Robust Distributed Routing in Dynamical Flow Networks

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Abstract

Robustness of distributed routing policies is studied for dynamical flow networks, with respect to adversarial disturbances that reduce the link flow capacities. A dynamical flow network is modeled as a system of ordinary differential equations derived from mass conservation laws on a directed acyclic graph with a single origin-destination pair and a constant inflow at the origin. Distributed routing policies regulate the way the incoming flow at a non-destination node gets split among its outgoing links as a function of the local information about the current particle density, while the outflow of a link is modeled to depend on the current particle density through a flow function. A dynamical flow network is called fully transferring if the outflow at the destination node is asymptotically equal to the inflow at the origin node, and partially transferring if the outflow at the destination node is asymptotically bounded away from zero. A class of distributed routing policies that are locally responsive is shown to yield the maximum possible resilience under local information constraint with respect to the two transferring properties, where resilience is measured as the minimum, among all the disturbances that make the network loose its transferring property, of the sum of the link-wise magnitude of disturbances. In particular, the maximum resilience of a dynamical flow network starting from an equilibrium condition, in order to remain fully transferring, is shown to equal its minimum node residual capacity. The latter is defined as the minimum, among all the non-destination nodes, of the sum of the difference between the maximum flow capacity and the initial equilibrium flow on all the links outgoing from the node. On the other hand, the maximum resilience of a dynamical flow network starting from an equilibrium condition, in order to remain partially transferring, is shown to be equal to the network's min-cut capacity and hence is independent of the initial equilibrium flow. Finally, a simple convex optimization problem is formulated for the most resilient initial equilibrium flow, and the use of tolls to induce such an initial equilibrium flow in transportation networks is discussed.

I. Introduction

Flow networks provide a fruitful modeling framework for many applications of interest such as transportation, data, or production networks. They entail a fluid-like description of the macroscopic motion of *particles*, which are routed from their origins to their destinations via intermediate nodes: we refer to standard textbooks, such as [2], for a thorough treatment. Robustness of routing policies for flow networks is a central problem which is gaining increased attention with a growing awareness to safeguard critical infrastructure networks against natural and man-induced disruptions. Information constraints limit the efficiency and resilience of such routing policies, and the possibility of cascaded failures through the network adds serious challenges to this problem. The difficulty is further magnified by the presence of dynamical effects [3].

This paper studies *dynamical flow networks*, modeled as systems of ordinary differential equations derived from mass conservation laws on directed acyclic graphs with a single origin-destination pair and a constant inflow at the origin. The rate of change of the particle density on each link of the network equals the difference between the *inflow* and the *outflow* of that link, while the way the incoming flow at an intermediate node gets split among its outgoing links depends on the current particle density on the outgoing links through the routing policy. We focus on *distributed routing policies* whereby the proportion of incoming flow routed to the outgoing links of a node is allowed to depend only on *local information*, consisting of the current particle densities on the outgoing links of the same node. We model the outflow of a link to be dependent on the current particle density on that

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link through a *flow function*. The inspiration for such a modeling paradigm comes from empirical findings from several application domains, such as *fundamental diagrams* in transportation networks [4], congestion-dependent *throughput* and *average delays* in data networks [5], and *clearing functions* in production networks [6].

Our objective is the design and analysis of distributed routing policies for dynamical flow networks that are *maximally robust* with respect to *adversarial disturbances* that reduce the link flow capacities. We define two notions of transfer efficiency in order to capture the extremes of the resilience of the network towards disturbances: we call the the dynamical flow network *fully transferring* if the outflow at the destination node asymptotically approaches the inflow at the origin node, and *partially transferring* if the outflow at the destination node is asymptotically bounded away from zero. We consider a setup where, before the disturbance, the network is operating at an *equilibrium flow* and hence is fully transferring: such an equilibrium flow might alternatively be thought of as the outcome of a slower time-scale learning process (e.g., see [7], [8] in case of transportation networks, or [9] in case of communication networks), or the outcome of the routing policies. We analyze the robustness of distributed routing policies, evaluating it in terms of the network's *strong* and *weak resilience*, which are defined as the minimum sum of link-wise magnitude of disturbances making the perturbed dynamical flow network not fully transferring, and, respectively, not partially transferring. We prove that the maximum possible resilience with respect to both notions is yielded by a class of *locally responsive* distributed routing policies, characterized by the property that the portion of its incoming flow that a node routes towards an outgoing link does not decrease as the particle density on any other outgoing link increases. Moreover, we show that the strong resilience of a dynamical flow network with such locally responsive distributed routing policies equals the *minimum node residual capacity*. The latter is defined as the minimum, among all the non-destination nodes, of the sum of the difference between the maximum flow capacity and the initial equilibrium flow on all the links outgoing from the node. On the other hand, the weak resilience of the dynamical flow network equals its *min-cut capacity* and hence is independent of the initial equilibrium flow. We also formulate a simple convex optimization problem to solve for the most strongly resilient initial equilibrium flow, and discuss the use of tolls to induce such an initial equilibrium flow in transportation networks. Our analysis assumes that every link has infinite capacity to hold particles and that the flow is a bounded, strictly increasing function of the particle density. We report results from numerical simulations illustrating how violation of these assumptions can result in cascaded spill-backs and possibly affect the network's resilience.

Stability analysis of network flow control policies under non-persistent disturbances, especially in the context of internet, has attracted a lot of attention, e.g., see [10], [11], [12], [13]. Recent work on robustness analysis of static flow networks under adversarial and probabilistic persistent disturbances in the spirit of this paper include [14], [15], [16]. It is also worth comparing the distributed routing policies studied in this paper with the backpressure policy [17], which is one of the most well-known robust distributed routing policy for queueing networks. While relying on local information in the same way as the distributed routing policies studied here, backpressure policies require the nodes to have, possibly unlimited, buffer capacity. In contrast, in our framework, the nodes have no buffer capacity. In fact, the distributed routing policies considered in this paper are closely related to the well-known *hot-potato* or deflection routing policies [18] [5, Sect. 5.1], where the nodes route incoming packets immediately to one of the outgoing links. However, to the best of our knowledge, the robustness properties of dynamical flow networks, where the outflow from a link is not necessarily equal to its inflow have not been studied before.

The contributions of this paper are as follows: (i) we formulate a novel dynamical system framework for robustness analysis of dynamical flow networks under local information constraint on the routing policies; (ii) we characterize a general class of distributed routing policies that yield the maximum strong and weak resilience under local information constraint; (iii) we provide a simple characterization of the resilience in terms of the topology and the pre-disturbance equilibrium flow of the network. For a given initial equilibrium flow, the class of locally responsive distributed routing policies can be interpreted as approximate Nash equilibria in an appropriate zero-sum game setting where the objective of the adversary inflicting the disturbance is to destabilize the network with a disturbance of minimum possible magnitude and the objective of the system planner is to design distributed routing policies that yield the maximum possible resilience. The technical results of this paper hinge on tools from several different fields. The upper bounds on the resilience for a given equilibrium flow use graph theory notions from flow networks (e.g., see [2]). The properties of the routing functions that give maximum resilience are reminiscent of cooperative dynamical systems in the sense of [19], [20]. The problem of determining tolls for a desired equilibrium flow exploits the fact that the associated congestion game is a potential game and that the extremum of the potential function corresponds to the equilibrium [21], [22].

The rest of the paper is organized as follows. In Section II, we formulate the problem by formally defining the notion of a dynamical flow network and its resilience. In Section III, we define the class of locally responsive distributed routing policies, state the main results on the network resilience, and provide discussions on the results. Section IV discusses the problem of selection of the most strongly resilient equilibrium flow of the network and the use of tolls to induce such a desired equilibrium in transportation networks. In Section V, we report illustrative numerical simulation results. In Sections VI, VII and VIII, we state proofs of the main results on network resilience. Finally, we conclude in Section IX with remarks on future research directions.

Before proceeding, we define some preliminary notation to be used throughout the paper. Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$ be the set of nonnegative real numbers. Let $\mathcal{A}$ and $\mathcal{B}$ be finite sets. Then, $|\mathcal{A}|$ will denote the cardinality of $\mathcal{A}$, $\mathbb{R}^{\mathcal{A}}$ (respectively, $\mathbb{R}_+^{\mathcal{A}}$) the space of real-valued (nonnegative-real-valued) vectors whose components are indexed by elements of $\mathcal{A}$, and $\mathbb{R}^{\mathcal{A}\times\mathcal{B}}$ the space of matrices whose real entries indexed by pairs of elements in $\mathcal{A}\times\mathcal{B}$. The transpose of a matrix $M \in \mathbb{R}^{\mathcal{A}\times\mathcal{B}}$, will be denoted by $M^T \in \mathbb{R}^{\mathcal{B}\times\mathcal{A}}$, while $\mathbf{1}$ the all-one vector, whose size will be clear from the context. Let $\mathrm{cl}(\mathcal{X})$ be the closure of a set $\mathcal{X} \subseteq \mathbb{R}^{\mathcal{A}}$. If $\mathcal{B} \subseteq \mathcal{A}$, $\mathbb{1}_{\mathcal{B}} : \mathcal{A} \to \{0, 1\}$ will stand for the indicator function of $\mathcal{B}$, with $\mathbb{1}_{\mathcal{B}}(a) = 1$ if $a \in B$, $\mathbb{1}_{\mathcal{B}}(a) = 0$ if $a \in \mathcal{A} \setminus \mathcal{B}$. For $p \in [1, \infty]$, $\|\cdot\|_p$ is the p-norm. By default, let $\|\cdot\| := \|\cdot\|_2$ denote the Euclidean norm. Let $\mathrm{sgn} : \mathbb{R} \to \{-1, 0, 1\}$ be the sign function, defined by $\mathrm{sgn}(x)$ is 1 if $x > 0$, $\mathrm{sgn}(x) = -1$ if $x < 1$, and $\mathrm{sgn}(x) = 0$ if $x = 0$. Conventionally, we shall assume the identity $\mathrm{d}|x|/\mathrm{d}x = \mathrm{sgn}(x)$ to be valid for every $x \in \mathbb{R}$, including $x = 0$.

II. Dynamical flow networks and their resilience

In this section, we introduce our model of dynamical flow networks and define the notions of transfer efficiency.

A. Dynamical flow networks

We start with the following definition of a flow network.

Definition 1 (Flow network): A *flow network* $\mathcal{N} = (\mathcal{T}, \mu)$ is the pair of a *topology*, described by a finite directed graph $\mathcal{T} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the node set and $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$ is the link set, and a family of *flow functions* $\mu := \{\mu_e : \mathbb{R}_+ \to \mathbb{R}_+\}_{e\in\mathcal{E}}$ describing the functional dependence $f_e = \mu_e(\rho_e)$ of the flow on the density of particles on every link $e \in \mathcal{E}$.

The *flow capacity* of a link $e \in \mathcal{E}$ is defined by

$$f_e^{\max} := \sup_{\rho_e \geq 0} \mu_e(\rho_e)\,. \tag{1}$$

For every node $v \in \mathcal{V}$, we shall denote by $\mathcal{E}_v^+ \subseteq \mathcal{E}$, and $\mathcal{E}_v^- \subseteq \mathcal{E}$, the set of its outgoing and incoming links, respectively. Moreover, we shall use the shorthand notation $\mathcal{R}_v := \mathbb{R}_+^{\mathcal{E}_v^+}$ for the set of nonnegative-real-valued vectors whose entries are indexed by elements of $\mathcal{E}_v^+$, $\mathcal{F}_v := \times_{e\in\mathcal{E}_v^+}[0, f_e^{\max})$ for the set of admissible flow vectors on outgoing links from node v, and $\mathcal{S}_v := \{p \in \mathcal{R}_v : \sum_{e\in\mathcal{E}_v^+} p_e = 1\}$ for the simplex of probability vectors over $\mathcal{E}_v^+$. We shall also use the notation $\mathcal{R} := \mathbb{R}_+^{\mathcal{E}}$ for the set of nonnegative-real-valued vectors whose entries are indexed by the links in $\mathcal{E}$, and $\mathcal{F} := \times_{e\in\mathcal{E}}[0, f_e^{\max})$ for the set of admissible flow vectors for the network. We shall write $f := \{f_e : e \in \mathcal{E}\} \in \mathcal{F}$, and $\rho := \{\rho_e : e \in \mathcal{E}\} \in \mathcal{R}$, for the vectors of flows and of densities, respectively, on the different links. The notation $f^v := \{f_e : e \in \mathcal{E}_v^+\} \in \mathcal{F}_v$, and $\rho^v := \{\rho_e : e \in \mathcal{E}_v^+\} \in \mathcal{R}_v$ will stand for the vectors of flows and densities, respectively, on the outgoing links of a node v. We shall compactly denote by $f = \mu(\rho)$ and $f^v = \mu^v(\rho^v)$ the functional relationships between density and flow vectors.

