mathcal{S}$. Hence, by Banach's fixed point theorem there exists a unique fixed point of Φ in $\mathcal{S}$. As observed, such a fixed point corresponds to a solution $\{\mu_t\}$ of the ODE (12) for $t \in [0,\tau]$, with the initial condition $\mu_0 = \mu$. We now show that indeed $\mu_t \in \mathcal{M}^+(\mathcal{X})$ for $t \in [0,\tau]$. By contradiction, assume that there exists $B \in \mathcal{B}(\mathcal{X})$ such that $\mu_t(B) < 0$ for some $t \in [0,\tau]$, and let $t^* := \sup\{s \in [0,t] : \mu_s(B) \geq 0\}$. By continuity, $\mu_{t^*}(B) = 0$ while $\mu_s(B) < 0$ for all $s \in]t^*, t]$. This implies that

$$F(\mu_s)(B) - G(\mu_s)(B) \geq -\mu_s^+(\mathcal{X})\mu_s(B) \geq 0\,, \qquad \forall s \in]t^*, t]\,.$$

But then

$$\mu_t(B) = \int_{t^*}^t (F(\mu_s)(B) - G(\mu_s)(B))\, \mathrm{d}s \geq 0$$

which is a contradiction. Hence, $\mu_t \in \mathcal{M}^+(\mathcal{X})$ for $t \in [0,\tau]$. Notice moreover that, because of property (11), $\mu_t(\mathcal{X}) = \mu(\mathcal{X})$ for all $t \in [0,\tau]$. Finally, a simple induction argument allows one to extend the existence and uniqueness of the solution to the whole interval $[0,+\infty)$. ■

Notice that, when considering an initial condition $\mu_0 \in \mathcal{P}(\mathcal{X})$, the solution of (12) satisfies $\mu_t \in \mathcal{P}(\mathcal{X})$ for all t, thus proving point (a) of Theorem 1.

3.2. Probability density solutions

We shall now investigate on the existence of density solutions when the initial condition μ_0 is absolutely continuous with respect to Lebesgue's measure.

Given the interaction kernel $\kappa(\,\cdot\,|\,\cdot\,,\,\cdot\,)$ and a non-negative measure $\mu \in \mathcal{M}^+(\mathcal{X})$, we put

$$\kappa_1(\mu)(B|y) := \int \kappa(B|x,y)\mathrm{d}\mu(x)\,, \qquad \kappa_2(\mu)(B|x) := \int \kappa(B|x,y)\mathrm{d}\mu(y)\,, \tag{20}$$

for all $B \in \mathcal{B}(\mathcal{X})$, $x, y \in \mathcal{X}$. The following result characterizes regularity properties of the solution of the initial value problem associated to the ODE (12).

Proposition 2. *Assume that $\mu_0 << \lambda$, and that*

$$\mu << \lambda \qquad \Longrightarrow \qquad \kappa_1(\mu)(\,\cdot\,|y),\, \kappa_2(\mu)(\,\cdot\,|x) << \lambda, \quad \forall x \in \mathcal{X}\,,\ \forall y \in \mathcal{X}\,. \quad (21)$$

Then, $\mu_t << \lambda$, for all $t \in [0,+\infty)$. Moreover, if there exists $C \in (0,+\infty)$ such that, for all $\mu << \lambda$,

$$\Big|\Big|\frac{\mathrm{d}\kappa_2(\mu)(\,\cdot\,|x)}{\mathrm{d}\lambda}\Big|\Big|_\infty \leq C\Big|\Big|\frac{\mathrm{d}\mu}{\mathrm{d}\lambda}\Big|\Big|_\infty\,, \qquad \forall x \in \mathcal{X}\,, \quad (22)$$

then, the density $f_t = \mathrm{d}\mu_t/\mathrm{d}\lambda$ satisfies the estimation:

$$||f_t||_\infty \leq ||f_0||_\infty e^{Ct}\,, \qquad \forall t \in [0,+\infty)\,. \quad (23)$$

Proof For every $t \in [0,+\infty)$, consider Lebesgue's decomposition $\mu_t = \mu_t^a + \mu_t^s$, where $\mu_t^a << \lambda$, and μ_t^s and λ are singular. It follows from (21) that, $\kappa_1(\mu_t^a)(\,\cdot\,|x) << \lambda$ for all $x \in \mathcal{X}$. Then, for any $B \in \mathcal{B}(\mathcal{X})$ such that $\lambda(B) = 0$, one has

$$\int\!\!\int \kappa(B|x,y)\mathrm{d}\mu_t^a(x)\mathrm{d}\mu_t(y) = \int \mathrm{d}\kappa_2(\mu_t^a)(B|x)\mathrm{d}\mu_t(x) = 0\,.$$

Similarly, one can show that $\int\!\!\int \kappa(B|x,y)\mathrm{d}\mu_t^s(x)\mathrm{d}\mu_t^a(y) = 0$. Hence,

$$\begin{aligned} F(\mu_t)(B) &= \int\!\!\int \kappa(B|x,y)\mathrm{d}\mu_t(x)\mathrm{d}\mu_t(y) \\ &= \int\!\!\int \kappa(B|x,y)\mathrm{d}\mu_t^a(x)\mathrm{d}\mu_t(y) + \int\!\!\int \kappa(B|x,y)\mathrm{d}\mu_t^s(x)\mathrm{d}\mu_t^a(y) \\ &\quad + \int\!\!\int \kappa(B|x,y)\mathrm{d}\mu_t^s(x)\mathrm{d}\mu_t^s(y) \\ &= \int\!\!\int \kappa(B|x,y)\mathrm{d}\mu_t^s(x)\mathrm{d}\mu_t^s(y) \\ &= F(\mu_t^s)(B)\,, \end{aligned}$$

for all $B \in \mathcal{B}(\mathcal{X})$ such that $\lambda(B) = 0$. This readily implies that μ_t^s satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu_t^s = F(\mu_t^s) - \mu_t^s\,.$$

Since $\mu_0^s = 0$ by assumption, it follows that $\mu_t^s = 0$ for all $t \geq 0$.

Assume now that (22) holds true. For any $\varphi \in \mathcal{C}_c(\mathcal{X})$, Holder's inequality, and (22) imply that

$$\begin{aligned} \langle F(\mu_t), \varphi\rangle &= \int\!\!\int\!\!\int \varphi(z)\mathrm{d}\kappa(z|x,y)\mathrm{d}\mu_t(x)\mathrm{d}\mu_t(y) \\ &= \int\!\!\int \varphi(z)\frac{\mathrm{d}\kappa_2(\mu_t)(z|x)}{\mathrm{d}\lambda}\mathrm{d}\lambda(z)\,\mathrm{d}\mu_t(x) \\ &\leq \int \Big|\Big|\frac{\mathrm{d}\kappa_2(\mu_t)(z|x)}{\mathrm{d}\lambda}\Big|\Big|_\infty ||\varphi||_1 \mathrm{d}\mu_t(x) \\ &\leq C||f_t||_\infty||\varphi||_1\,. \end{aligned}$$

It follows that, for all non-negative-valued $\varphi \in \mathcal{C}_c(\mathcal{X})$,

$$\begin{aligned}\int \varphi(x) f_t(x) \mathrm{d}\lambda(x) &= \int \varphi(x) f_0(x) \mathrm{d}\lambda(x) + \int_0^t \left(\langle F(\mu_s), \varphi\rangle - \langle \mu_s, \varphi\rangle\right) \mathrm{d}s \\ &\leq ||f_0||_\infty ||\varphi||_1 + \int_0^t \langle F(\mu_s), \varphi\rangle \mathrm{d}s \\ &\leq ||\varphi||_1 \left(||f_0||_\infty + C\int_0^t ||f_s||_\infty \mathrm{d}s\right).\end{aligned}$$

Then, by the isometry of $L^\infty(\mathcal{X})$ with the dual of $L^1(\mathcal{X})$, the fact that f_t is non-negative valued, and the density of $\mathcal{C}_c(\mathcal{X})$ in $L^1(\mathcal{X})$, one gets that

$$\begin{aligned}||f_t||_\infty &= \sup\left\{\int \varphi(x) f_t(x) \mathrm{d}x : \varphi \in L^1(\mathcal{X}),\ ||\varphi||_1 \leq 1\right\} \\ &= \sup\left\{\int \varphi(x) f_t(x) \mathrm{d}x : \varphi \in \mathcal{C}_c(\mathcal{X}),\ \varphi \geq 0,\ ||\varphi||_1 \leq 1\right\} \\ &\leq ||f_0(x)||_\infty + C\int_0^t ||f_s||_\infty \mathrm{d}s\,.\end{aligned}$$