Throughout this paper, we shall restrict ourselves to network topologies satisfying the following:

Assumption 1: The topology $\mathcal{T}$ contains no cycles, has a unique origin (i.e., a node $v \in \mathcal{V}$ such that $\mathcal{E}_v^-$ is empty), and a unique destination (i.e., a node $v \in \mathcal{V}$ such that $\mathcal{E}_v^+$ is empty). Moreover, there exists a path in $\mathcal{T}$ to the destination node from every other node in $\mathcal{V}$.

Assumption 1 implies that one can find a (not necessarily unique) topological ordering of the node set $\mathcal{V}$ (see, e.g., [23]). We shall assume to have fixed one such ordering, identifying $\mathcal{V}$ with the integer set $\{0, 1, \ldots, n\}$, where $n := |\mathcal{V}| - 1$, in such a way that

$$\mathcal{E}_v^- \subseteq \bigcup_{0\leq u<v} \mathcal{E}_u^+\,, \qquad \forall v = 0, \ldots, n\,. \tag{2}$$

In particular, (2) implies that 0 is the origin node, and n the destination node in the network topology $\mathcal{T}$. An *origin-destination cut* (see, e.g., [2]) of $\mathcal{T}$ is a partition of $\mathcal{V}$ into $\mathcal{U}$ and $\mathcal{V}\setminus\mathcal{U}$ such that $0\in\mathcal{U}$ and $n\in\mathcal{V}\setminus\mathcal{U}$. Let $\mathcal{E}_{\mathcal{U}}^+=\{(u,v)\in\mathcal{E}:\,u\in\mathcal{U},v\in\mathcal{V}\setminus\mathcal{U}\}$ be the set of all the links pointing from some node in $\mathcal{U}$ to some node in $\mathcal{V}\setminus\mathcal{U}$. The *min-cut capacity* of a flow network $\mathcal{N}$ is defined as

$$C(\mathcal{N}):=\min_{\mathcal{U}}\sum\nolimits_{e\in\mathcal{E}_{\mathcal{U}}^+}f_{\mathrm{e}}^{\max}\,, \tag{3}$$

where the minimization runs over all the origin-destination cuts of $\mathcal{T}$. Throughout this paper, we shall assume a constant inflow $\lambda_0\geq 0$ at the origin node. Let us define the set of *admissible equilibrium flows* associated to an inflow λ_0 as

$$\mathcal{F}^*(\lambda_0):=\left\{f^*\in\mathcal{F}:\,\sum\nolimits_{e\in\mathcal{E}_0^+}f_e^*=\lambda_0,\,\sum\nolimits_{e\in\mathcal{E}_v^+}f_e^*=\sum\nolimits_{e\in\mathcal{E}_v^-}f_e^*,\,\forall\,0<v<n\right\}.$$

Then, it follows from the max-flow min-cut theorem (see, e.g., [2]), that $\mathcal{F}^*(\lambda_0)\neq\emptyset$ whenever $\lambda_0<C(\mathcal{N})$. That is, the min-cut capacity equals the maximum flow that can pass from the origin to the destination while satisfying capacity constraints on the links, and conservation of mass at the intermediate nodes.

Throughout the paper, we shall make the following assumption on the flow functions:

Assumption 2: For every link $e\in\mathcal{E}$, the map $\mu_e:\mathbb{R}_+\to\mathbb{R}_+$ is continuously differentiable, strictly increasing, such that $\mu_e(0)=0$, and $f_e^{\max}<+\infty$.

Thanks to Assumption 2, one can define the *median density* on link $e\in\mathcal{E}$ as the unique value $\rho_e^\mu\in\mathbb{R}_+$ such that

$$\mu_e(\rho_e^\mu)=f_e^{\max}/2. \tag{4}$$

Example 1 (Flow function): For every link $e\in\mathcal{E}$, let a_e and $f_e^{\max}$ be positive real constants. Then, a simple example of flow function satisfying Assumption 2 is given by

$$\mu_e(\rho_e)=f_e^{\max}\left(1-\exp(-a_e\rho_e)\right).$$

It is easily verified that the flow capacity is $f_e^{\max}$, while the median density for such a flow function is $\rho_e^\mu=a_e^{-1}\log 2$.

We now introduce the notion of a distributed routing policy used in this paper.

Definition 2 (Distributed routing policy): A *distributed routing policy* for a flow network $\mathcal{N}$ is a family of functions $\mathcal{G}:=\{G^v:\mathcal{R}_v\to\mathcal{S}_v\}_{0\leq v<n}$ describing the ratio in which the particle flow incoming in each non-destination node v gets split among its outgoing link set $\mathcal{E}_v^+$, as a function of the observed current particle density ρ^v on the outgoing links themselves.

The salient feature of Definition 2 is that the routing policy $G^v(\rho^v)$ depends only on the *local information* on the particle density ρ^v on the set $\mathcal{E}_v^+$ of outgoing links of the non-destination node v. On the other hand, the structural form of the routing policy $\mathcal{G}$ may depend on some global information on the flow network which might have been accumulated through a slower time-scale evolutionary dynamics. A two time-scale process of this sort has been analyzed in our related work [8] in the context of transportation networks.

We are now ready to define a dynamical flow network.

Definition 3 (Dynamical flow network): A *dynamical flow network* associated to a flow network $\mathcal{N}$ satisfying Assumption 1, a distributed routing policy $\mathcal{G}$, and an inflow $\lambda_0\geq 0$, is the dynamical system

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_e(t)=\lambda_v(t)G_e^v(\rho^v(t))-f_e(t)\,,\qquad\forall\,0\leq v<n\,,\quad\forall\,e\in\mathcal{E}_v^+\,, \tag{5}$$

where

$$f_e(t):=\mu_e(\rho_e(t))\,,\qquad\lambda_v(t):=\left\{\begin{array}{lll}\lambda_0 & \text{if} & v=0\\ \sum_{e\in\mathcal{E}_v^-}f_e(t) & \text{if} & 0<v\leq n.\end{array}\right. \tag{6}$$

Equation (5) states that the rate of variation of the particle density on a link e outgoing from some non-destination node v is given by the difference between $\lambda_v(t)G_e^v(\rho^v(t))$, i.e., the portion of the incoming particle flow of node v which is routed to link e, and $f_e(t)$, i.e., the particle flow on link e. Observe that the distributed routing policy

$G^v(\rho^v)$ induces a local feedback which couples the dynamics of the particle flow on the outgoing links of each non-destination node v.

We can now introduce the following notion of transfer efficiency of a dynamical flow network.

Definition 4 (Transfer efficiency of a dynamical flow network): Consider a dynamical flow network $\mathcal{N}$ satisfying Assumptions 1 and 2. Given some flow vector $\hat{f} \in \mathcal{F}$, and $\alpha \in [0,1]$, the dynamical flow network (5) is said to be *α-transferring* with respect to $\hat{f}$ if the solution of (5) with initial condition $\rho(0) = \mu^{-1}(\hat{\rho})$ satisfies

$$\liminf_{t \to +\infty} \lambda_n(t) \geq \alpha \lambda_0 \,. \tag{7}$$

Definition 4 states that a dynamical flow network is α-transferring when the outflow is asymptotically not smaller than α times the inflow. In particular, a fully transferring dynamical flow network is characterized by the property of having outflow asymptotically equal to its inflow, so that there is no throughput loss. On the other hand, a partially transferring dynamical flow network might allow for some throughput loss, provided that some fraction of the flow is still guaranteed to be asymptotically transferred.

Observe that a fully transferring dynamical flow network does not necessarily imply that the link-wise flows necessarily converge to an equilibrium, for it might in principle have a persistently oscillatory or more complex behavior. Nevertheless, it will prove useful to introduce the notions of equilibrium and limit flow as follows.

Definition 5 (Equilibrium and limit flow of a dynamical flow network): An *equilibrium flow* for the dynamical flow network (5) is a vector $f^* \in \mathcal{F}^*(\lambda_0)$ such that

$$\lambda_v^* G_e^v(\rho^v) = f_e^* \,, \qquad \forall e \in \mathcal{E}_v^+ \,,\ \forall 0 \leq v < n \,, \tag{8}$$

where $\rho_e^v := \mu_e^{-1}(f_e^*)$, and $\lambda_v^* = \lambda_0$ for $v = 0$ and $\lambda_v^* = \sum_{e \in \mathcal{E}_v^-} f_e^*$ for $0 < v < n$. A *limit flow* for the dynamical flow network (5) is a vector $f^* \in \mathrm{cl}(\mathcal{F})$ such that, for some initial condition $\rho(0) \in \mathcal{R}$, the flow $f(t)$ converges to f^* as t grows large.

Remark 1: Observe that an equilibrium flow $f^* \in \mathcal{F}^*(\lambda_0)$ is always a limit flow, since the solution of the dynamical flow network (5) with initial condition $\rho(0) = \mu^{-1}(f^*)$ stays put for all $t \geq 0$, and hence it is trivially convergent to f^*. On the other hand, if a limit flow $f^* \in \mathrm{cl}(\mathcal{F})$ satisfies all the capacity constraints with strict inequality, i.e., if $f^* \in \mathcal{F}$, then necessarily $f^* \in \mathcal{F}^*(\lambda_0)$ is also an equilibrium flow for (5), i.e., it satisfies mass conservation equations at all the non-destination nodes. In particular, if a dynamical flow network admits an equilibrium flow f^*, then it is necessarily fully transferring with respect to f^*, as well as with respect to all the initial flows $f(0) \in \mathcal{F}$ which are attracted by f^*.

In contrast, if $f^* \in \mathrm{cl}(\mathcal{F}) \setminus \mathcal{F}$, i.e., if at least one of the capacity constraints is satisfied with equality, then f^* is not an equilibrium flow for (5). In fact, in this case one has that $\sum_{e \in \mathcal{E}_v^+} f_e^* \leq \lambda_v^*$ with possibly strict inequality for some non-destination node $0 \leq v < n$. Hence, the dynamical flow network might still be non fully transferring. Finally, observe that a limit flow $f^* \in \mathrm{cl}(\mathcal{F})$ (and, *a fortiori*, an equilibrium flow) may not exist for general flow networks $\mathcal{N}$, and distributed routing policies $\mathcal{G}$.

Remark 2: Standard definitions in the literature are typically limited to static flow networks describing the particle flow at equilibrium via conservation of mass. In fact, they usually consist (see e.g., [2]) in the specification of a topology $\mathcal{T}$, a vector of flow capacities $f^{\max} \in \mathcal{R}$, and an admissible equilibrium flow vector $f^* \in \mathcal{F}^*(\lambda_0)$ for $\lambda_0 < C(\mathcal{N})$ (or, often, $f^* \in \mathrm{cl}(\mathcal{F}^*(\lambda_0))$ for $\lambda_0 \leq C(\mathcal{N})$).

In contrast, in our model we focus on the off-equilibrium particle dynamics on a flow network $\mathcal{N}$, induced by a distributed routing policy $\mathcal{G}$. Existence of an equilibrium of the dynamical flow network (5) depends on the topology $\mathcal{T}$, the structural form of the flow functions μ and of the distributed routing policy $\mathcal{G}$, as well as on the inflow λ_0. A necessary condition for that is $\lambda_0 < C(\mathcal{N})$. In contrast, simple, locally verifiable, sufficient conditions on $\mathcal{G}$ for the existence of an equilibrium flow might be hard to find for complex flow networks. However, in some cases, it is reasonable to assume the distributed routing policy $\mathcal{G}$ to be the outcome of a slow time-scale evolutionary dynamics with global feedback which can naturally lead to an equilibrium flow $f^* \in \mathcal{F}^*(\lambda_0)$. This has been shown, e.g., in our related work [8] on transportation networks, where the emergence of Wardrop equilibria is proven using tools from singular perturbation theory and evolutionary dynamics.

On the other hand, as shown in Section III, there is a class of locally responsive distributed routing policies (as characterized by Definition 8) such the associated dynamical flow network (5) always has a unique limit flow

$f^* \in \mathrm{cl}(\mathcal{F})$ such that, from any initial condition $\rho(0) \in \mathcal{R}$, the flow $f(t)$ associated to (5) converges to f^* as t grows large. Provided that such $f^* \in \mathcal{F}$, i.e., such limit flow satisfies the capacity constraints with strict inequality, this will prove that $f^* \in \mathcal{F}^*(\lambda_0)$, and it is a globally attractive equilibrium for the dynamical flow network (5).

B. Examples

We now present three illustrative applications of the dynamical flow network framework.

(i) *Transportation networks*: In transportation networks, particles represent drivers and distributed routing policies correspond to their local route choice behavior in response to the locally observed link congestions. A desired route choice behavior from a social optimization perspective may be achieved by appropriate incentive mechanisms. However, we do not address the issue of mechanism design in this paper. Section IV, however, discusses the use of tolls in influencing the long-term global route choice behavior of drivers to get a desired initial equilibrium state for the network. The robust distributed routing policies designed in this paper would correspond to the *ideal* node-wise route choice behavior of the drivers. The flow function $\mu_e(\rho_e)$ presented in this paper is related to the notion of fundamental diagram in traffic theory, e.g., see [4]. Note that in our formulation, we assume that the density of drivers is homogeneous over a link. One can refer to [4] for models that incorporate inhomogeneity, although the models and their analysis in [4] are developed under static routing policies. We shall refer to the transportation network setup frequently in the course of the paper.

(ii) *Data networks*: In data networks, the particles represent data packets that are to be routed from sources to destinations by routers placed at the nodes (see, e.g., [5, Ch. 5]). Typically the average packet delay from one router to the other increases with the increase in queue length on the link between the two routers. Hence, one has that such average delay is given by $d_e(\rho_e)$, where $d_e(\rho_e)$ is an increasing function. If one further assumes that the delay function $d_e(\rho_e)$ is concave and such that $d_e(\rho_e) = \Omega(\rho_e)$[1] as ρ_e grows large, then the relationship between the throughput and the queue length, $f_e \propto \rho_e/d_e(\rho_e)$, can be easily shown to satisfy Assumption 2. Therefore, in analogy with the general framework, ρ_e and f_e denote the queue length and the throughput, respectively, and $\mu_e(\rho_e)$ represents the throughput functions on the links of data networks.

(iii) *Production networks*: In production networks, the particles represent goods that need to be processed by a series of production modules represented by nodes. It is known, e.g., see [6], that the rate of doing work decreases with the amount of work in progress at a production module. This relationship is formalized by the concept of *clearing functions*. In this context, production networks have a clear analogy with our setup where ρ_e represents the work-in-progress, f_e represents the rate of doing work, and $\mu_e(\rho_e)$ represents the clearing function.

Remark 3: While there are many examples of congestion-dependent throughput functions and clearing functions that satisfy Assumption 2, typical fundamental diagrams in transportation systems have a $\cap$-shaped profile. While we do not study the implications of this analytically, we provide some simulations in Section V to illustrate how the results of this paper could be extended to this case.

C. Perturbed network and resilience

We shall consider persistent perturbations of the dynamical flow network (5) that reduce the flow functions on the links, as per the following:

Definition 6 (Admissible perturbation): An *admissible perturbation* of a flow network $\mathcal{N} = (\mathcal{T}, \mu)$, satisfying Assumptions 1 and 2, is a flow network $\tilde{\mathcal{N}} = (\mathcal{T}, \tilde{\mu})$, with the same topology $\mathcal{T}$, and a family of perturbed flow functions $\tilde{\mu} := \{\tilde{\mu}_e : \mathbb{R}_+ \to \mathbb{R}_+\}_{e \in \mathcal{E}}$, such that, for every $e \in \mathcal{E}$, $\tilde{\mu}_e$ satisfies Assumption 2, as well as

$$\tilde{\mu}_e(\rho_e) \leq \mu_e(\rho_e)\,, \qquad \forall \rho_e \geq 0\,.$$

We accordingly let $\tilde{f}_{\mathrm{e}}^{\max} := \sup\{\tilde{\mu}_e(\tilde{\rho}_e) : \tilde{\rho}_e \geq 0\}$. The *magnitude* of an admissible perturbation is defined as

$$\delta := \sum\nolimits_{e \in \mathcal{E}} \delta_e\,, \qquad \delta_e := \sup\{\mu_e(\rho_e) - \tilde{\mu}_e(\rho_e) :\ \rho_e \geq 0\}\,. \tag{9}$$

[1]Here, we use the Landau notation $f(x) = \Omega(g(x))$ as $x \to +\infty$ to mean that there exists positive constants K and x_0, such that $f(x) \geq Kg(x)$ for all $x \geq x_0$.

The *stretching coefficient* of an admissible perturbation is defined as

$$\theta := \max\{\tilde{\rho}^\mu_e/\rho^\mu_e : e \in \mathcal{E}\}\,, \tag{10}$$

where ρ^μ_e, and $\tilde{\rho}^\mu_e$ are the median densities respectively associated to the unperturbed and the perturbed flow function on link $e \in \mathcal{E}$, as defined in (4).

Given a dynamical flow network as in Definition 3, and an admissible perturbation as in Definition 6, we shall consider the *perturbed dynamical flow network*

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\rho}_e(t) = \tilde{\lambda}_v(t) G^v_e(\tilde{\rho}^v(t)) - \tilde{f}_e(t)\,, \qquad \forall\, 0 \le v < n\,, \quad \forall\, e \in \mathcal{E}^+_v\,, \tag{11}$$

where

$$\tilde{f}_e(t) := \tilde{\mu}_e(\tilde{\rho}_e(t))\,, \qquad \tilde{\lambda}_v(t) := \begin{cases} \sum_{e \in \mathcal{E}^-_v} \tilde{f}_e(t) & \text{if} \quad 0 < v < n \\ \lambda_0 & \text{if} \quad v = 0\,. \end{cases} \tag{12}$$

Observe that the perturbed dynamical flow network (11) has the same structure of the original dynamical flow network (5), as it describes the rate of variation of the particle density on each link e outgoing from some non-destination node v as the difference between $\tilde{\lambda}_v(t) G^v_e(\tilde{\rho}^v(t))$, i.e., the portion of the incoming perturbed flow of node v routed to link e, minus the perturbed flow on link e itself. Notice that the only difference with respect to the original dynamical flow network (5) is in the perturbed flow function $\tilde{\mu}_e(\rho_e)$ on each link $e \in \mathcal{E}$, which replaces the original one, $\mu_e(\rho_e)$. In particular, the distributed routing policy $\mathcal{G}$ is the same for the unperturbed and the perturbed dynamical flow networks. In this way, we model a situation in which the routers are not aware of the fact that the flow network has been perturbed, but react to this change only indirectly, in response to variations of the local density vectors $\tilde{\rho}^v(t)$.

We are now ready to define the following notion of resilience of a dynamical flow network as in Definition 3 with respect to an initial equilibrium flow f^*.

Definition 7 (Resilience of a dynamical flow network): Let $\mathcal{N}$ be a flow network satisfying Assumptions 1 and 2, $\mathcal{G}$ be a distributed routing policy, and $\lambda_0 \ge 0$ be a constant inflow at the origin node. Assume that the associated dynamical flow network (5) admits $f^* \in \mathcal{F}^*(\lambda_0)$ as an equilibrium flow. For every $\alpha \in (0,1]$, $\theta \ge 1$, let $\gamma_{\alpha,\theta}(f^*, \mathcal{G})$ be equal to the infimum magnitude of all the admissible perturbations of stretching coefficient less than or equal to θ for which the perturbed dynamical flow network (11) is not α-transferring with respect to f^*. Also, define $\gamma_{0,\theta}(f^*) := \lim_{\alpha \downarrow 0} \gamma_{\alpha,\theta}(f^*)$. For $\alpha \in [0,1]$, the *α-resilience* with respect to f^* is defined as[2] $\gamma_\alpha(f^*, \mathcal{G}) := \lim_{\theta \uparrow \infty} \gamma_{\alpha,\theta}$. The 1-resilience will be referred to as the *strong resilience*, while the 0-resilience will be referred to as the *weak resilience*.

In the remainder of the paper, we shall focus on the characterization of the strong and weak resilience of dynamical flow networks. Before proceeding, let us elaborate a bit on Definition 7. Notice that, for every $\alpha \in (0,1]$, the α-resilience $\gamma_\alpha(f^*, \mathcal{G})$ is simply the infimum magnitude of all the admissible perturbations such that the perturbed dynamical network (11) is not α-transferring with respect to the equilibrium flow f^*. In fact, one might think of $\gamma_\alpha(f^*, \mathcal{G})$ as the minimum effort required by a hypothetical adversary in order to modify the dynamical flow network from (5) to (11), and make it not α-transferring, provided that such an effort is measured in terms of the magnitude of the perturbation $\delta = \sum_{e \in \mathcal{E}} ||\mu_e(\,\cdot\,) - \tilde{\mu}_e(\,\cdot\,)||_\infty$. For $\alpha = 0$, trivially the perturbed network flow is always 0-transferring with respect to any initial flow. For this reason, the definition of the weak resilience $\gamma_0(f^*)$ involves the double limit $\lim_{\theta \uparrow \infty} \lim_{\alpha \downarrow 0} \gamma_{\alpha,\theta}$: the introduction of the bound on the stretching coefficient of the admissible perturbation is a mere technicality whose necessity will become clear in Section VIII-B.

Remark 4 (Zero-sum game interpretation): The notions of resilience are with respect to adversarial perturbations. Therefore, one can provide a zero-sum game interpretation as follows. Let the strategy space of the system planner be the class of distributed routing policies and the strategy space of an adversary be the set of admissible perturbations. Let the utility function of the adversary be $M\Theta - \delta$, where M is a large quantity, e.g., $\sum_{e \in \mathcal{E}} f^{\max}_{\mathrm{e}}$, and Θ takes the value 1 if the network is not α-transferring under given strategies of the system planner and the adversary, and

[2] It is easily seen that the limits involved in this definition always exist, as $\gamma_{\alpha,\theta}$ is clearly nonincreasing in α (the higher α, the more stringent the requirement of α-transfer) and θ (the higher θ, the more admissible perturbations are considered that may potentially make the dynamical flow network to be not α-transferring).

zero otherwise. Let the utility function of the system planner be $\delta - M\Theta$. As stated in Section III, a certain class of *locally responsive* distributed routing policies, characterized by Definition 8, is maximally robust with respect to both notions of weak and strong resilience. This will then show that the locally responsive distributed routing policies correspond to approximate Nash equilibria in this zero-sum game setting.

III. Main results and discussion

In this paper, we shall be concerned with the characterization of *maximally robust* distributed routing policies. That is, for a given flow network $\mathcal{N}$, and inflow $\lambda_0 \in [0, C(\mathcal{N}))$, we shall study a class of distributed routing policies $\mathcal{G}$ which have the maximum margin both weak and strong resilience under local information constraint.

The candidate class of such maximally robust distributed routing policy is characterized by the following.

Definition 8 (Locally responsive distributed routing policy): A *locally responsive* distributed routing policy for a flow network topology $\mathcal{T} = (\mathcal{V}, \mathcal{E})$ with node set $\mathcal{V} = \{0, 1, \dots, n\}$ is a family of continuously differentiable distributed routing functions $\mathcal{G} = \{G^v : \mathcal{R}_v \to \mathcal{S}_v\}_{v \in \mathcal{V}}$ such that, for every non-destination node $0 \leq v < n$:

(a) $\dfrac{\partial}{\partial \rho_e} G^v_j(\rho^v) \geq 0\,, \qquad \forall j, e \in \mathcal{E}^+_v\,, j \neq e\,, \rho^v \in \mathcal{R}^v\,;$

(b) $G^v_e(\rho^v) > 0$, for every $e \in \mathcal{E}^+_v$, $\rho^v \in \mathcal{R}_v$;

(c) for every nonempty proper subset $\mathcal{J} \subsetneq \mathcal{E}^+_v$, there exists a continuously differentiable map $G^{\mathcal{J}} : \mathcal{R}_{\mathcal{J}} \to \mathcal{S}_{\mathcal{J}}$, where $\mathcal{R}_{\mathcal{J}} := \mathbb{R}^{\mathcal{J}}_+$, and $\mathcal{S}_{\mathcal{J}} := \{p \in \mathcal{R}_{\mathcal{J}} : \sum_{j \in \mathcal{J}} p_j = 1\}$ is the simplex of probability vectors over $\mathcal{J}$, such that, for every $\rho^{\mathcal{J}} \in \mathcal{R}_{\mathcal{J}}$, if $\rho^v_e \to +\infty$ for all $e \in \mathcal{E}^+_v \setminus \mathcal{J}$ and $\rho_j \to \rho^{\mathcal{J}}_j$ for all $j \in \mathcal{J}$, then

$$G^v_e(\rho^v) \to 0, \quad \forall e \in \mathcal{E}^+_v \setminus \mathcal{J}\,, \qquad G^v_j(\rho) \to G^{\mathcal{J}}_j(\rho^{\mathcal{J}}), \quad \forall j \in \mathcal{J}\,.$$

Property (a) in Definition 8 states that, as the particle density on an outgoing link $e \in \mathcal{E}^+_v$ increases while the particle density on all the other outgoing links remains constant, the fraction of incoming particle flow of node v routed to any link $j \in \mathcal{E}^+_v \setminus \{e\}$ does not decrease, and hence the fraction of incoming particle flow routed to link e itself does not increase. In fact, Property (a) in Definition 8 is reminiscent of the definition of *cooperative dynamical systems* in the sense of [19], [20]. Property (b), instead, implies that for every observed local density $\rho^v \in \mathcal{R}_v$, every link $e \in \mathcal{E}^+_v$ gets a nonzero fraction of the incoming particle flow of node v. On the other hand, Property (c) implies that the fraction of incoming particle flow routed to a subset of outgoing links $\mathcal{J} \subset \mathcal{E}^+_v$ vanishes as the density on links in $\mathcal{J}$ grows unbounded while the density on the remaining outgoing links remains bounded.