By Gronwall's lemma, this readily implies (23). ■

The technical condition on the stochastic kernel κ is actually verified in many important cases. Suppose that κ is the form (3) with $\theta^i(\,\cdot\,|x, y) = \delta_{\omega(|x-y|)}$ for some non-increasing function $\omega : \mathbb{R}^+ \to [0, \omega_0]$, $\omega_0 \in [0, 1[$, which is piecewise Lipschitz-continuous. In this case, which includes the bounded confidence dynamics (1) as well as the Gaussian interaction model (2) as special instances, for all $y \in \mathcal{X}$, there is a finite partition $\mathcal{X} = \bigcup_i \mathcal{X}_i$ such that the function $x \mapsto \overline{\omega}(|x-y|)x + \omega(|x-y|)y$ is locally invertible, with absolutely continuous inverse $g_i(\cdot, y)$. Similarly, assume that Then, if μ is absolutely continuous with density f, one has for all nonnegative-valued $\varphi \in L^1(\mathcal{X})$, $\theta^e(\,\cdot\,|x, y) = \delta_{\upsilon(|x-y|)}$ for some non-increasing function $\upsilon : \mathbb{R}^+ \to [0, \upsilon_0]$, $\upsilon_0 \in [0, 1[$, which is piecewise Lipschitz-continuous, and for $y \in \mathcal{X}$,let $\mathcal{X} = \bigcup_j \mathcal{Y}_j$ a finite partition such that the function $x \mapsto \overline{\upsilon}(|x-y|)x + \upsilon(|x-y|)y$ is locally invertible, with absolutely continuous inverse $h_i(\cdot, y)$.

$$\begin{aligned}\langle \nu_2(\,\cdot\,|y), \varphi\rangle &= \alpha \sum_i \int \int_{\mathcal{X}_i} \varphi\left(\overline{\omega}(|x-y|)x + \omega(|x-y|)y\right) f(x) \mathrm{d}\lambda(x) \mathrm{d}\mu(y) \\ &\quad + \overline{\alpha} \sum_j \int \int_{\mathcal{Y}_j} \varphi(\overline{\upsilon}(|x-y|)x + \upsilon(|x-y|)w) f(x) \mathrm{d}\lambda(x) \mathrm{d}\psi(w) \\ &= \alpha \sum_i \int \int_{\mathcal{X}_i} \varphi(z)\, f(g_i(z, y)) |D_z g_i(z, y)| \mathrm{d}\lambda(z) \mathrm{d}\mu(y) \\ &\quad + \overline{\alpha} \sum_j \int \int_{\mathcal{Y}_j} \varphi(z)\, f(h_j(z, y)) |D_z h_j(z, y)| \mathrm{d}\lambda(z) \mathrm{d}\psi(w) \\ &\leq C ||f||_\infty ||\varphi||_1\,,\end{aligned}$$

where $C := \alpha \sum_i ||D_z g_i(\,\cdot\,, y)||_\infty + \overline{\alpha} \sum_j ||D_z h_j(\,\cdot\,, y)||_\infty$. Similarly, $\langle \nu_1(\,\cdot\,|y), \varphi\rangle \leq C||f||_\infty ||\varphi||_1$. As a consequence, $\nu_i(\,\cdot\,|y) << \lambda$, for all $y \in \mathcal{X}$, and $i = 1, 2$, and

(22) holds. Then, Proposition 2 applies, e.g., to the interaction kernels described in (1) and (2).

4. Behavior of the mean-field dynamics

This section is devoted to a deeper analysis of the ODE (12) for the gossip model with heterogenous influential environment, and the bounded-confidence opinion dynamics. In particular, we shall investigate the asymptotic behavior of μ_t in the limit as t tends to infinity.

4.1. Gossip model with heterogenous influential environment

We start by analyzing the case when the stochastic kernel $\kappa(\,\cdot\,|\,\cdot\,,\,\cdot\,)$ has the form (3), with constant weights: $\theta^i(\,\cdot\,|x,y) = \delta_\omega(\,\cdot\,)$, $\theta^e(\,\cdot\,|x,y) = \delta_\upsilon(\,\cdot\,)$, for some fixed $\omega, \upsilon \in [0,1]$. Throughout this subsection, we shall assume an exponential bound on the moments of both μ_0 and ν, i.e.

$$\sup_{k\in\mathbb{N}} \left(\int |x|^k \mathrm{d}\mu_0(x) \right)^{1/k} < +\infty \,, \qquad \sup_{k\in\mathbb{N}} \left(\int |x|^k \mathrm{d}\psi(x) \right)^{1/k} < +\infty \,. \tag{24}$$

Clearly, (24) is authomatically satisfied when $\mathcal{X}$ is bounded. Let us fix some $z \in \mathbb{R}^d$, and consider the z-weighted moments of μ_t and ψ, respectively:

$$m_t^{(k)} := \int (x\cdot z)^k \mathrm{d}\mu_t(x) \,, \qquad n_t^{(k)} := \int (x\cdot y)^k \mathrm{d}\psi(y) \,, \qquad k \in \mathbb{Z}^+ \,.$$

Straightforward computation shows that the first z-weighted moment satisfies the autonomous differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} m_t^{(1)} = \overline{\alpha}\upsilon \left(n^{(1)} - m_t^{(1)} \right) \,, \tag{25}$$

whereas the higher moments satisfy the differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t} m_t^{(k)} = -\gamma_k m_t^{(k)} + f_k \left(m_t^{(1)}, \dots, m_t^{(k-1)} \right) + \overline{\alpha}\upsilon^k n^{(k)} \,, \tag{26}$$

where

$$\gamma_k := 1 - \alpha \left(\overline{\omega}^k + \omega^k \right) - \overline{\alpha\upsilon}^k \,,$$

$$f_k \left(m_t^{(1)}, \dots, m_t^{(k-1)} \right) := \sum_{j=1}^{k-1} \binom{k}{j} \left(\alpha \overline{\omega}^j \omega^{k-j} m_t^{(j)} m_t^{(k-j)} + \overline{\alpha\upsilon}^j \upsilon^{k-j} m_t^{(j)} n^{(k-j)} \right) \,.$$

Example 1. *In the special case when $\alpha = 1$, namely when there is no influential environment, we obtain from (25) that $\frac{\mathrm{d}}{\mathrm{d}t} \int x \mathrm{d}\mu_t(x) = 0$, so that the first moment is constant. On the other hand, the variance*

$$v_t := \int \left| x - \int y \mathrm{d}\mu_0(y) \right|^2 \mathrm{d}\mu_t(x)$$

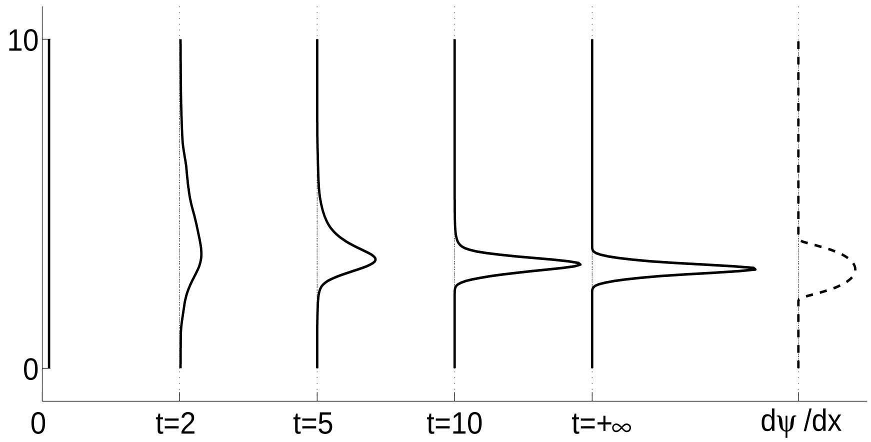

FIG 1. *Behaviour in time of the ODE solution in $d = 1$, with initial condition μ_0 uniform over $(0, 10)$, heterogenous environment $\mathrm{d}\psi(x) = \exp(-(1-(x-3)^2)^{-1})\mathbb{1}_{(2,4)}(x)\mathrm{d}x$, and parameters $\alpha = 0.5$, $\omega = 0.5$, and $\upsilon = 0.5$. The Radon-Nikodym derivates of the asymptotic measure μ_∞, and of the influential environment ψ (dashed) are plotted as a reference.*

satisfies $\frac{\mathrm{d}}{\mathrm{d}t}v_t = -2\omega\overline{\omega}v_t$. *Hence*

$$v_t = v_0 e^{-\omega\overline{\omega}t} ,$$

i.e. μ_t *converges to a delta centered in the average initial opinion exponentially fast in* t.