Example 2 (Locally responsive distributed routing policy): An example of a locally responsive distributed routing policy corresponding to an equilibrium flow vector $f^* = \mu(\rho^*) \in \mathcal{F}^*(\lambda_0)$ is given by

$$G^v_e(\rho) = \frac{f^*_e \exp(-\eta(\rho_e - \rho^*_e))}{\sum_{j \in \mathcal{E}^+_v} f^*_j \exp(-\eta(\rho_j - \rho^*_j))}\,, \qquad \forall e \in \mathcal{E}^+_v\,, \quad \forall 0 \leq v < n\,, \tag{13}$$

where $\eta > 0$ is a constant. Computing partial derivatives one gets

$$\frac{\partial}{\partial \rho_j} G_e(\rho^v) = \eta \frac{f^*_e f^*_j \exp(-\eta(\rho_e - \rho^*_e)) \exp(-\eta(\rho_j - \rho^*_j))}{\left(\sum_{i \in \mathcal{E}^+_v} f^*_i \exp(-\eta(\rho_i - \rho^*_i))\right)^2} \geq 0 \qquad \forall e, j \in \mathcal{E}^+_v, \quad e \neq j\,, \tag{14}$$

so that Property (a) of Definition 8 holds true. Properties (b) and (c) are also easily verified. In the context of transportation networks, the example in (13) is a variant of the logit function from discrete choice theory emerging from utilization maximization perspective of drivers, where the utility associated with link e is the sum of $\rho^*_e - \rho_e + \log f^*_e/\eta$ and a double exponential random variable with parameter η (see, e.g., [24]).

We are now ready to state our main results. The first one shows that, when the distributed routing policy $\mathcal{G}$ is locally responsive, the dynamical flow network (5) always admits a unique, globally attractive limit flow $f^* \in \mathrm{cl}(\mathcal{F})$.

Theorem 1 (Existence of a globally attractive limit flow under locally responsive routing policies): Let $\mathcal{N}$ be a flow network satisfying Assumptions 1 and 2, $\lambda_0 \geq 0$ a constant inflow, and $\mathcal{G}$ a locally responsive distributed routing policy. Then, there exists a unique limit flow $f^* \in \mathrm{cl}(\mathcal{F})$ such that, for every initial condition $\rho(0) \in \mathcal{R}$, the flow $f(t)$ associated to the dynamical flow network (5) converges to f^* as t grows large.

Proof: See Section VI. ■

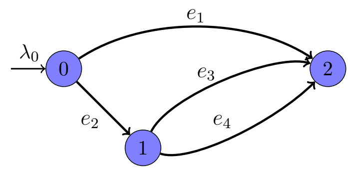

Fig. 1. The network topology used in Example 3.

The following is an immediate consequence of Theorem 1 and Remarks 1 and 2.

Corollary 1: Let $\mathcal{N}$ be a flow network satisfying Assumptions 1 and 2, $\lambda_0 \geq 0$ a constant inflow, and $\mathcal{G}$ a locally responsive distributed routing policy. If the limit flow $f^* \in \mathcal{F}$, then it is a globally attractive equilibrium flow for the dynamical network flow (5).

We start by providing a characterization of the strong resilience of the dynamical flow network. Towards this, for a flow network $\mathcal{N}$, and an equilibrium flow vector $f^* \in \mathcal{F}^*$, define the *minimum node residual capacity* as

$$R(\mathcal{N}, f^*) := \min_{0 \leq v < n} \left\{ \sum\nolimits_{e \in \mathcal{E}_v^+} \left(f_{\mathrm{e}}^{\max} - f_e^* \right) \right\} . \tag{15}$$

Theorem 2 (Strong resilience): Let $\mathcal{N}$ be a flow network satisfying Assumptions 1 and 2, $\lambda_0 \geq 0$ a constant inflow, and $\mathcal{G}$ a distributed routing policy. Assume that the associated dynamical flow network (5) admits an equilibrium flow $f^* \in \mathcal{F}^*(\lambda_0)$. Then, its strong resilience satisfies $\gamma_1(f^*, \mathcal{G}) \leq R(\mathcal{N}, f^*)$. Moreover, if $\mathcal{G}$ is locally responsive, then $\gamma_1(f^*, \mathcal{G}) = R(\mathcal{N}, f^*)$.

Proof: See Section VII. ■

For a given flow network $\mathcal{N}$, a constant inflow λ_0, Theorem 2 shows that any locally responsive distributed routing policy $\mathcal{G}$ such that the associated dynamical flow network (5) admits an equilibrium flow $f^* \in \mathcal{F}^*$, has strong resilience $R(\mathcal{N}, f^*)$ larger than or equal to that of any distributed routing policy whose associated dynamical flow network admits the same equilibrium flow f^*. It is worth stressing that the minimum node residual capacity $R(\mathcal{N}, f^*)$ depends both on the flow network $\mathcal{N}$, and on the equilibrium flow f^*. While the proof of upper bound on $\gamma_1(f^*, \mathcal{G})$ for an arbitrary distributed routing policy $\mathcal{G}$ is relatively straightforward (see Lemma 5 in Section VII), the fact that $\gamma_1(f^*, \mathcal{G}) = R(\mathcal{N}, f^*)$ when the distributed routing policy responds to local variations in the density is nontrivial, as illustrated in the following Example.

Example 3: Consider the topology illustrated in Figure 1, with $\lambda_0 = 2$, flow functions as in Example 1 with $a_1 = a_2 = a_3 = a_4 = 1$ and $f_{e_1}^{\max} = f_{e_2}^{\max} = 2$, $f_{e_3}^{\max} = f_{e_4}^{\max} = 0.75$. First consider the case when $G_{e_1}^0(\rho^0) = 1 - G_{e_2}^0(\rho^0) \equiv 0.75$, and $G_{e_3}^1(\rho^1) = 1 - G_{e_4}^1(\rho^1) \equiv 0.5$. One can verify that the associated dynamical flow network has a unique equilibrium flow f^* with $f_{e_1}^* = 1.5$, $f_{e_2}^* = 0.5$, and $f_{e_3}^* = f_{e_3}^* = 0.25$. Now, consider an admissible perturbation such that $\tilde{\mu}_{e_1} = 0.7\mu_{e_1}$ and $\tilde{\mu}_{e_k} = \mu_{e_k}$ for $k = 2, 3, 4$. The magnitude of such perturbation is $\delta = \delta_{e_1} = 0.6$. It is easy to see that in this case $\lim_{t\to\infty} \tilde{f}_{e_1}(t) = 1.4 = \tilde{f}_{e_1}^{\max}$ which is less than 1.5, which is the the flow routed to it. Therefore, $\lim_{t\to\infty} \tilde{\lambda}_2(t) = 1.9 < \lambda_0$, and hence the network is not fully transferring.

Now, consider the same (unperturbed) flow network as before, but with distributed routing policies such that $G_{e_1}^0(\rho^0) = 1 - G_{e_2}^0(\rho^0) = 2e^{-0.031\rho_1}/(2e^{-0.031\rho_1} + e^{0.7196\rho_2})$ and $G_{e_3}^1(\rho^1) = 1 - G_{e_4}^1(\rho^1) \equiv 0.5$. One can verify that the associated dynamical flow network again admits the same f^* as before as an equilibrium flow. Let us consider the same admissible perturbation as before. One can verify that, for the corresponding perturbed dynamical flow network, $\lim_{t\to\infty} \tilde{f}_{e_1}(t) = 0.4 < \tilde{f}_{e_1}^{\max} = 1.4$ and $\lim_{t\to\infty} \tilde{f}_{e_2}(t) = 1.6 < \tilde{f}_{e_2}^{\max} = 2$. However, with an asymptotic arrival rate of 1.6 at node 1, we have that $\lim_{t\to\infty} \tilde{f}_{e_3}(t) = 0.75 = \tilde{f}_{e_3}^{\max}$ and $\lim_{t\to\infty} \tilde{f}_{e_4}(t) = 0.75 = \tilde{f}_{e_4}^{\max}$. Therefore, $\lim_{t\to\infty} \tilde{\lambda}_2(t) = 1.9 < \lambda_0$, and hence the network is not fully transferring.

In both the cases, $R(\mathcal{N}, f^*) = 1$ and a disturbance of magnitude 0.6 is enough to ensure that the perturbed dynamical flow network is not fully transferring. However, note that in the second case, unlike the first case, the routing policy at node 0 responds to variations in the local flow densities by sending more flow to link e_2, but it is *overly* responsive in the sense that it sends more flow downstream than the cumulative flow capacity of the links outgoing from node 1. However, by Definition 2, a distributed routing policy is not allowed any information about

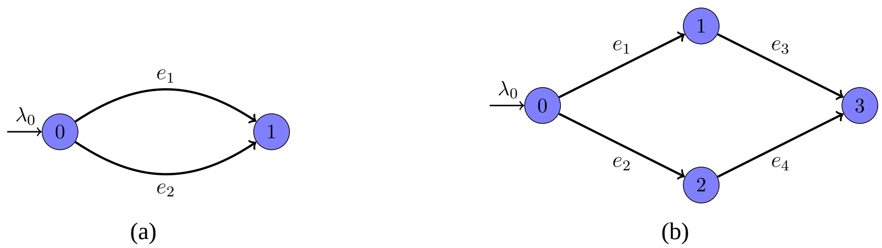

Fig. 2. (a) A parallel link topology. (b) A topology to illustrate arbitrarily large $C(\mathcal{N}) - R(\mathcal{N}, f^*)$.

any other link other than the current flow densities of its outgoing links. This illustrates one of the challenges in designing distributed routing policies which yield $R(\mathcal{N}, f^*)$ as the strong resilience. One can verify that G^0 used in the first case, does not satisfy Property (c) of Definition 8 and, in the second case, it does not satisfy Properties (a) and (c).

Example 3 illustrates that a candidate maximally robust distributed routing policy has to respond to variations in the local flow densities, but not respond excessively. We will formalize these features and show that they are satisfied by locally responsive distributed routing policies. We now pass to the characterization of the weak resilience.

Theorem 3 (Weak resilience): Let $\mathcal{N}$ be a flow network satisfying Assumptions 1 and 2, $\lambda_0 \geq 0$ a constant inflow, and $\mathcal{G}$ a distributed routing policy. Assume that the associated dynamical flow network (5) admits an equilibrium flow $f^* \in \mathcal{F}^*(\lambda_0)$. Then, the weak resilience satisfies $\gamma_w(f^*, \mathcal{G}) \leq C(\mathcal{N})$. Moreover, if $\mathcal{G}$ is locally responsive, then $\gamma_w(f^*, \mathcal{G}) = C(\mathcal{N})$.

Proof: See Section VIII. ■

Theorem 3 shows that, given a flow network $\mathcal{N}$, a constant inflow λ_0, the min-cut capacity $C(\mathcal{N})$ is the maximum weak resilience over all distributed routing policies. It also shows that locally responsive distributed routing policies, as in Definition 8, are maximally robust with respect to the weak resilience notion. Notice that the maximum weak resilience coincides with the min-cut capacity, and hence it depends on the flow network $\mathcal{N}$ only, and not on the initial equilibrium f^*.

A few remarks are in order. First, it is worth comparing strong and weak resilience. Clearly, the former cannot exceed the latter, as can be also directly verified from the definitions (15) and (3): for this, it is sufficient to consider

$$\mathcal{U}^* \in \underset{\mathcal{U} \text{ origin-destination cut}}{\operatorname{argmin}} \left\{ \sum\nolimits_{e \in \mathcal{E}^+_{\mathcal{U}}} f_e^{\max} \right\}, \qquad v^* \in \underset{v \in \mathcal{U}^*}{\operatorname{argmax}} \left\{ \sum\nolimits_{e \in \mathcal{E}^+_v} (f_e^{\max} - f_e^*) \right\},$$

and observe that, since $\sum_{e \in \mathcal{E}^+_{\mathcal{U}^*}} f_e^* = \lambda_0$ by conservation of mass, and $\mathcal{E}^+_{v^*} \subseteq \mathcal{E}^+_{\mathcal{U}^*}$, one has

$$C(\mathcal{N}) - \lambda_0 = \sum\nolimits_{e \in \mathcal{E}^+_{\mathcal{U}^*}} f_{\mathrm{e}}^{\max} - \lambda_0 = \sum\nolimits_{e \in \mathcal{E}^+_{\mathcal{U}^*}} (f_{\mathrm{e}}^{\max} - f_e^*) \geq \sum\nolimits_{e \in \mathcal{E}^+_{v^*}} (f_{\mathrm{e}}^{\max} - f_e^*) = R(\mathcal{N}, f^*).$$

We provide below two examples to illustrate the difference between the two quantities.

Example 4: For parallel link topologies, an example of which is illustrated in Figure 2 (a), one has that $R(\mathcal{N}, f^*) = \sum_{e \in \mathcal{E}} f_{\mathrm{e}}^{\max} - \lambda_0 = C(\mathcal{N}) - \lambda_0$.

Example 5: Consider the topology shown in Figure 2 (b) with $\lambda_0 = 1$, $f^* = [\epsilon, 1-\epsilon, \epsilon, 1-\epsilon]$ and $f_{\mathrm{e}}^{\max} = [1/\epsilon, 1, 1/\epsilon, 1]$ for some $\epsilon \in (0,1)$. In this case, we have that $C(\mathcal{N}) = 1 + 1/\epsilon$ and $R(\mathcal{N}, f^*) = \epsilon$. Therefore, $C(\mathcal{N}) - R(\mathcal{N}, f^*) = 1 + 1/\epsilon - \epsilon$, and hence $C(\mathcal{N}) - R(\mathcal{N}, f^*)$ grows unbounded as ϵ vanishes.

We conclude this section with the following observation. Using arguments along the lines of those employed in [8], it is not hard to show that $C(\mathcal{N}) - \lambda_0$ provides an upper bound on the strong resilience even if the locality constraint on the information used by the routing policies is removed, i.e., if one allows G^v to depend on the full vector of current densities ρ, rather than on the local density vector ρ^v only. Indeed, one might exhibit routing policies which are functions of the global density information ρ, for which the strong resilience is exactly $C(\mathcal{N}) - \lambda_0$ using ideas developed in the companion paper [8]. Hence, one may interpret the gap $C(\mathcal{N}) - \lambda_0 - R(\mathcal{N}, f^*)$ as the strong resilience loss due to the locality constraint on the information available to the distributed routing policies.

One could use Example 5 to again demonstrate arbitrarily large such loss due to the locality constraint on the information available to the routing policies. In fact, it is possible to consider intermediate levels of information available to the routing policies, which interpolate between the one-hop information of our current modeling of the network, and the global information described above.

IV. Robust equilibrium selection

In this section, for a given flow network $\mathcal{N}$ satisfying Assumptions 1 and 2, a constant inflow $\lambda_0 \in [0, C(\mathcal{N}))$, and locally responsive distributed routing policies, we shall address the issue of optimizing the maximum strong resilience of the associated dynamical flow network, $R(\mathcal{N}, f^*)$ with respect to the initial equilibrium flow f^* (recall that the corresponding weak resilience $C(\mathcal{N})$ is independent of f^*). First, in Section IV-A, we shall address the issue of maximizing $R(f^*) := R(\mathcal{N}, f^*)$ over all admissible equilibrium flow vectors $f^* \in \mathcal{F}^*(\lambda_0)$, i.e., with the only constraints given by the link capacities and the conservation of mass. Then, in Section IV-B we shall focus on the transportation network case of Section II-B, and address the problem of optimizing $R(f^*)$ indirectly, assuming that f^* satisfies the additional constraint of being an equilibrium influenced by some static tolls. Finally, in Section IV-C, we shall evaluate the gap between the strong resilience associated to the maximizer of $R(f^*)$ and a generic equilibrium f^*, and interpret it as the robustness price of anarchy with respect to f^*.

A. *Robust equilibrium selection as an optimization problem*

The robust initial equilibrium condition selection problem can be posed as an optimization problem as follows:

$$R^* := \sup_{f^* \in \mathcal{F}^*(\lambda_0)} R(f^*) \,, \tag{16}$$

where we recall that $\mathcal{F}^*(\lambda_0)$ is the set of admissible equilibrium flow vectors corresponding to the inflow $\lambda_0 \in [0, C(\mathcal{N}))$. Equation (15) implies that $R(f^*)$ is the minimum of a set of functions linear in f^*, and hence is concave in f^*. Since the closure of the constraint set $\mathcal{F}^*(\lambda_0)$ is a polytope, we get that the optimization problem stated in (16) is equivalent to a simple convex optimization problem. However, note that the objective function, $R(f^*)$ is non-smooth and one needs to use sub gradient techniques, e.g., see [25], for finding the optimal solution.

B. *Using tolls for equilibrium implementation in transportation networks*

In this section, we study the use of static tolls to influence the decisions of the drivers in order to get a desired emergent equilibrium condition for (unperturbed) transportation networks. The static tolls affect the driver decisions over a slower time scale at which the drivers update their preferences for global paths through the network. These global decisions are complemented by the *fast-scale* node-wise route choice decisions characterized by Definition 2 and 8. The details of the analysis of the transportation network with such two time-scale driver decisions can be found in our companion paper [8]. In particular, we show that when the time scales are sufficiently separated apart, then the network densities converge to a neighborhood of Wardrop equilibrium. In this section, in order to highlight the relationship between static tolls and the resultant equilibrium point, we assume that the fast scale dynamics equilibrates quickly and focus only on the slow scale dynamics.

We briefly describe the congestion game framework for transportation networks to formalize the equilibrium corresponding to the slow scale driver decision dynamics. Let $\Upsilon \in \mathcal{R}$ be the link-wise vector of tolls, with Υ_e denoting the toll on link e. Assuming that Υ is rescaled in such a way that one unit of toll corresponds to a unit amount of delay, the utility of a driver associated with link e when the flow on it is f_e is $-\left(T_e(f_e) + \Upsilon_e\right)$, where $T_e(f_e)$ is the delay on link $e \in \mathcal{E}$ when the flow through it is f_e. Let $\mathcal{P}$ be the set of distinct *paths* from node 0 to node n. The utility associated with a path $p \in \mathcal{P}$ is $-\sum_{e \in p}\left(T_e\left(f_e\right) + \Upsilon_e\right)$. In order to formally describe the functions $T_e(f_e)$, we shall assume that each flow function μ_e satisfies Assumption 2, and additionally is strictly concave and satisfies $\mu_e'(0) < +\infty$. Observe that the flow function described in Example 1 satisfies these additional assumptions. Since the flow on a link is the product of speed and density on that link, one can define the link-wise delay functions T_e by

$$T_e(f_e) := \begin{cases} +\infty & \text{if } f_e \geq f_{\mathrm{e}}^{\max}, \\ \mu_e^{-1}(f_e)/f_e & \text{if } f_e \in (0, f_{\mathrm{e}}^{\max}), \\ 1/\mu_e'(0) & \text{if } f_e = 0, \end{cases} \qquad \forall e \in \mathcal{E}. \tag{17}$$

Let $T(f) = \{T_e(f_e) : e \in \mathcal{E}\}$ be the vector of link-wise delay functions. We are now ready to define a *toll-induced* equilibrium.

Definition 9 (Toll-induced equilibrium): For a given $\Upsilon \in \mathcal{R}$, a toll-induced equilibrium is a vector $f^*(\Upsilon) \in \mathcal{F}^*$ that satisfies the following for all $p \in \mathcal{P}$:

$$f_e > 0 \quad \forall e \in p \Longrightarrow \sum_{e \in p} \left(T_e\left(f_e\right) + \Upsilon_e\right) \leq \sum_{e \in q} \left(T_e\left(f_e\right) + \Upsilon_e\right) \quad \forall q \in \mathcal{P}.$$

Note that, $f^*(\mathbf{0})$ corresponds to a Wardrop equilibrium, e.g., see [26], [21], where $\mathbf{0}$ is a vector all of whose entries are zero. For brevity in notation, we shall denote the Wardrop equilibrium. The following result guarantees the existence and uniqueness of a toll-induced equilibrium.

Proposition 1 (Existence and uniqueness of toll-induced equilibrium): Let $\mathcal{N}$ be a flow network satisfying Assumptions 1 and 2 and $\lambda_0 \in [0, C(\mathcal{N}))$ a constant inflow. Assume additionally that the flow function μ_e is strictly concave and satisfies $\mu_e'(0) < +\infty$ for every link $e \in \mathcal{E}$. Then, for every toll vector $\Upsilon \in \mathcal{R}$, there exists a unique toll-induced equilibrium $f^*(\Upsilon) \in \mathcal{F}^*$.

Proof: It follows from Assumption 2, strict concavity and the assumption $\mu_e'(0) < +\infty$ on the flow functions that, for all $e \in \mathcal{E}$, the delay function $T_e(f_e)$, as defined by (17), is continuous, strictly increasing, and is such that $T_e(0) > 0$. The Proposition then follows by applying Theorems 2.4 and 2.5 from [27]. ■

In this subsection, to illustrate the proof of concept, we will focus on equilibrium flows f^* each of whose components is strictly positive and less than the flow capacities of the corresponding links. Let $A \in \{0,1\}^{\mathcal{P} \times \mathcal{E}}$ be the path-link incidence matrix, i.e., for all $e \in \mathcal{E}$ and $p \in \mathcal{P}$, $A_{p,e} = 1$ if $e \in p$ and zero otherwise. The results for a generic $f^* \in \mathcal{F}^*$ follow along similar lines. Definition 9 implies that for $f^*(\Upsilon) \in \mathcal{R}$, with $f_e^*(\Upsilon) > 0$ for all $e \in \mathcal{E}$, to be the toll-induced equilibrium corresponding to the toll vector $\Upsilon \in \mathcal{R}$ is equivalent to $A\left(T(f^*(\Upsilon)) + \Upsilon\right) = \nu \mathbf{1}$, for some $\nu > 0$. We shall use this fact in the next result, where we compute tolls to get a desired equilibrium.

Proposition 2 (Tolls for desired equilibrium): Let $\mathcal{N}$ be a flow network satisfying Assumptions 1 and 2 and $\lambda_0 \in [0, C(\mathcal{N}))$ a constant inflow. Assume additionally that the flow function μ_e is strictly concave and satisfies $\mu_e'(0) < +\infty$ for every link $e \in \mathcal{E}$. Assume that the Wardrop equilibrium f^{W} is such that $f_e^{\mathrm{W}} > 0$ for all $e \in \mathcal{E}$. Let $f^* \in \mathcal{F}^*$, with $f_e^* \in (0, f_{\mathrm{e}}^{\max})$ for all $e \in \mathcal{E}$, be the desired toll-induced equilibrium flow vector. Define $\Upsilon(f) \in \mathcal{R}$ by

$$\Upsilon(f) = \left(\max_{e \in \mathcal{E}} \frac{T_e(f_e)}{T_e(f_e^{\mathrm{W}})}\right) T(f^{\mathrm{W}}) - T(f)\,. \tag{18}$$

Then f^* is the desired toll-induced equilibrium associated to the toll vector $\Upsilon(f^*)$.

Proof: Since f^W is the Wardrop equilibrium, corresponding to the toll vector $\Upsilon = \mathbf{0}$, we have that

$$AT(f^{\mathrm{W}}) = \nu_1 \mathbf{1}, \tag{19}$$

for some $\nu_1 > 0$. For f^* to be the toll-induced equilibrium associated to the toll vector $\Upsilon \in \mathcal{R}$, one needs to find $\nu_2 > 0$ such that

$$A\left(T(f^*) + \Upsilon\right) = \nu_2 \mathbf{1}. \tag{20}$$

Using (19) and simple algebra, one can verify that (20) is satisfied with $\Upsilon(f^*)$ as defined in (18) and $\nu_2 = \nu_1 \cdot \left(\max_{e \in \mathcal{E}} \frac{T_e(f_e^*)}{T_e(f_e^{\mathrm{W}})}\right)$. ■

Remark 5: The toll vector yielding a desired equilibrium operating condition is not unique. In fact, any toll of the form $\Upsilon(f^*) = cT(f^{\mathrm{W}}) - T(f^*)$, with $c \geq \max\{T_e(f_e^*)/T_e(f_e^{\mathrm{W}}) : e \in \mathcal{E}\}$ would induce f^* as the toll-induced equilibrium. Proposition 2 gives just one such toll vector.

C. *The robustness price of anarchy*

Conventionally, transportation networks have been viewed as static flow networks, where a given equilibrium traffic flow is an outcome of driver's selfish behavior in response to the delays associated with various paths and the incentive mechanisms in place. The price of anarchy [28] has been suggested as a metric to measure how sub-optimal a given equilibrium is with respect to the societal optimal equilibrium, where the societal optimality is related to the average delay faced by a driver. In the context of robustness analysis of transportation networks, it is natural to consider societal optimality from the robustness point of view, thereby motivating a notion of the robustness price of anarchy. Formally, for a $f^* \in \mathcal{F}^*(\lambda_0)$, define the robustness price of anarchy as $P\left(f^*\right) := R^* - R\left(f^*\right)$. It is worth noting that, for a parallel topology, we have that $R^* = R\left(f^*\right) = \sum_{e \in \mathcal{E}} f_{\mathrm{e}}^{\max} - \lambda_0$ for all f^*. That is, the strong resilience is independent of the equilibrium operating condition and hence, for a parallel topology, $P\left(f^*\right) \equiv 0$. However, for a general topology and a general equilibrium, this quantity is non-zero. This can be easily justified, for example, for robustness price of anarchy with respect to the Wardrop equilibrium: a Wardrop equilibrium is determined by the delay functions $T_e(f_e)$ as well as the topology of the network, whereas the maximizer of $R(f^*)$ depends only on the topology and the link-wise flow capacities of the network, as implied by the optimization problem in (16). In fact, as the following example illustrates, for a non-parallel topology, the robustness price of anarchy with respect to Wardrop equilibrium can be arbitrarily large.