We now focus on the limit as $t \to +\infty$ for the general case. An inductive argument proves the following result.

Lemma 1. *Assume* $\alpha < 1$. *Then, for every* $z \in \mathbb{R}^d$, *the* z*-weighted moments of* μ_t *satisfy*

$$\lim_{t\to\infty} m_t^{(k)} = m_\infty^{(k)} , \qquad k \in \mathbb{Z}^+ , \tag{27}$$

where $m_\infty^{(k)}$ *can be recursively evaluated by*

$$m_\infty^{(1)} := n^{(1)} , \qquad m_\infty^{(k+1)} = \gamma_{k+1}^{-1}\left[f_{k+1}\left(m_\infty^{(1)}, \dots, m_\infty^{(k)}\right) + \overline{\alpha}\upsilon^k n^{(k+1)}\right] . \tag{28}$$

Proof For $k = 1$, the solution of the ODE (25) is easily found to be

$$m_t^{(1)} = e^{-\overline{\alpha}\upsilon t} m_0^{(1)} + \left(1 - e^{-\overline{\alpha}\upsilon t}\right) n^{(1)} , \tag{29}$$

so that (27) clearly holds. Moreover, assume that (27) holds for all $k \in \{1, \dots, j-1\}$, and define $\chi_t^{(j)} := f_j\left(m_t^{(1)}, \dots, m_t^{(j-1)}\right)$ for $t \in [0, +\infty]$. Then, the continuity of f_j implies that $\lim_{t\to\infty} \chi_t^{(j)} = \chi_\infty^{(j)}$. Solving the ODE (26) gives

$$m_t^{(j)} = \int_0^t e^{-\gamma_j(t-s)}\left(\chi_t^{(j)} + \overline{\alpha}\upsilon^j n^{(j)}\right)\mathrm{d}s + e^{-\gamma_j t} m_0^{(j)} . \tag{30}$$

Clearly, the second addend of the right-hand side of (30) converges to zero for $t \to \infty$. On the other hand, the convergence of $\chi_t^{(j)}$ implies that

$$\begin{aligned}\lim_{t\to\infty}\int_0^t e^{-\gamma_j(t-s)}\left(\chi_t^{(j)}+\overline{\alpha}v^j n^{(j)}\right)\mathrm{d}s &= \left(\chi_\infty^{(j)}+\overline{\alpha}v^j n^{(j)}\right)\lim_{t\to\infty}\int_0^t e^{-\gamma_j(t-s)}\mathrm{d}s\\ &= \gamma_j^{-1}\left(\chi_\infty^{(j)}+\overline{\alpha}v^j n^{(j)}\right).\end{aligned}$$

The forgoing, together with (30), implies the claim. ■

We are now in a position to prove the following result for the convergence of μ_t.

Proposition 3. *Assume that (24) holds. Then,*

$$\lim_{t\to\infty}\mu_t=\mu_\infty\,,$$

weakly, where $\mu_\infty \in \mathcal{P}(\mathcal{X})$ is uniquely characterized by its moments $m_\infty^{(k)}$.

Proof It follows from (24) that there exists some finite $M \in \mathbb{R}^+$ such that

$$|m_0^{(k)}| \leq |z|^k M^k\,, \qquad |n^{(k)}| \leq |z|^k M^k\,, \tag{31}$$

for all $z \in \mathbb{R}^d$, and $k \in \mathbb{N}$. Now, an inductive argument shows that

$$|m_t^{(k)}| \leq |z|^k M^k\,, \qquad \forall t\in[0,+\infty]\,,\ z\in\mathbb{R}^d\,, \tag{32}$$

for all $k \in \mathbb{N}$. In fact, (29) and (31) immediately imply that (32) holds for $k = 1$. Moreover, if (32) holds for all $k \in \{1, \dots, j-1\}$, then (30), and (32) give

$$\begin{aligned}|m_t^{(j)}| &\leq \int_0^t e^{-\gamma_j(t-s)}\left(\left|f_j\left(m_t^{(1)},\dots,m_t^{(j-1)}\right)\right|+\overline{\alpha}v^j|n^{(j)}|\right)\mathrm{d}s+e^{-\gamma_j t}|m_0^{(j)}|\\ &\leq \int_0^t e^{-\gamma_j(t-s)}M^j|z|^j\gamma_j\mathrm{d}s+e^{-\gamma_j t}M^j|z|^j\\ &= M^j|z|^j\,.\end{aligned}$$

Let us consider the characteristic functions $\phi_t(z) := \int \exp(iz\cdot x)\mathrm{d}\mu_t(x)$, and, for $k \in \mathbb{Z}^+$, define $a_t(k) := i^k m_t^{(k)}/k!$, $b(k) := M^k|z|^k/k!$, and observe that $\sum_{k\in\mathbb{Z}^+} b(k) = \exp(M|z|)$. One has that

$$\phi_t(z)=\int\sum_{k\in\mathbb{Z}^+}\frac{(iz\cdot x)^k}{k!}\mathrm{d}\mu_t(x)=\sum_{k\in\mathbb{Z}^+}\frac{i^k}{k!}\int(x\cdot z)^k\mathrm{d}\mu_t(x)=\sum_{k\in\mathbb{Z}^+}a_t(k)\,,$$

where the exchange between the series and the integral is justified by Lebesgue's dominated convergence theorem, since

$$|\sum_{0\leq k\leq n}\frac{i^k}{k!}(x\cdot z)^k|\leq\sum_{0\leq k\leq n}b(k)\leq\exp(Mz)\,.$$

Moreover, observe that, since $|a_t(k)| \leq b(k)$, another application of Lebesgue's dominated convergence theorem gives

$$\lim_{t\to\infty} \phi_t(z) = \lim_{t\to\infty} \sum_{k\in\mathbb{Z}^+} a_t(k) = \sum_{k\in\mathbb{Z}^+} a_\infty(k) =: \phi_\infty(z)\,.$$

Hence, $\phi_t(z)$ converges pointwise to $\phi_\infty(z)$, which in turn can be easily verified to be continuous at 0. Then, the claim follows from Lévy's continuity theorem [7, Th. 2.5.1]. ■

Observe that, for all $\alpha \in (0,1)$, the limit measure μ_∞ is independent of the initial condition μ_0, and depends only on the influential environment ψ, and on the parameters α, ω, and υ. Notice that the first moment satisfies $m_\infty^{(1)} = n^{(1)}$. In contrast, if $\psi \neq \delta_{x_0}$, it easily seen that $m_\infty^{(k)} \neq n^{(k)}$ for $k \geq 2$, so that in particular $\mu_\infty \neq \psi$. On the other hand, it follows from (28) that, if $\psi \neq \delta_{x_0}$, then the variance of μ_∞ is positive, so that $\mu_\infty \neq \delta_{x_0}$. This result may be interpreted as showing that the presence of an heterogeneous influential environment prevents the population from achieving an asymptotic opinion agreement. In fact, as shown in the following Proposition, the asymptotic opinion distribution μ_∞ is absolutely continuous whenever so is the influential environment ψ.

Proposition 4. *Assume* $\psi << \lambda$. *Then,* $\mu_\infty << \lambda$ *for all* $\alpha \in [0,1)$.

Proof For $\mu, \nu \in \mathcal{P}(\mathcal{X})$, $\gamma \in [0,1]$, define $\overline{\gamma} := 1-\gamma$, and

$$L_\gamma(\mu,\nu) \in \mathcal{P}(\mathcal{X}), \qquad \langle L_\gamma(\mu,\nu), \varphi\rangle = \iint \varphi(\overline{\gamma}x+\gamma y)\mathrm{d}\mu(x)\mathrm{d}\nu(y)\,, \quad \forall \varphi \in \mathcal{C}_b(\mathcal{X})\,.$$

Since L_γ is a rescaled convolution operator, and since $\psi << \lambda$, one has that $L_\upsilon(\mu,\psi) << \lambda$. Similarly, $L_\omega(\mu_\infty,\mu_\infty) = \alpha L(\mu_\infty^s,\mu_\infty^s)$, where μ_∞^s is the singular part of μ_∞. Combining this with the fact that the asymptotic measure satisfies $\mu_\infty = F(\mu_\infty) = \alpha L_\omega(\mu,\mu) + \overline{\alpha} L_\upsilon(\mu,\psi)$, one gets that $\mu_\infty^s(\mathcal{X}) = \alpha\left(L_\omega(\mu_\infty^s,\mu_\infty^s)\right)(\mathcal{X}) = \alpha\left(\mu^s(\mathcal{X})\right)^2$. Therefore, $\mu_\infty^s(\mathcal{X})\left(1-\alpha\mu_\infty^s(\mathcal{X})\right) = 0$. But, since $\mu_\infty^s(\mathcal{X}) \leq 1$ and $\alpha < 1$, this necessarily implies that $\mu_\infty^s(\mathcal{X}) = 0$. ■

Fig. 4.1 reports numerical simulations of the mean-field dynamics, when started from a uniform distribution over an interval, and influenced by an absolutely continuous environment. Coherently with Proposition 2, the solution remains absolutely continuous during its evolution, and converges to a limit measure which is absolutely continuous, as predicted by Propositions 3, and 4, respectively. Such a limit density may be interpreted as resulting from a tension between the aggregating forces represented by the first addend in the right-hand side of (3), and the environment's influence captured by the second addend in the right-hand side of (3).