Example 6 (Arbitrarily large robustness price of anarchy with respect to Wardrop equilibrium): Consider the network topology shown in Figure 1. Let the link-wise flow functions be the one given by Example 1. The delay function is then given by $T_e(0) = (a_e f_{\mathrm{e}}^{\max})^{-1}$, $T_e(f_e) = -\frac{1}{a_e f_e} \log(1 - f_e / f_{\mathrm{e}}^{\max})$ for $f_e \in (0, f_{\mathrm{e}}^{\max})$ and $T_e(f_e) = +\infty$ for $f_e \geq f_{\mathrm{e}}^{\max}$. Fix some $\epsilon \in (0,1)$ and let $\lambda_0 = 1/\epsilon$. Let the parameters of the flow functions be given by $f_{e_1}^{\max} = f_{e_2}^{\max} = 1/\epsilon + \epsilon$, $f_{e_3}^{\max} = f_{e_4}^{\max} = 1/(2\epsilon) + \epsilon/2$, $a_1 = 1$, $a_2 = a_3 = a_4 = \left(\frac{3\epsilon}{1-\epsilon}\right) \log\left(\frac{\epsilon + \epsilon^2}{1+\epsilon^2}\right) / \log\left(\frac{1+\epsilon^2-\epsilon}{1+\epsilon^2}\right)$. For these values of the parameters, one can verify that the unique Wardrop equilibrium is given by $f^{\mathrm{W}} = [1 \quad 1/\epsilon - 1 \quad 1/(2\epsilon) - 1/2 \quad 1/(2\epsilon) - 1/2]^T$. The strong resilience of f^W is then given by $R(\mathcal{N}, f^{\mathrm{W}}) = \min\{2/\epsilon + 2\epsilon - 1/\epsilon, 1/\epsilon + \epsilon - (1/\epsilon - 1)\} = 1 + \epsilon$. One can also verify that, for this case, $R^* = 1/\epsilon + 2\epsilon$ which would correspond to $f^* = [1/\epsilon \quad 0 \quad 0 \quad 0]^T$. Therefore, $P(f^{\mathrm{W}}) = 1/\epsilon + 2\epsilon - (1+\epsilon) = 1/\epsilon + \epsilon - 1$ which tends to $+\infty$ as $\epsilon \to 0^+$.

V. Simulations

In this section, through numerical experiments, we study the case when the flow functions are set to the ones commonly accepted in the transportation literature, e.g., see [4]. In transportation literature, the flow functions are defined over a finite interval of the form $[0, \rho_e^{\max}]$, where $\rho_e^{\max}$ is the maximum traffic density that link e can handle. Additionally, μ_e is assumed to be strictly concave and achieves its maximum in $(0, \rho_e^{\max})$. For example, consider the following:

$$\mu_e(\rho_e) = \frac{4 f_{\mathrm{e}}^{\max} \rho_e (\rho_e^{\max} - \rho_e)}{(\rho_e^{\max})^2}, \quad \rho_e \in [0, \rho_e^{\max}]. \tag{21}$$

An important implication of the finite capacity on the traffic densities is the possibility of cascaded *spill-backs* traveling upstream as follows. When the density on a link reaches its capacity, its outflow permanently becomes zero and hence the link is effectively cut out from the network. When all the outgoing links from a particular node are cut out, it makes the outflow on all the incoming links to that node zero. Eventually, these *upstream* links might possibly reach their capacity on the density and cutting themselves off permanently and cascading the effect further upstream. We shall show how such cascaded effects possibly reduce the resilience.

Another important differentiating feature of the flow functions given by (21) with respect to the flow functions satisfying Assumption 2 is that the flow functions corresponding to (21) are not strictly increasing. As a result, one cannot readily claim that the locally responsive distributed routing policies are maximally robust for this case. However, we illustrate via simulations that, with additional assumptions, the locally responsive distributed routing policies considered in this paper could possibly be maximally robust. However, one can show that the upper bound on the strong and weak resiliences, as given by Theorems 2 and 3 hold true even in this case. For the simulations, we selected the following parameters:

- the graph topology $\mathcal{T}$ shown in Figure 3.
- $\lambda_0 = 3$.
- let $\rho_e^{\max} = 3$ for all $e \in \mathcal{E}$, and flow capacities given by $f_{e_1}^{\max} = f_{e_2}^{\max} = f_{e_3}^{\max} = 2.5$, $f_{e_4}^{\max} = 0.9$, $f_{e_5}^{\max} = 1.75$, $f_{e_6}^{\max} = f_{e_{11}}^{\max} = f_{e_{13}}^{\max} = 1$, $f_{e_7}^{\max} = f_{e_8}^{\max} = 0.7$, $f_{e_9}^{\max} = 0.4$, $f_{e_{10}}^{\max} = f_{e_{12}}^{\max} = 1.5$, $f_{e_{14}}^{\max} = 2$, and $f_{e_{15}}^{\max} = 1.6$. The link-wise flow functions are as given in (21), if $e \in \mathcal{E}_n^-$ or if $\rho < \rho_{e'}^{\max}$ for at least one *downstream* edge e', i.e., $e' \in \mathcal{E}$ such that $e \in \mathcal{E}_v^-$ and $e' \in \mathcal{E}_v^+$ for some $v \in \{1, \ldots, n-1\}$, and the flow functions are uniformly zero otherwise;
- the equilibrium flow f^* has components $f_{e_1}^* = f_{e_3}^* = f_{e_6}^* = 0.5$, $f_{e_2}^* = 2$, $f_{e_4}^* = f_{e_{13}}^* = 0.3$, $f_{e_5}^* = 1.5$, $f_{e_7}^* = f_{e_8}^* = 0.25$, $f_{e_9}^* = 0.2$, $f_{e_{10}}^* = f_{e_{12}}^* = 0.9$, $f_{e_{11}}^* = 0.2$, $f_{e_{13}}^* = 0.3$, $f_{e_{14}}^* = 1.1$, and $f_{e_{15}}^* = 0.7$;

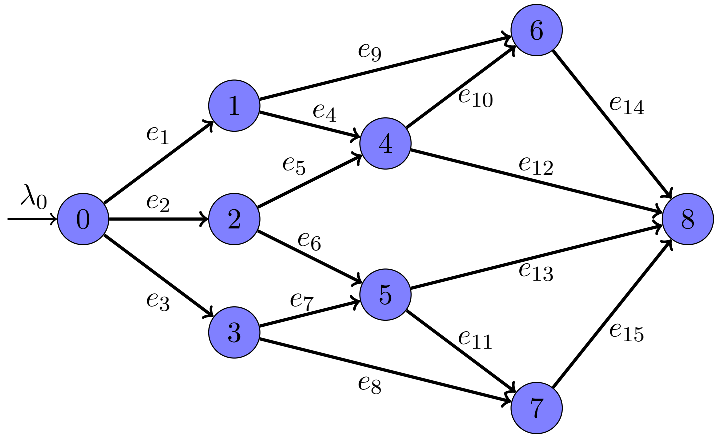

Fig. 3. The graph topology used in simulations.

- for the route choice function, a modified version of (13) is used. The modification is done to respect the finite traffic density constraint on the links. The modified route choice policy is

$$G_e^v(\rho^v) = \frac{f_e^* \exp(-\eta(\rho_e - \rho_e^*))\mathbb{1}_{[0,\rho_e^{\max}]}(\rho_e)}{\sum_{j\in\mathcal{E}_v^+} f_j^* \exp(-\eta(\rho_j - \rho_j^*))\mathbb{1}_{[0,\rho_j^{\max}]}(\rho_j)},$$

  where η will be a variable parameter for the simulations.

One can verify that, with these parameters, the minimum node residual capacity, and hence an upper bound on the strong resilience, as defined by (15) is 0.75. One can also verify that the maximum flow capacity of the network, and hence an upper bound on the weak resilience, is 5.2.

A. Effect of η on the strong resilience

Consider an admissible perturbation such that $\tilde{\mu}_{e_{10}} = \frac{8}{15}\mu_{e_{10}}$ and $\tilde{\mu}_{e_k} = \mu_k$ for all $k \in \{1,\ldots,15\} \setminus \{10\}$. As a result, $\delta_{e_{10}} = 0.7$ and $\delta_{e_k} = 0$ for all $k \in \{1,\ldots,15\} \setminus \{10\}$. Therefore, the magnitude of the perturbation is $\delta = 0.7$. Note that this value is less than the minimum node residual capacity of the network. We found that $\lim_{t\to\infty}\lambda_{e_8}(t) = 0$ for all $\eta < 0.25$, and $\lim_{t\to\infty}\lambda_{e_8}(t) = \lambda_0 = 3$ for all $\eta \geq 0.25$. The role of η in the strong resilience is best understood by concentrating on a parallel topology consisting of edges e_{10} and e_{12} with arrival rate $\lambda_{e_4}^*$. Using similar techniques as in the proof of Theorem 2, one can show the existence of a new equilibrium for this *local* system. However, this equilibrium is not attractive from a configuration where at least one of $\tilde{\rho}_{e_{10}}$ or $\tilde{\rho}_{e_{12}}$ is at $\rho_{e_{10}}^{\max}$ or $\rho_{e_{12}}^{\max}$, respectively. For $\eta < 0.25$, $\tilde{\rho}_{e_{10}}$ reaches $\rho_{e_{10}}^{\max}$, whereas for $\eta \geq 0.25$, neither $\tilde{\rho}_{e_{10}}$ nor $\tilde{\rho}_{e_{12}}$ hit the maximum density capacity and the system is attracted towards the new equilibrium.

B. Effect of cascaded shutdowns on the weak resilience

Consider an admissible disturbance such that $\tilde{\mu}_{e_4} = \frac{2}{9}\mu_{e_4}$, $\tilde{\mu}_{e_5} = \frac{23}{35}\mu_{e_5}$, $\tilde{\mu}_{e_6} = \frac{4}{5}\mu_6$, $\tilde{\mu}_{e_7} = \frac{2}{7}\mu_{e_7}$, $\tilde{\mu}_{e_8} = \frac{2}{7}\mu_{e_8}$, $\tilde{\mu}_{e_9} = \frac{1}{2}\mu_{e_9}$, $\tilde{\mu}_{e_{10}} = \frac{3}{5}\mu_{e_{10}}$, $\tilde{\mu}_{e_{12}} = \frac{8}{15}\mu_{e_{12}}$ and $\tilde{\mu}_k = \mu_k$ for $k = \{1,2,3,11,13,14,15\}$. As result, $\delta_{e_4} = 0.7$, $\delta_{e_5} = 0.6$, $\delta_{e_6} = 0.2$, $\delta_{e_7} = 0.5$, $\delta_{e_8} = 0.5$, $\delta_{e_9} = 0.2$, $\delta_{e_{10}} = 0.6$, $\delta_{e_{12}} = 0.7$ and $\delta_{e_k} = 0$ for $k = \{1,2,3,11,13,14,15\}$. Therefore, $\delta = 4$, which is less than the min-cut flow capacity of the network. For this case, it is observed that, $\lim_{t\to\infty}\lambda_{e_8}(t) = 0$ independent of the value of η. This can be explained as follows. For the given disturbance, we have that $\tilde{f}_{e_{10}}^{\max} + \tilde{f}_{e_{12}}^{\max} = 1.7 < 1.8 = f_{e_{10}}^* + f_{e_{12}}^*$. Therefore, after finite time t_1, we have that $\tilde{\rho}_{e_{10}}(t) = \rho_{e_{10}}^{\max}$ and $\tilde{\rho}_{e_{12}}(t) = \rho_{e_{12}}^{\max}$ for all $t \geq t_1$. As a consequence, we have that, $\tilde{f}_{e_4}(t) = 0$ and $\tilde{f}_{e_5}(t) = 0$ for all $t \geq t_1$. One can repeat this argument to conclude that, for the given disturbance, after finite time, $\tilde{\rho}_{e_k}$ for $k = 1,\ldots,9$ reach and remain at their maximum density capacities. As a consequence, after such a finite time, $\tilde{f}_{e_1}(t) + \tilde{f}_{e_2}(t) + \tilde{f}_{e_3}(t) = 0$ and hence, $\lim_{t\to\infty}\lambda_{e_8}(t) = 0$, i.e., the network is not partially transferring. This example illustrates that the cascaded effects can potentially reduce the weak resilience of a dynamical flow network.

VI. Proof of Theorem 1

Let $\mathcal{N}$ be a flow network satisfying Assumptions 1 and 2, $\mathcal{G}$ a locally responsive distributed routing policy, and $\lambda_0 \geq 0$ a constant inflow. We shall prove that there exists a unique $f^* \in \mathrm{cl}(\mathcal{F})$ such that the flow $f(t)$ associated to the solution of the dynamical flow network (5) converges to f^* as t grows large, for every initial condition $\rho(0) \in \mathcal{R}$. We shall proceed by proving a series of intermediate results some of which will prove useful also in the following sections.