4.2. Bounded confidence opinion dynamics

We analyze now the case when $\kappa(\,\cdot\,|\,\cdot\,,\,\cdot\,)$ is in the form (3) with $\alpha = 1$, and weight distribution $\theta(\,\cdot\,|x,y) := \theta^i(\,\cdot\,|x,y)$ supported on $[0,\omega_0]$ for some $\omega_0 \in [0,1[$, and

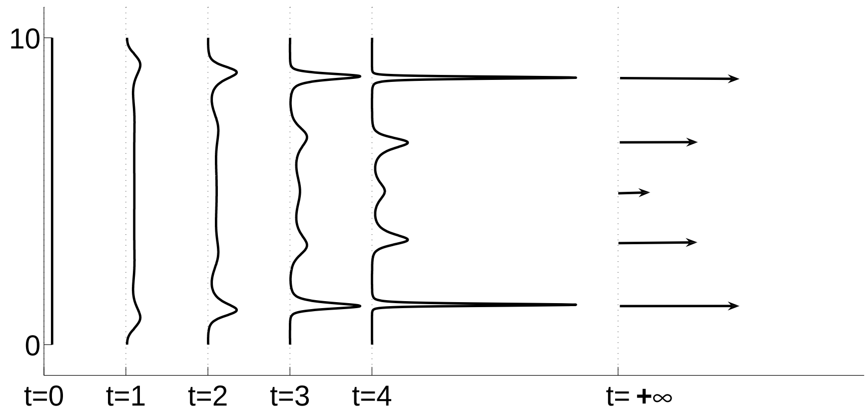

FIG 2. *Behaviour in time of the ODE solution in $d = 1$, with initial condition μ_0 uniform over $(0, 10)$, and $\theta(\cdot|x, y) = \delta_{1/2}\mathbb{1}_{[0,1]}(|x - y|) + \delta_0\mathbb{1}_{(1,+\infty)}(|x - y|)$.*

satisfying the symmetry assumption

$$\theta(\,\cdot\,|x, y) = \theta(\,\cdot\,|y, x)\,, \tag{33}$$

for all $x, y \in \mathcal{X}$. The following result states weak convergence of μ_t.

Proposition 5. *Assume that $\int |x|^2 \mathrm{d}\mu_0(x) < \infty$. Then, there exists $\mu_\infty \in \mathcal{P}(\mathcal{X})$ such that*

$$\lim_{t\to\infty} \mu_t = \mu_\infty\,,$$

weakly.

Proof We start by proving that the second moment $m_t^{(2)} := \int |x|^2 \mathrm{d}\mu_t(x)$ is a Lyapunov function for the system. Observe that, for all $x, y \in \mathbb{R}^d$, $\omega \in [0, 1]$, $\bar{\omega} = 1 - \omega$, one has

$$|x + \omega(y - x)|^2 + |y + \omega(x - y)|^2 = \left(\overline{\omega}^2 + \omega^2\right)\left(|x|^2 + |y^2|\right) + 4\omega\overline{\omega}x \cdot y\,,$$

so that

$$\begin{aligned} 2\omega\overline{\omega}|x - y|^2 &= 2\omega\overline{\omega}\left(|x|^2 + |y|^2 - 2x \cdot y\right) \\ &= \left(1 - \omega^2 - \overline{\omega}^2\right)\left(|x|^2 + |y|^2\right) - 4\omega\overline{\omega}x \cdot y \\ &= |x|^2 + |y^2| - |x + \omega(y - x)|^2 - |y + \omega(x - y)|^2\ . \end{aligned}$$

From the foregoing, and the symmetry of $\theta(\,\cdot\,|x,y)$, it follows that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}m_t^{(2)} &= \frac{\mathrm{d}}{\mathrm{d}t}\int |x|^2\mathrm{d}F(\mu_t)(x) - m_t^{(2)} \\
&= \iiint \left(|x+\omega(y-x)|^2 - |x|^2\right)\mathrm{d}\theta(\omega|x,y)\mathrm{d}\mu_t(x)\mathrm{d}\mu_t(y) \\
&= \frac{1}{2}\iiint \left(|x+\omega(y-x)|^2 + |y+\omega(x-y)|^2 - |x|^2 - |y|^2\right)\mathrm{d}\theta(\omega|x,y)\mathrm{d}\mu_t(x)\mathrm{d}\mu_t(y) \\
&= -\iiint \omega\,(1-\omega)\,|x-y|^2\mathrm{d}\theta(\omega|x,y)\mathrm{d}\mu_t(x)\mathrm{d}\mu_t(y) \\
&\leq -(1-\omega_0)\iiint \omega|x-y|^2\mathrm{d}\theta(\omega|x,y)\mathrm{d}\mu_t(x)\mathrm{d}\mu_t(y)\,.
\end{aligned} \tag{34}$$

Hence, in particular, $\frac{\mathrm{d}}{\mathrm{d}t}m_t^{(2)} \leq 0$, so that $m_t^{(2)}$ is nonincreasing, and therefore convergent. Define $m_\infty^{(2)} := \lim_{t\to\infty} m_t^{(2)}$ and observe that (34) implies that

$$\begin{aligned}
\lim_{t\to\infty}\int_0^t \iiint \omega|x-y|^2\mathrm{d}\theta(\omega|x,y)\mathrm{d}\mu_s(x)\mathrm{d}\mu_s(y)\mathrm{d}s &\leq \lim_{t\to\infty} -\frac{1}{1-\omega_0}\int_0^t \frac{\mathrm{d}}{\mathrm{d}s}m_s^{(2)}\mathrm{d}s \\
&= \lim_{t\to\infty}\frac{m_0^{(2)} - m_t^{(2)}}{1-\omega_0} \\
&= \frac{m_0^{(2)} - m_\infty^{(2)}}{1-\omega_0}\,.
\end{aligned} \tag{35}$$

Now, for any smooth and compact-supported test function $\varphi \in C_c^\infty(\mathbb{R}^d)$, we can write

$$\varphi(x+\omega(y-x)) - \varphi(x) = \omega(y-x)\cdot\nabla\varphi(x) + r(x,y) \tag{36}$$

with $|r(x,y)| \leq \omega^2|y-x|^2\Phi$ where $\Phi := ||D^2\varphi||$. Moreover, again from the symmetry of $\theta(\,\cdot\,|x,y)$, one has

$$\begin{aligned}
&\left|\iiint \omega(y-x)\cdot\nabla\varphi(x)\mathrm{d}\theta(\omega|x,y)\mathrm{d}\mu_t(x)\mathrm{d}\mu_t(y)\right| \\
=\;& \frac{1}{2}\left|\iiint \omega(y-x)\cdot(\nabla\varphi(x)-\nabla\varphi(y))\,\mathrm{d}\theta(\omega|x,y)\mathrm{d}\mu_t(x)\mathrm{d}\mu_t(y)\right| \\
\leq\;& \frac{1}{2}\iiint \omega|y-x|\,|\nabla\varphi(x)-\nabla\varphi(y)|\,\mathrm{d}\theta(\omega|x,y)\mathrm{d}\mu_t(x)\mathrm{d}\mu_t(y) \\
\leq\;& \frac{\Phi}{2}\iiint \omega|x-y|^2\mathrm{d}\theta(\omega|x,y)\mathrm{d}\mu_t(x)\mathrm{d}\mu_t(y)\,.
\end{aligned} \tag{37}$$

From (36) and (37) it follows that

$$\begin{aligned}|\langle F(\mu_t)-\mu_t,\varphi\rangle| &= \left|\iiint (\varphi\left(x+\omega(y-x)\right)-\varphi(x))\,\mathrm{d}\theta(\omega|x,y)\mathrm{d}\mu_t(x)\mathrm{d}\mu_t(y)\right| \\ &\leq \left|\iiint \omega(y-x)\cdot\nabla\varphi(x)\mathrm{d}\theta(\omega|x,y)\mathrm{d}\mu_t(x)\mathrm{d}\mu_t(y)\right| \\ &\quad +\Phi\iiint \omega^2|x-y|^2\mathrm{d}\theta(\omega|x,y)\mathrm{d}\mu_t(x)\mathrm{d}\mu_t(y) \\ &\leq \frac{3\Phi}{2}\iiint \omega|x-y|^2\mathrm{d}\theta(\omega|x,y)\mathrm{d}\mu_t(x)\mathrm{d}\mu_t(y)\,,\end{aligned}$$

so that

$$\begin{aligned}\lim_{t\to\infty}\int_0^t|\langle F(\mu_s)-\mu_s,\varphi\rangle|\,\mathrm{d}s &\leq \frac{3\Phi}{2}\lim_{t\to\infty}\int_0^t\iiint \omega|x-y|^2\mathrm{d}\theta(\omega|x,y)\mathrm{d}\mu_s(x)\mathrm{d}\mu_s(y)\mathrm{d}s \\ &\leq \frac{3\Phi}{2(1-\omega_0)}(m_0^{(2)}-m_\infty^{(2)})\,.\end{aligned}$$

Therefore, in particular, the limit

$$\lim_{t\to\infty}\langle\mu_t,\varphi\rangle=\lim_{t\to\infty}\int_0^t\langle F(\mu_s)-\mu_s,\varphi\rangle\mathrm{d}s$$

exists and it is finite. From the arbitrariness of $\varphi\in\mathcal{C}_c^\infty(\mathcal{X})$, it follows that μ_t converges in the sense of distributions. Finally, μ_t converges in $\mathcal{P}(\mathcal{X})$, by tightness. ■

If we make the further assumption that the weight $\omega\sim\theta(\,\cdot\,|x,y)$ is strictly positive in a neighborhood of the diagonal $\{(x,x):\,x\in\mathcal{X}\}$, we have the following characterization of the equilibrium points.

Proposition 6. *Let $R>0$ be such that,*

$$\delta(R):=\inf\{\omega\,:\,\mathrm{supp}(\theta(\,\cdot\,|x,y))\subseteq[\omega,1]\;\forall\,x,y\in\mathcal{X},\;|x-y|<R\}>0$$

Then μ_∞ is a convex combination of Dirac's deltas centered in points separated by a distance not smaller then R.

Proof Assume by contradiction that $x^*,y^*\in\mathrm{supp}(\mu_\infty)$ and $|x^*-y^*|<R$. We can find suitable neighborhoods A and B of x^* and y^*, respectively, such that $|x-y|<R$ for all $x\in A$ and $y\in B$. Hence, $\mathrm{supp}(\theta(\,\cdot\,|x,y))\subseteq[\delta(R),1]$ for all $x\in A$, and $y\in B$. Then,

$$\iiint |x-y|^2\omega\mathrm{d}\theta(\omega|x,y)\mathrm{d}\mu_\infty(x)\mathrm{d}\mu_\infty(y)\geq\delta\int_A\int_B|x-y|^2\mathrm{d}\mu_\infty(x)\mathrm{d}\mu_\infty(y)>0\,.$$

This clearly contradicts (35). ■

It is worth stating the following simple, though important, consequence of Proposition 6, which, in particular, applies to the Gaussian interaction model (2).

Corollary 1. *Suppose that* $\cup_{\omega_0>0}\{(x,y):\mathrm{supp}(\theta(\cdot|x,y)\subseteq[\omega_0,1])\}=\mathcal{X}\times\mathcal{X}$. *Then,* $\mu_\infty=\delta_{x_0}$ *where* $x_0=\int x\mathrm{d}\mu_0(x)$.

Fig. 4.2 reports numerical simulations of the mean-field ODE associated to the bounded-confidence model of Deffuant-Weisbuch, in dimension $d=1$, starting from an initial condition uniform over the interval $(0,10)$. Observe that, as predicted by Proposition 2, the solution remains absolutely continuous, with bounded density, at any finite time t. It is possible to appreciate the effect of local aggregation forces, which first lead to the formation of two peaks around the opinion points $x=1,9$, then of other two smaller peaks around the points $x=3,7$, and finally of a smaller peak in $x=5$. As t grows large, the opinion density converges to a convex combination of Dirac's deltas, as predicted by Proposition 5. Notice that such deltas seem to be centered in opinion points separated by a distance of about 2, whereas an inter-cluster distance of at least 1 is predicted by Proposition 6. These results may be interpreted as explaining how locally aggregating interactions modeling homophily can generate global fragmentation.

5. Concentration around the solution of the ODE

In this section, we finally show that, as the population size n grows, the stochastic process $\{\mu_t^n\}$ concentrates around the solution $\{\mu_t\}$ of the ODE (7), at an exponential probability rate. Throughout this section, we shall assume that $\mathcal{X}\subseteq\mathbb{R}^d$ is bounded, with Δ denoting its diameter, and that the stochastic kernel $\kappa(\,\cdot\,|\,\cdot\,,\,\cdot\,)$ is globally Lipschitz in the Kantorovich-Wasserstein metric, i.e. that

$$W_1\left(\kappa(\,\cdot\,|x,y),\kappa(\,\cdot\,|x',y')\right)\leq\frac{L_F}{2}\left|(x,y)-(x',y')\right|\,,\qquad\forall x,x',y,y'\in\mathcal{X}\tag{38}$$

holds for some finite positive constant L_F. Our first step consists in showing that the operator F inherits the Lipschitz property from the stochastic kernel $\kappa(\,\cdot\,|\,\cdot\,,\,\cdot\,)$. The proof of the next result relies on the duality formula [2, (7.1.2)]

$$W_1(\mu,\nu)=\sup\left\{\langle\mu,\varphi\rangle-\langle\nu,\varphi\rangle:\,\varphi\in\mathrm{Lip}_1(\mathcal{X})\right\}\,,\tag{39}$$

where $\mathrm{Lip}_1(\mathcal{X})$ denotes the set of 1-Lipschitz functions on $\mathcal{X}$.

Lemma 2. *If (38) holds, then*

$$W_1\left(\mu,\nu\right)\leq L_FW_1(F(\mu),F(\nu))\,,\qquad\forall\mu,\nu\in\mathcal{P}_1(\mathcal{X})\,.$$

Proof First, observe that, for arbitrary $\varphi\in\mathrm{Lip}_1(\mathcal{X})$, and $x,y,x',y'\in\mathcal{X}$,

$$\begin{aligned}\int\varphi(z)\mathrm{d}\kappa(z|x,y)-\int\varphi(z)\mathrm{d}\kappa(z|x',y')\quad&\leq\quad W_1\left(\kappa(\,\cdot\,|x,y),\kappa(\,\cdot\,|x',y')\right)\\&\leq\quad\frac{L_F}{2}\left|(x,y)-(x',y')\right|\\&\leq\quad\frac{L_F}{2}\left(|x-x'|+|y-y'|\right)\,,\end{aligned}$$

by (39), and (38). For $\mu, \nu \in \mathcal{P}(\mathcal{X})$, let $\xi \in \mathcal{P}_1(\mathcal{X}\times\mathcal{X})$ be their optimal coupling, i.e. the one such that $\int\int |x-y| d\xi(x,y) = W_1(\mu,\nu)$. Then,

$$\begin{aligned}
\langle F(\mu), \varphi\rangle - \langle F(\nu), \varphi\rangle &= \int\!\!\int\!\!\int\!\!\int\!\!\int \varphi(z)\,[d\kappa(z|x,y)d\mu(x)d\mu(y) - d\kappa(z|x',y')d\nu(x')d\nu(y')] \\
&= \int\!\!\int\!\!\int\!\!\int\!\!\int \varphi(z)\,[d\kappa(z|x,y) - d\kappa(z|x',y')]\,d\xi(x,y)d\xi(x',y') \\
&\leq \frac{L_F}{2}\int\!\!\int\!\!\int\!\!\int (|x-x'| + |y-y'|)\,d\xi(x,y)d\xi(x',y') \\
&= L_F W_1(\mu,\nu)\,.
\end{aligned}$$

Therefore, the claim follows by applying the duality formula (39) once more. ■

Observe that there are three sources of randomness in the system: the empirical measure of the initial opinions μ_0^n, the update times $\{T_k\}$, and the agents' interaction. The first two can be easily dealt with by appealing to the following classical large deviations results.