First, given an arbitrary non-destination node $0 \leq v < n$, we shall focus on the input-output properties of the *local system*

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_e(t) = \lambda(t) G_e^v(\rho^v(t)) - f_e(t)\,, \qquad f_e(t) = \mu_e(\rho_e(t))\,, \qquad \forall e \in \mathcal{E}_v^+\,, \tag{22}$$

where $\lambda(t)$ is a nonnegative-real-valued, Lipschitz continuous input, and $f^v(t) := \{f_e(t) : e \in \mathcal{E}_v^+\}$ is interpreted as the output. We shall first prove existence (and uniqueness) of a globally attractive limit flow for the system (22) under constant input. We shall then extend this result to show the existence and attractivity of a local equilibrium point under time-varying, convergent local input. Finally, we shall exploit this local input-output property, and the assumption of acyclicity of the network topology in order to establish the main result.

The following is a simple technical result, which will prove useful in order to apply Property (a) of Definition 8.

Lemma 1: Let $0 \leq v < n$ be a nondestination node, and $G^v : \mathcal{R}_v \to \mathcal{S}_v$ a continuously differentiable function satisfying Property (a) of Definition 8. Then, for any $\sigma, \varsigma \in \mathcal{R}_v$,

$$\sum\nolimits_{e \in \mathcal{E}_v^+} \mathrm{sgn}(\sigma_e - \varsigma_e)\left(G_e^v(\sigma) - G_e^v(\varsigma)\right) \leq 0. \tag{23}$$

Proof: Consider the sets $\mathcal{K} := \{e \in \mathcal{E}_v^+ : \sigma_e > \varsigma_e\}$, $\mathcal{J} := \{e \in \mathcal{E}_v^+ : \sigma_e \leq \varsigma_e\}$, and $\mathcal{L} := \{e \in \mathcal{E}_v^+ : \sigma_e < \varsigma_e\}$. Define $G_{\mathcal{K}}(\zeta) := \sum_{k \in \mathcal{K}} G_k^v(\zeta)$, $G_{\mathcal{L}}(\zeta) := \sum_{l \in \mathcal{L}} G_l^v(\zeta)$, and $G_{\mathcal{J}}(\zeta) := \sum_{j \in \mathcal{J}} G_j^v(\zeta)$. We shall show that, for any $\sigma, \varsigma \in \mathcal{R}_v$,

$$G_{\mathcal{K}}(\sigma) \leq G_{\mathcal{K}}(\varsigma)\,, \qquad G_{\mathcal{L}}(\sigma) \geq G_{\mathcal{L}}(\varsigma)\,. \tag{24}$$

Let $\xi \in \mathcal{R}_v$ be defined by $\xi_k = \sigma_k$ for all $k \in \mathcal{K}$, and $\xi_e = \varsigma_e$ for all $e \in \mathcal{E}_v^+ \setminus \mathcal{K}$. We shall prove that $G_{\mathcal{K}}(\sigma) - G_{\mathcal{K}}(\varsigma) \leq 0$ by writing it as a path integral of $\nabla G_{\mathcal{K}}(\zeta)$ first along the segment $S_{\mathcal{K}}$ from ς to ξ, and then along the segment $S_{\mathcal{L}}$ from ξ to σ. Proceeding in this way, one gets

$$G_{\mathcal{K}}(\sigma) - G_{\mathcal{K}}(\varsigma) = \int_{S_{\mathcal{K}}} \nabla G_{\mathcal{K}}(\zeta) \cdot \mathrm{d}\zeta + \int_{S_{\mathcal{L}}} \nabla G_{\mathcal{K}}(\zeta) \cdot \mathrm{d}\zeta = -\int_{S_{\mathcal{K}}} \nabla G_{\mathcal{J}}(\zeta) \cdot \mathrm{d}\zeta + \int_{S_{\mathcal{L}}} \nabla G_{\mathcal{K}}(\zeta) \cdot \mathrm{d}\zeta\,, \tag{25}$$

where the second equality follows from the fact that $G_{\mathcal{K}}(\zeta) = 1 - G_{\mathcal{J}}(\zeta)$ since $G^v(\zeta) \in \mathcal{S}_v$. Now, Property (a) of Definition 8 implies that $\partial G_{\mathcal{K}}(\zeta)/\partial \zeta_l \geq 0$ for all $l \in \mathcal{L}$, and $\partial G_{\mathcal{J}}(\zeta)/\partial \zeta_k \geq 0$ for all $k \in \mathcal{K}$. It follows that $\nabla G_{\mathcal{J}}(\zeta) \cdot \mathrm{d}\zeta \geq 0$ along $S_{\mathcal{K}}$, and $\nabla G_{\mathcal{K}}(\zeta) \cdot \mathrm{d}\zeta \leq 0$ along $S_{\mathcal{L}}$. Substituting in (25), one gets the first inequality in (24). The second inequality in (24) follows by similar arguments. Then, one has

$$\sum\nolimits_{e \in \mathcal{E}_v^+} \mathrm{sgn}(\sigma_e - \varsigma_e)\left(G_e^v(\sigma) - G_e^v(\varsigma)\right) = G_{\mathcal{K}}(\sigma) - G_{\mathcal{K}}(\varsigma) + G_{\mathcal{L}}(\varsigma) - G_{\mathcal{L}}(\sigma) \leq 0\,,$$

which proves the claim. ■

We can now exploit Lemma 1 in order to prove the following key result guaranteeing that the solution of the local dynamical system (22) with constant input $\lambda(t) \equiv \lambda$ converges to a limit point which depends on the value of λ but not on the initial condition. For the ease of presentation, let us define

$$\lambda^{\max} := \sum\nolimits_{e \in \mathcal{E}_v^+} f_e^{\max}\,.$$

Lemma 2: Let $0 \leq v < n$ be a non-destination node, and λ a nonnegative-real constant. Assume that $G^v : \mathcal{R}_v \to \mathcal{S}_v$ is continuously differentiable and satisfies Properties (a) and (c) of Definition 8. Then, there exists a unique $f^*(\lambda) \in \mathrm{cl}(\mathcal{F}_v)$ such that, for every initial condition $\rho^v(0) \in \mathcal{R}_v$, the solution of the dynamical system (22) with constant input $\lambda(t) \equiv \lambda$ satisfies

$$\lim_{t \to +\infty} f_e(t) = f_e^*(\lambda)\,, \qquad \forall e \in \mathcal{E}_v^+\,.$$

Moreover, if $\lambda < \lambda^{\max}$, then $f_e^*(\lambda) < \lambda_e^{\max}$, and $\lambda G_e^v(\mu^{-1}(f^*(\lambda))) = f_e^*$, for every $e \in \mathcal{E}_v^+$; if $\lambda \geq \lambda^{\max}$, then $f_e^* = f_e^{\max}$, for every $e \in \mathcal{E}_v^+$. Finally, $f^*(\lambda)$ is continuous as a map from $\mathbb{R}_+$ to $\mathrm{cl}(\mathcal{F}_v)$.

Proof: Let us fix some $\lambda \in \mathbb{R}_+$. For every initial condition $\sigma \in \mathcal{R}_v$, and time $t \geq 0$, let $\Phi^t(\sigma) := \rho^v(t)$ be the value of the solution of (22) with constant input $\lambda(t) \equiv \lambda$ and initial condition $\rho(0) = \sigma$, at time $t \geq 0$. Also, let $\Psi^t(\sigma) \in \mathcal{R}_v$ be defined by $\Psi^t_e(\sigma) = \mu_e(\Phi^t_e(\sigma))$, for every $e \in \mathcal{E}^+_v$. Now, fix two initial conditions $\sigma, \varsigma \in \mathcal{R}_v$, and define $\chi(t) := ||\Phi^t(\sigma) - \Phi^t(\varsigma)||_1$ and $\xi(t) := ||\Psi^t(\sigma) - \Psi^t(\varsigma)||_1$. Since $\mu_e(\rho_e)$ satisfies Assumption 2, one has that

$$\mathrm{sgn}(\Phi^t_e(\sigma) - \Phi^t_e(\varsigma)) = \mathrm{sgn}(\Psi^t_e(\sigma) - \Psi^t_e(\varsigma))\,. \tag{26}$$

On the other hand, using Lemma 1, one gets

$$\sum\nolimits_{e\in\mathcal{E}^+_v} \mathrm{sgn}(\Phi^t_e(\sigma) - \Phi^t_e(\varsigma)) \left(G^v_e(\Phi^t(\sigma)) - G^v_e(\Phi^t(\varsigma))\right) \leq 0\,, \qquad \forall\, t \geq 0\,. \tag{27}$$

From (26) and (27), it follows that, for all $t \geq 0$,

$$\begin{array}{rcl} \chi(t) & = & ||\Phi^t(\varsigma) - \Phi^t(\sigma)||_1 \\ & = & \chi(0) + \displaystyle\int_0^t \sum\nolimits_{e\in\mathcal{E}^+_v} \mathrm{sgn}(\Phi^s_e(\sigma) - \Phi^s_e(\varsigma))\big(G^v_e(\Phi^s(\sigma)) - G^v_e(\Phi^s(\varsigma)) - \Psi^s_e(\sigma) + \Psi^s_e(\varsigma)\big)\mathrm{d}s \\ & \leq & \chi(0) - \displaystyle\int_0^t ||\Psi^s(\sigma) - \Psi^s(\varsigma)||_1 \mathrm{d}s \\ & = & \chi(0) - \displaystyle\int_0^t \xi(s)\mathrm{d}s\,. \end{array} \tag{28}$$

Since $\chi(t) \geq 0$, (28) implies that $\int_0^t \xi(s)\mathrm{d}s \leq \chi(0)$ for all $t \geq 0$. Passing to the limit of large t, one gets $\int_0^{+\infty} \xi(s)\mathrm{d}s \leq \chi(0) < +\infty$. This, and the fact that $\xi(s) \geq 0$ for all $s \geq 0$, readily imply that $\xi(t)$ converges to 0, as t grows large. That is,

$$\lim_{t\to+\infty} ||\Psi^t(\sigma) - \Psi^t(\varsigma)||_1 = 0\,, \qquad \forall \sigma, \varsigma \in \mathcal{R}_v\,. \tag{29}$$

Now, for any $\sigma \in \mathcal{R}_v$, one can apply (29) with $\varsigma := \Phi^\tau(\sigma)$, and get that

$$\lim_{t\to+\infty} ||\Psi^t(\sigma) - \Psi^{t+\tau}(\sigma)||_1 = \lim_{t\to+\infty} ||\Psi^t(\sigma) - \Psi^t(\Phi^\tau(\sigma))||_1 = 0\,, \qquad \forall \tau \geq 0\,.$$

The above implies that, for any initial condition $\rho^v(0) = \sigma \in \mathcal{R}_v$, the flow $\Psi^t(\sigma)$ is Cauchy, and hence convergent to some $f^*(\lambda, \sigma) \in \mathrm{cl}(\mathcal{F}_v)$. Define $\rho^*(\lambda, \sigma) = \rho^* \in \mathcal{R}_v$ by

$$\rho^*_e := \begin{cases} \mu_e^{-1}(f^*_e(\lambda, \sigma)) & \text{if} \quad f^*_e(\lambda, \sigma) < f^{\max}_e \\ +\infty & \text{if} \quad f^*_e(\lambda, \sigma) = f^{\max}_e\,. \end{cases}$$

Now, by contradiction, assume that there exists a nonempty proper subset $\mathcal{J} \subset \mathcal{E}^+_v$ such that $\rho^*_j < +\infty$ for every $j \in \mathcal{J}$, and $\rho^*_e = +\infty$ for every $k \in \mathcal{K} := \mathcal{E}^+_v \setminus \mathcal{J}$. Thanks to Property (c) of Definition 8, one would have that

$$\lim_{t\to+\infty} \sum\nolimits_{k\in\mathcal{K}} \lambda G^v_k(\rho^v(t)) - f_k(t) = -\sum\nolimits_{k\in\mathcal{K}} f^{\max}_k < 0\,,$$

so that there exists some $\tau \geq 0$ such that $\sum_{k\in\mathcal{K}} (\lambda G^v_k(\rho^v(t)) - f_k(t)) \leq 0$ for all $t \geq \tau$. Hence,

$$\sum\nolimits_k \rho_k(t) = \sum\nolimits_k \rho_k(\tau) + \int_\tau^t \sum\nolimits_k \left(\lambda G^v_k(\rho^v(s)) - f_k(s)\right) \mathrm{d}s \leq \sum\nolimits_k \rho_k(\tau) < +\infty\,, \quad \forall t \geq \tau,$$

which would contradict the assumption that $\rho^*_k = +\infty$ for every $k \in \mathcal{K}$. Therefore, either ρ^*_e is finite for every $e \in \mathcal{E}^+_v$, or ρ^*_e is infinite for every $e \in \mathcal{E}^+_v$.