Lemma 3. *For all $\mu_0 \in \mathcal{P}(\mathcal{X})$, $\varepsilon > 0$, it holds*

$$\lim_n n^{-1}\log \mathbb{P}\left(W_1(\mu_0^n, \mu_0) \geq \varepsilon\right) \leq -\varepsilon^2/2\,.$$

Proof Sanov's theorem [23, Th. 2.14], and the Csiszar-Kullback-Pinsker inequality [24, pag. 580] imply that

$$\begin{aligned}
\lim_n n^{-1}\log \mathbb{P}\left(W_1\left(M_0, \mu_0\right) \geq \varepsilon\right) &= \inf\left\{\mathrm{H}\left(\nu||\mu_0\right) : \nu \in \mathcal{P}(\mathcal{X}),\, W_1(\nu,\mu_0) \geq \varepsilon\right\} \\
&\geq \inf\left\{\tfrac{1}{2}||\nu-\mu_0||^2 : \nu \in \mathcal{P}(\mathcal{X}),\, W_1(\nu,\mu_0) \geq \varepsilon\right\} \\
&\geq \varepsilon^2/(2\Delta^2)\,,
\end{aligned}$$

where $\mathrm{H}\left(\nu||\mu\right)$ denoted the relative entropy, and the last inequality follows from the estimate $W_1(\nu,\mu) \leq \Delta||\nu-\mu||$ [24, Th. 6.15]. ■

Lemma 4. *For $t \in \mathbb{R}^+$, let $\varsigma(t) := \sup\{k \in \mathbb{Z}^+ : T_k \leq t\}$. For all $\tau \in \mathbb{R}^+$, $a \geq 1$ it holds*

$$\limsup_n n^{-1}\log \mathbb{P}\left(\sup\{t - T_{\varsigma(t)} : 0 \leq t \leq \tau\} \geq \varepsilon\right) \leq -\varepsilon^2/\tau\,,$$

$$\limsup_n n^{-1}\log \mathbb{P}(\varsigma(\tau) \geq \alpha\tau n) \leq -(a-1)^2\tau\,.$$

Proof The first statement follows, e.g., from [23, Th. 5.1]. The second one, e.g., from [23, Ex. 1.13]. ■

We are now left with the third source of randomness, originated by the selection of the interacting agents. Observe that, in the right-hand side of the duality formula (39), one may restrict the supremization to the test functions φ

belonging to $\mathrm{Lip}_1^\Delta := \{\varphi \in \mathrm{Lip}_1(\mathcal{Y}) : |\varphi(x)| \leq \Delta/2\}$, where $\mathcal{Y}$ is an hypercube of edge-length Δ containing $\mathcal{X}$, and μ, ν are naturally identified as elements of $\mathcal{P}(\mathcal{Y})$. The following result shows that the set Lip_1^Δ can be approximated in the infinity norm by not-too-large a set of functions.

Lemma 5. *Let $\mathcal{X} \subseteq \mathbb{R}^d$ be compact and convex. Then, for all $\varepsilon \in]0, \Delta/2]$, there exists a finite set $\mathcal{H}_\varepsilon \subseteq \mathrm{Lip}_1^\Delta$ such that $|\mathcal{H}_\varepsilon| \leq \frac{2\sqrt{d}+1}{6}\frac{\Delta}{\varepsilon}3^{\left(\frac{\Delta}{\varepsilon}(\sqrt{d}+1)\right)^d}$, and*

$$\min\{||h - \varphi|| : h \in \mathcal{H}_\varepsilon\} \leq \varepsilon\,, \quad \forall \varphi \in \mathrm{Lip}_1^\Delta\,.$$

Proof With no loss of generality we shall restrict to the case $\mathcal{X} \subseteq \mathcal{Y} = [0, \Delta]^d$. We introduce a discretization operator $\Phi : \mathrm{Lip}_1^\Delta \to \mathrm{Lip}_1^\Delta$ as follows. Let $\eta := \varepsilon/(\sqrt{d}+1/2)$ and define $\mathcal{J} := \{0, 1, \dots, \lfloor \Delta/\eta \rfloor\}$. For any $\varphi \in \mathrm{Lip}_1^\Delta$, and $\boldsymbol{j} \in \mathcal{J}^d$, let $k(\boldsymbol{j}) = i \in \mathcal{J}$ iff $\varphi(\boldsymbol{j}\eta) \in [-1/2 + \eta i, -1/2 + \eta(i+1)[$. Observe that, since φ is 1-Lipschitz, one has

$$\sum_{1 \leq l \leq d} |j_l - j_l'| \leq 1 \qquad \Longrightarrow \qquad |k(\boldsymbol{j}) - k(\boldsymbol{j}')| \leq 1\,. \tag{40}$$

Then, define $\Phi(\varphi) = h$, by putting, for all $x \in \prod_{1 \leq l \leq d} [j_l\eta, (j_l+1)\eta]$,

$$h(x) = \prod_{1 \leq l \leq d} \left((k(\boldsymbol{j} + \delta_l) - k(\boldsymbol{j}))\,(x_l - j_l\eta) + \eta k(\boldsymbol{j}) - \tfrac{1}{2} + \tfrac{\eta}{2} \right)\,.$$

Thanks to (40), one has that $\Phi(\varphi) \in \mathrm{Lip}_1^\Delta$ for all $\varphi \in \mathrm{Lip}_1^\Delta$. Moreover, for all $\boldsymbol{j} \in \mathcal{J}^d$, one has $|\Phi(\varphi)(\boldsymbol{j}\eta) - \varphi(\boldsymbol{j}\eta)| \leq \frac{\eta}{2}$. Observe that, for all $x \in [0, \Delta]^d$, there exists $\boldsymbol{j}(x) \in \mathcal{J}^d$ such that $|x - \eta\boldsymbol{j}| \leq \sqrt{d}\eta/2$. Therefore,

$$\begin{aligned} |\Phi(\varphi)(x) - \varphi(x)| &\leq |\Phi(\varphi)(\boldsymbol{j}\eta) - \varphi(\boldsymbol{j}\eta)| + |\Phi(\varphi)(\boldsymbol{j}\eta) - \Phi(\varphi)(x)| + |\varphi(\boldsymbol{j}\eta) - \varphi(x)| \\ &\leq \eta/2 + 2|\boldsymbol{j}\eta - x| \\ &\leq \eta(\sqrt{d} + 1/2)\,, \end{aligned}$$

so that the second part of the claim follows by substituting the value of η.

It remains to estimate the cardinality of $\mathcal{H}_\varepsilon := \Phi(\mathrm{Lip}_1^\Delta)$. To see that, first observe that $k(\mathbf{0})$ can take at most Δ/η values. On the other hand, it follows from (40) that, given $k(\boldsymbol{j})$, $k(\boldsymbol{j} + \delta_l)$ can assume at most three different values, for all $1 \leq l \leq d$. This implies that

$$|\mathcal{H}_\varepsilon| \leq \frac{\Delta}{\eta}3^{(\Delta/\eta+1)^d - 1} = \frac{\Delta}{\varepsilon}\frac{2\sqrt{d}+1}{6}3^{\left((\sqrt{d}+1/2)\Delta/\varepsilon+1\right)^d} \leq \frac{\Delta}{\varepsilon}\frac{2\sqrt{d}+1}{6}3^{\left((\sqrt{d}+1)\Delta/\varepsilon\right)^d}\,,$$

the last inequality following since $1 \leq \Delta/(2\varepsilon)$. ■

We can now estimate the error incurred when when using an Euler approximation of some future value of the empirical density process, centered on its current value.

Lemma 6. *For $k \in \mathbb{Z}^+$, $n \in \mathbb{N}$, and $\sigma \in [0,1]$,*

$$\mathbb{P}\left(W_1\left(\overline{\sigma} M_k + \sigma F(M_k), M_{k+\lfloor \sigma n \rfloor}\right) \geq K\Delta\sigma^2\right) \leq \rho\,,$$

where $\overline{\sigma} = 1-\sigma$, $K = K_F + 1$, with K_F being the Lipschitz constant of F on $\mathcal{P}(\mathcal{X})$ in the variational distance, and

$$\rho := \frac{4\sqrt{d}+2}{K\sigma^2}\exp\left(\left(\frac{12}{K\sigma^2}(\sqrt{d}+1)\right)^d \log 3 - \frac{K^2\sigma^3}{2^7}n\right)\,. \tag{41}$$

Proof First, observe that the following control of the increments holds:

$$||M_{k+1} - M_k|| \leq 2/n\,. \tag{42}$$

Define $w := \lfloor \sigma n \rfloor$, and $\varepsilon := K\Delta\sigma^2$. Also, for $\varphi \in \mathrm{Lip}_1^\Delta$, define

$$Z_j^{(\varphi)} := \langle M_{k+j} - M_k, \varphi\rangle - \frac{1}{n}\sum_{0\leq i<j}\langle F\left(M_{k+i}\right) - M_{k+i}, \varphi\rangle\,,$$

for $j = 0, \ldots, w$, and

$$V^{(\varphi)} := \langle M_{k+w} - (1-\frac{w}{n})M_k - \frac{w}{n}F(M_k), \varphi\rangle - Z_w^{(\varphi)}\,.$$

It follows from (42) that $||M_{k+j} - M_k|| \leq 2j/n$. Hence,

$$\begin{array}{rcl}\left|V^{(\varphi)}\right| & = & n^{-1}\Big|\sum\limits_{0\leq j<w}\langle F(M_{k+j}) - F(M_k), \varphi\rangle - \sum\limits_{0\leq j<w}\langle M_{k+j} - M_k, \varphi\rangle\Big| \\ & \leq & n^{-1}\sum\limits_{0\leq j<w}\left(||F(M_{k+j}) - F(M_k)|| + ||M_{k+j} - M_k||\right)||\varphi|| \\ & \leq & n^{-1}\sum\limits_{0\leq j<w}K\dfrac{2j}{n}||\varphi|| \\ & \leq & \varepsilon/2\,,\end{array} \tag{43}$$

the last inequality following from the fact that $||\varphi|| \leq \Delta/2$. Observe that, for all $\varphi \in \mathrm{Lip}_1(\mathcal{X})$, $Z_0^{(\varphi)} = 0$, while $\{Z_j^{(\varphi)} : 0 \leq j \leq w\}$ is a martingale. Moreover, (42) provides the following control on the increments:

$$\begin{array}{rcl}|Z_{j+1}^{(\varphi)} - Z_j^{(\varphi)}| & \leq & |\langle M_{k+j+1} - M_{k+j}, \varphi\rangle| + n^{-1}\left|\langle F(M_{k+j}) - M_{k+j}, \varphi\rangle\right| \\ & \leq & ||M_{k+j+1} - M_{k+j}||||\varphi|| + n^{-1}||F(M_{k+j}) - M_{k+j}||||\varphi|| \\ & \leq & 4n^{-1}||\varphi||\,.\end{array} \tag{44}$$

Let $\mathcal{H} := \mathcal{H}_{\varepsilon/12} \subseteq \mathrm{Lip}_1(\mathcal{X})$ be as in Lemma 5. By first applying the union bound, and then the Hoeffding-Azuma inequality [1, Th. 7.2.1], the probability of the event $E := \bigcup_{h\in\mathcal{H}}\left\{|Z_w^{(h)}| \geq \varepsilon/4\right\}$ can be estimated as follows:

$$\mathbb{P}\left(E\right) \leq |\mathcal{H}|\mathbb{P}\left(|Z_w^{(h)}| \geq \varepsilon/4\right) \leq 2|\mathcal{H}|\exp\left(-\frac{\varepsilon^2 n^2}{2^7 w\Delta^2}\right)\,. \tag{45}$$

Now, Lemma 5 and (44) imply that,

$$Z_w^{(\varphi-h)} \leq 3\frac{w}{n}||\varphi - h|| \leq 3\sigma\frac{\varepsilon}{12} \leq \frac{\varepsilon}{4}\,,$$

for some $h \in \mathcal{H}_{\varepsilon/12}$. Hence, if E does not occur, then

$$|Z_w^{(\varphi)}| \leq \min\left\{|Z_w^{(h)}| + |Z_w^{(\varphi-h)}| : \, h \in \mathcal{H}\right\} \leq \frac{\varepsilon}{2}\,, \tag{46}$$

for every $\varphi \in \mathrm{Lip}_1^{\Delta}$. By combining (43), (45), and (46), one gets

$$\begin{aligned} \mathbb{P}\left(W_1\left(M_{k+w}, \overline{\sigma}M_k + \sigma F(M_k)\right) \geq \varepsilon\right) &= \mathbb{P}\left(\sup\{Z_w^{(\varphi)} + V^{(\varphi)}\} \geq \varepsilon\right) \\ &\leq \mathbb{P}\left(\sup\{Z_w^{(\varphi)}\} \geq \tfrac{3}{4}\varepsilon\right) \\ &\leq 2|\mathcal{H}|\exp\left(-\tfrac{\varepsilon^2 n^2}{2^7 w \Delta^2}\right)\,, \end{aligned}$$

and the claim follows upon substituting the expressions for w and ε, and applying Lemma 5. ■

We are now ready to prove point (b) of Theorem 1. Let $L := L_F - 1$, and $K = K_F + 1$, where L_F and K_F are the global Lipschitz constants of F on $\mathcal{P}(\mathcal{X})$ in the Kantorovich-Wasserstein distance, and in the variational distance, respectively. Let us fix some $\varepsilon > 0$, $\tau > 0$, and introduce the quantities

$$\sigma := \frac{L\varepsilon}{2\Delta L + 3K\Delta e^{2L\tau}}\,, \qquad w = \lfloor \sigma/n \rfloor\,.$$

With no loss of generality let us assume that $\sigma \in]0,1]$, and put $\overline{\sigma} = 1 - \sigma$. Further, let ρ be as in (41), and define

$$\alpha_0 = e^{-2L\tau}\varepsilon/2, \qquad \alpha_{i+1} = (1+\sigma L)\,\alpha_i + \frac{3}{2}K\Delta\sigma^2\,, \quad i \in \mathbb{Z}^+\,. \tag{47}$$

Solving the iterative equation above, one obtains the estimate

$$\alpha_i = (1+\sigma L)^i\left(\alpha_0 + \frac{3K\Delta\sigma}{2L}\right) - \frac{3K\Delta\sigma}{2L} \leq e^{\sigma L i}\left(\alpha_0 + \frac{3K\Delta\sigma}{2L}\right)\,. \tag{48}$$

For $i \in \mathbb{Z}^+$, consider the random variable $\Gamma_i^n := W_1\left(M_{iw}, \mu_{\sigma i}\right)$, and the events $A_i := \{\Gamma_j^n \geq \alpha_i\}$, $B_i := \bigcup_{0 \leq j \leq i} A_j$. We shall prove by induction that

$$\mathbb{P}\left(B_i\right) \leq (i+1)\rho\,, \tag{49}$$

for all $i \in \mathbb{Z}_+$. First, it follows from Lemma 3 that (49) holds with $i = 0$, for sufficiently small ε, and sufficiently large n. Then, for any nonnegative integer i, consider the intermediate measures

$$\lambda := \overline{\sigma}M_{wi} + \sigma F\left(M_{wi}\right)\,, \qquad \nu := \overline{\sigma}\mu_{\sigma i} + \sigma F\left(\mu_{\sigma i}\right)\,.$$

From the duality formula (39), and Lemma 2, one has

$$W_1(\lambda,\nu) \le (\overline{\sigma} + \sigma L_F)\Gamma_i^n = (1+\sigma L)\Gamma_i^n\,. \tag{50}$$

Furthermore, since $\{\mu_t\}$ is a solution of the ODE (12), it follows from (15), and the estimate $W_1(\mu,\nu) \le \Delta/2||\mu-\nu||$,

$$||\mu_t - \mu_{\sigma i}|| \le 2\,(t-s)\ , \qquad W_1(\mu_t,\mu_s) \le \Delta(t-s)\,, \tag{51}$$

for all $t \ge s$. From the duality formula (39), the fact that $\{\mu_t\}$ solves the ODE (12), and (51), one gets the estimate

$$\begin{array}{rcl} W_1\left(\nu,\mu_{\sigma(i+1)}\right) & = & \sup\left\{\langle\mu_{\sigma(i+1)},\varphi\rangle - \langle\nu,\varphi\rangle : \varphi\in \mathrm{Lip}_1^\Delta\right\} \\ & \le & \displaystyle\int_{\sigma i}^{\sigma(i+1)} \sup\left\{\langle F(\mu_t)-\mu_t - F(\mu_{\sigma i}) + \mu_{\sigma i},\varphi\rangle : \varphi\in\mathrm{Lip}_1^\Delta\right\}\mathrm{d}t \\ & \le & \displaystyle\frac{\Delta}{2}K\int_{\sigma i}^{\sigma(i+1)} ||\mu_t-\mu_{\sigma i}||\mathrm{d}t \\ & \le & \displaystyle\Delta K\int_{\sigma i}^{\sigma(i+1)} (t-\sigma i)\,\mathrm{d}t \\ & = & \Delta K\sigma^2/2\,. \end{array} \tag{52}$$

From the triangle inequality, (50), and (52), one finds that

$$\begin{array}{rcl} \Gamma_{i+1}^n & \le & W_1(M_{w(i+1)},\lambda) + W_1(\lambda,\nu) + W_1\left(\nu,\mu_{\sigma(i+1)}\right) \\ & \le & W_1(M_{w(i+1)},\lambda) + K\Delta\sigma^2/2 + (1+\sigma L)\,\Gamma_i^n\,. \end{array} \tag{53}$$

Therefore, (53), the inductive hypothesis (49), (47), and Lemma 6, imply that

$$\begin{array}{rcl} \mathbb{P}(B_{i+1}) & = & \mathbb{P}\left(B_i^c\cap A_{i+1}\right) + \mathbb{P}(B_i) \\ & \le & \mathbb{P}\left(B_i^c \cap \left\{W_1(M_{w(i+1)},\lambda) > \alpha_{i+1} - K\Delta\sigma^2/2 - (1+\sigma L)\,\alpha_i\right\}\right) + (i+1)\rho \\ & \le & \mathbb{P}(W_1(M_{w(i+1)},\lambda) > K\Delta\sigma^2) + (i+1)\rho \\ & \le & (i+2)\rho\,. \end{array}$$

Hence, (49) holds for all $i\in\mathbb{Z}^+$.

Observe that, if $iw - w/2 \le k \le iw + w/2$, then

$$W_1(M_k,\mu_{k/n}) \le W_1(M_{wi},\mu_{\sigma i}) + W_1(M_{wi},M_k) + W_1(\mu_{k/n},\mu_{\sigma i}) \le \Gamma_i^n + \Delta\sigma\,. \tag{54}$$

Now, recall the definition of $\varsigma(t)$ given in Lemma 4, and consider the events $C := \{\varsigma(\tau) \le \frac{3}{2}n\tau\}$, and $D := \left\{\sup\{|t - T_{\varsigma(t)} : t\in[0,\tau]\} \le \varepsilon/(4\Delta)\right\}$. Observe that C implies that, for all $t\le\tau$,

$$\iota(t) := \left\lfloor \frac{\varsigma(t)}{\lfloor\sigma n\rfloor} + \frac{1}{2}\right\rfloor \le \frac{3\tau n/2}{\sigma n - 1} + \frac{1}{2} \le \frac{2\tau}{\sigma}\,. \tag{55}$$

It follows from (54), (51), (48), and (55), that, if the event $B^c_{\lfloor 2\tau\sigma\rfloor} \cap D \cap C$ occurs, then, for all $t \in [0, \tau]$, the following estimate holds

$$\begin{array}{rcl} W_1\left(\mu^n_t, \mu_t\right) & = & W_1\left(M_{\varsigma(t)}, \mu_t\right) \\ & \leq & W_1\left(M_{\varsigma(t)}, \mu_{T_{\varsigma(t)}}\right) + W_1\left(\mu_{T_{\varsigma(t)}}, \mu_t\right) \\ & \leq & \Gamma^n_{\iota(t)} + \Delta\sigma + \Delta|t - T_{\varsigma(t)}| \\ & \leq & \alpha_{\iota(t)} + \Delta\sigma + \varepsilon/4 \\ & \leq & e^{\sigma L \iota(t)} + \Delta\sigma + \varepsilon/4 \\ & \leq & e^{2L\tau}\left(\alpha_0 + \frac{3K\Delta\sigma}{2L}\right) + \Delta\sigma + \varepsilon/4 \\ & = & \varepsilon\,, \end{array}$$

where the last equality follows by substituting the expressions for σ and α_0. For sufficiently small ε, and large n, Lemma 4 implies that $\mathbb{P}(C \cap D) \geq 1 - \rho$. Therefore, using (49), one gets that

$$\mathbb{P}\left(\sup\{W_1\left(\mu^n_t, \mu_t\right) : t \in [0, \tau]\} > \varepsilon\right) \leq \mathbb{P}(B_{\iota(\tau)}) + \mathbb{P}(C^c \cup D^c) \leq (2\tau/\sigma + 2)\rho\,,$$

from which point (b) of Theorem 1 follows.

References

[1] N. Alon, and J.H. Spencer, *The probabilistic method*, 3rd edition, Wiley, Hoboken, NJ, 2008.
[2] L. Ambrosio, N. Gigli, and G. Savarè, *Gradient flows in metric spaces and in the space of probability measures*, Lectures in Mathematics ETH Zurich, Birkhauser Verlag, Basel, 2005.
[3] R. Axelrod, "The dissemination of culture", *J. of Conflict Resolution*, vol. 42(2), pp. 203–226, 1997.
[4] E. Ben-Naim, P.L. Krapivsky, and S. Redner. "Bifurcations and patterns in compromise processes", *Physica D*, vol. 183(3), pp. 190–204, 2003.
[5] V.D. Blondel, J.M. Hendrickx, and J.N. Tsitsiklis, "On Krause's consensus formation model with state-dependent connectivity", *IEEE Trans. Automat. Control*, vol. 54(11), pp. 2586–2597, 2009.
[6] V.D. Blondel, J.M. Hendrickx, and J.N. Tsitsiklis, "Continuous-time average-preserving opinion dynamics with opinion-dependent communications", submitted, 2009.
[7] V.S. Borkar, *Probability theory: An advanced course*, Springer-Verlag, New York, 1995.
[8] S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah, "Randomized Gossip Algorithms", *IEEE Trans. Inf. Theory*, vol. 52(6), pp. 2508–2530, 2006.
[9] C. Canuto, F. Fagnani, and P. Tilli, "A Eulerian approach to the analysis of rendez-vous algorithms", in Proc. of 2008 IFAC Conf., Seoul, Korea, July 6-11, pp. 9039–9044, 2008. Extended version [online] available at http://calvino.polito.it/∼fagnani/coordincontrol/IFAC08CFT.pdf.

[10] C. Castellano, S. Fortunato, V. Loreto, "Statistical physics of social dynamics", *Rev. Mod. Phys.*, vol. 81, pp. 591–646, 2009.

[11] F. Cucker, and S. Smale, "The mathematics of emergence", *Japan J. Math.* , vol. 2, pp. 197–227, 2007.

[12] F. Cucker, and S. Smale, "Emergent behavior in flocks", *IEEE Trans. Automat. Control*, vol. 52, pp. 852–862, 2007.

[13] G. Deffuant, D. Neau, F. Amblard, and G. Weisbuch, "Mixing beliefs among interacting agents", *Adv. Complex Syst.*, vol. 3, pp. 87–98, 2000.

[14] F. Fagnani, and S. Zampieri, "Randomized Consensus Algorithms over Large Scale Networks", *IEEE J. Select. Aereas Commun.* , vol. 26(4), pp. 634–649, 2008.

[15] F. Fagnani, and S. Zampieri, "Asymmetric Randomized Gossip Algorithms for Consensus", in Proc. of 2008 IFAC Conf., Seoul, Korea, July 6-11, pp. 9052-9056, 2008.

[16] S. Fortunato, "On the consensus threshold for the opinion dynamics of Krause-Hegselmann", *Int. J. Mod. Phys.*, vol. 16(2), pp. 259-270, 2005.

[17] S.-Y. Ha, and J.-G. Liu, "*A simple proof of the Cucker-Smale flocking dynamics and mean-field limit*", *Commun. Math. Sci.*, vol. 7(2), pp. 297–325, 2009.

[18] U. Krause, "A discrete nonlinear and non-autonomous model of consensus formation", *Communications in Difference Equations*, pp. 227–236, S. Elaydi, G. Ladas, J. Popenda, and J. Rakowski editors, Gordon and Breach, Amsterdam, 2000.

[19] T.M. Liggett, *Interacting particle systems*, Springer, Berlin Heidelberg, 2005.

[20] J. Lorenz, "A stabilization theorem for continuous opinion dynamics", *Physica A*, vol. 355(1), pp. 217–223, 2005.

[21] J. Lorenz, "Continuous opinion dynamics under bounded confidence: A survey", *Internat. J. Modern Phys. C*, vol. 18(12), pp. 1819–1838, 2007.

[22] D. Shah, "Gossip algorithms", *Foundations and Trends in Networking*, vol. 3(1), pp. 1–125, 2008.

[23] A. Shwartz, and A. Weiss, *Large deviations for performance analysis*, Chapman and Hall, London, 1995.

[24] C. Villani, *Optimal transport: old and new*, Springer Verlag, Berlin Heidelberg, 2009.

[25] N.C. Wormald, "The differential equation method for random graph processes and greedy algorithms", *Ann. Appl. Probab.*, vol. 5(4), pp. 1217–1235, 1995.