In order to distinguish between the two cases, let $\zeta(t) := \sum_{e\in\mathcal{E}^+_v} \rho_e(t)$, $\vartheta(t) := \sum_{e\in\mathcal{E}^+_v} f_e(t)$. Observe that, for all $t \geq \tau \geq 0$,

$$\zeta(t) = \zeta(\tau) + \int_\tau^t (\lambda - \vartheta(s))\,\mathrm{d}s\,. \tag{30}$$

First, consider the case when $\lambda < \lambda^{\max}$, and assume by contradiction that $\rho^*_e = +\infty$, and hence $f^*_e = f^{\max}_e$, for every $e \in \mathcal{E}^+_v$. This would imply that

$$\lim_{t\to\infty} \vartheta(t) = \lambda^{\max} > \lambda\,,$$

so that there would exist some $\tau \geq 0$ such that $\lambda - \vartheta(t) \leq 0$ for every $t \geq \tau$, and hence (30) would imply that $\zeta(t) \leq \zeta(\tau) < +\infty$ for all $t \geq \tau$, thus contradicting the assumption that $\rho_e(t)$ converges to $\rho_e^* = +\infty$ as t grows large. Hence, for every $\sigma \in \mathcal{R}_v$, $f^*(\lambda, \sigma) \in \mathcal{F}_v$, and hence it is necessarily an equilibrium flow for the local dynamical system (22). It follows that, for every $\sigma, \varsigma \in \mathcal{R}_v$, and $t \geq 0$, $\Psi^t(\rho^*(\lambda, \sigma)) = f^*(\lambda, \sigma)$, and $\Psi^t(\rho^*(\lambda, \zeta)) = f^*(\lambda, \zeta)$. Then, taking the limit of large t in (29) readily implies that $f^*(\lambda, \sigma) = f^*(\lambda, \varsigma)$, so that the limit flow $f^*(\lambda) \in \mathcal{F}^*(\lambda_0)$ does not depend on the initial condition.

On the other hand, when $\lambda \geq \lambda^{\max}$, (30) shows that $\zeta(t)$ is non-decreasing, hence convergent to some $\zeta(\infty) \in [0, +\infty]$ at t grows large. Assume, by contradiction, that $\zeta(\infty)$ is finite. Then, passing to the limit of large t in (30), one would get

$$\int_{\tau}^{+\infty} (\lambda - \vartheta(s)) \mathrm{d}s = \zeta(\infty) - \zeta(\tau) \leq \zeta(\infty) < +\infty \,.$$

This, and the fact that $\vartheta(t) < \lambda^{\max} \leq \lambda$ for all $t \geq 0$, would imply that

$$\lim_{t \to +\infty} \vartheta(t) = \lambda \,. \tag{31}$$

Since $f_e(t) < f_e^{\max}$, (31) is impossible if $\lambda > \lambda^{\max}$. On the other hand, if $\lambda = \lambda^{\max}$, then (31) implies that, for every $e \in \mathcal{E}_v^+$, $f_e(t)$ converges to $f_e^{\max}$, and hence $\rho_e(t)$ grows unbounded as t grows large, so that $\zeta(\infty)$ would be infinite. Hence, if $\lambda \geq \lambda^{\max}$, then necessarily $\zeta(\infty)$ is infinite, and thanks to the previous arguments this implies that $\rho_e^*(\lambda, \sigma) = \rho_e^*(\lambda) = +\infty$, and hence $f_e^*(\lambda, \sigma) < f_e^{\max}$ for all $\sigma \in \mathcal{R}_v$, $e \in \mathcal{E}_v^+$.

Finally, it remains to prove continuity of $f^*(\lambda)$ as a function of λ. For this, consider the function $H : (0, +\infty)^{\mathcal{E}_v^+} \times (0, \lambda^{\max}) \to \mathbb{R}^{\mathcal{E}_v^+}$ defined by

$$H_e(\rho^v, \lambda) := \lambda G_e^v(\rho^v) - \mu_e(\rho_e) \,, \qquad \forall e \in \mathcal{E}_v^+ \,.$$

Clearly, H is differentiable and such that

$$\frac{\partial}{\partial_{\rho_e}} H_e(\rho^v, \lambda) = \lambda \frac{\partial}{\partial_{\rho_e}} G_e^v(\rho^v) - \mu_e'(\rho_e) = -\sum_{j \neq e} \lambda \frac{\partial}{\partial_{\rho_e}} G_j^v(\rho^v) - \mu_e'(\rho_e) < -\sum_{j \neq e} \frac{\partial}{\partial_{\rho_e}} H_j(\rho^v, \lambda) \,, \tag{32}$$

where the inequality follows from the strict monotonicity of the flow function (see Assumption 2). Property (a) in Definition 8 implies that $\partial H_j(\rho^v, \lambda)/\partial \rho_e \geq 0$ for all $j \neq e \in \mathcal{E}_v^+$. Hence, from (32), we also have that $\partial H_e(\rho^v, \lambda)/\partial \rho_e < 0$ for all $e \in \mathcal{E}_v^+$. Therefore, for all $\rho^v \in (0, +\infty)^{\mathcal{E}_v^+}$, and $\lambda \in (0, \lambda^{\max})$, the Jacobian matrix $\nabla_{\rho^v} H(\rho^v, \lambda)$ is strictly diagonally dominant, and hence invertible by a standard application of the Gershgorin Circle Theorem, e.g., see Theorem 6.1.10 in [29]. It then follows from the implicit function theorem that $\rho^*(\lambda)$, which is the unique zero of $H(\,\cdot\,, \lambda)$, is continuous on the interval $(0, \lambda^{\max})$. Hence, also $f^*(\lambda) = \mu(\rho^*(\lambda))$ is continuous on $(0, \lambda^{\max})$, since it is the composition of two continuous functions. Moreover, since

$$\textstyle\sum_{e \in \mathcal{E}_v^+} f_e^*(\lambda) = \lambda \,, \qquad 0 \leq f_e^*(\lambda) \leq f_e^{\max} \,, \qquad \forall e \in \mathcal{E}_v^+ \,, \qquad \forall \lambda \in (0, \lambda^{\max}) \,,$$

one gets that $\lim_{\lambda \downarrow 0} f_e^*(\lambda) = 0$, and $\lim_{\lambda \uparrow \lambda^{\max}} f_e^*(\lambda) = f_e^{\max}$, for all $e \in \mathcal{E}_v^+$. Now, one has that $\sum_{e \in \mathcal{E}_v^+} f_e^*(0) = 0$, so that $0 = f_e^*(0) = \lim_{\lambda \downarrow 0} f_e^*(\lambda)$ for all $e \in \mathcal{E}_v^+$. Moreover, as previously shown, $f_e^*(\lambda) = f_e^{\max} = \lim_{\lambda \uparrow \lambda^{\max}} f_e^*(\lambda)$ for all $\lambda \geq \lambda^{\max}$. This completes the proof of continuity of $f^*(\lambda)$ on $[0, +\infty)$. ■

While Lemma 2 ensures existence of a unique limit point for the local system (22) with constant input $\lambda(t) \equiv \lambda$, the following lemma establishes a monotonicity property with respect to a time-varying input $\lambda(t)$.

Lemma 3 (Monotonicity of the local system): Let $0 \leq v < n$ be a nondestination node, $G^v : \mathcal{R}_v \to \mathcal{S}_v$ a continuously differentiable map, satisfying Properties (a) and (c) of Definition 8, and $\lambda^-(t)$, and $\lambda^+(t)$ be two nonnegative-real valued Lipschitz-continuous functions such that $\lambda^-(t) \leq \lambda^+(t)$ for all $t \geq 0$. Let $\rho^-(t)$ and $\rho^+(t)$ be the solutions of the local dynamical system (22) corresponding to the inputs $\lambda^-(t)$, and $\lambda^+(t)$, respectively, with the same initial condition $\rho^-(0) = \rho^+(0)$. Then

$$\rho_e^-(t) \leq \rho_e^+(t) \,, \qquad \forall e \in \mathcal{E}_v^+ \,,\ \forall t \geq 0 \,. \tag{33}$$

Proof: For $e \in \mathcal{E}_v^+$, define $\tau_e := \inf\{t \geq 0 : \rho_e^+(t) > \rho_e^-(t)\}$, and let $\tau := \min\{\tau_e : e \in \mathcal{E}_v^+\}$. Assume by contradiction that $\rho_e^-(t) > \overline{\rho}_e^+(t)$ for some $t \geq 0$, and $e \in \mathcal{E}_v^+$. Then, $\tau < +\infty$, and $\mathcal{I} := \operatorname{argmin}\{\tau_e : e \in \mathcal{E}_v^+\}$ is a well defined nonempty subset of $\mathcal{E}_v^+$. Moreover, by continuity, one has that there exists some $\varepsilon > 0$ such that,

$\rho_i^-(\tau) = \rho_i^+(\tau)$, $\rho_i^-(t) > \rho_i^+(t)$, and $\rho_j^-(t) < \rho_j^+(t)$ for all $i \in \mathcal{I}$, $j \in \mathcal{J}$, and $t \in (\tau, \tau + \varepsilon)$, where $\mathcal{J} := \mathcal{E}_v^+ \setminus \mathcal{I}$. Using Lemma 1, one gets, for every $t \in (\tau, \tau + \varepsilon)$,

$$\begin{aligned} 0 &\geq \tfrac{1}{2} \textstyle\sum_e \operatorname{sgn}(\rho_e^-(t) - \rho_e^+(t)) \left(G_e^v(\rho^-(t)) - G_e^v(\rho^+(t)) \right) \\ &= \tfrac{1}{2} \left(\textstyle\sum_i G_i^v(\rho^-(t)) - \sum_i G_i^v(\rho^+(t)) - \sum_j G_j^v(\rho^-(t)) + \sum_j G_j^v(\rho^+(t)) \right) \\ &= \textstyle\sum_i G_i^v(\rho^-(t)) - \sum_i G_i^v(\rho^+(t)) , \end{aligned}$$

where the summation indices e, i, and j run over $\mathcal{E}_v^+$, $\mathcal{I}$, and $\mathcal{J}$, respectively. On the other hand, Assumption 2 implies that $\mu_i(\rho_i^-(t)) \geq \mu_i(\rho_i^+(t))$ for all $i \in \mathcal{I}$, $t \in [\tau, \tau + \varepsilon)$. Now, let $\chi(t) := \sum_{i \in \mathcal{I}} \left(\rho_i^-(t) - \rho_i^+(t) \right)$. Then, for every $t \in (\tau, \tau + \varepsilon)$, one has

$$\begin{aligned} 0 &< \chi(t) - \chi(\tau) \\ &= \int_\tau^t \lambda^-(s) \sum\nolimits_{i \in \mathcal{I}} \left(G_i^v(\rho^-(s)) - G_i^v(\rho^-(s)) \right) \mathrm{d}s \\ &\quad - \int_\tau^t (\lambda^+(s) - \lambda^-(s)) \sum\nolimits_{i \in \mathcal{I}} G_i^v(\rho^+(s)) \mathrm{d}s - \int_\tau^t \sum\nolimits_{i \in \mathcal{I}} \left(\mu_i(\rho_i^-(s)) - \mu_i(\rho_i^+(s)) \right) \mathrm{d}s \quad \leq \quad 0 , \end{aligned}$$

which is a contradiction. Then, necessarily (33) has to hold true. ■

The following lemma establishes that the output of the local system (22) is convergent, provided that the input is convergent.

Lemma 4 (Attractivity of the local dynamical system): Let $0 \leq v < n$ be a nondestination node, $G^v : \mathcal{R}_v \to \mathcal{S}_v$ a continuously differentiable map, satisfying Properties (a) and (c) of Definition 8, and $\lambda(t)$ a nonnegative-real-valued Lipschitz function such that

$$\lim_{t \to +\infty} \lambda(t) = \lambda . \tag{34}$$

Then, for every initial condition $\rho(0) \in \mathcal{R}$, the solution of the local dynamical system (22) satisfies

$$\lim_{t \to +\infty} f_e(t) = f_e^*(\lambda) , \qquad \forall e \in \mathcal{E}_v^+ , \tag{35}$$

where $f^*(\lambda)$ is as defined in Lemma 2.

Proof: Fix some $\varepsilon > 0$, and let $\tau \geq 0$ be such that $|\lambda(t) - \lambda| \leq \varepsilon$ for all $t \geq \tau$. For $t \geq \tau$, let $f^-(t)$ and $f^+(t)$ be the flow associated to the solutions of the local dynamical system (22) with initial condition $\rho^-(\tau) = \rho^+(\tau) = \rho^v(\tau)$, and constant inputs $\lambda^-(t) \equiv \lambda^- := \max\{\lambda - \varepsilon, 0\}$, and $\lambda^+(t) \equiv \lambda + \varepsilon$, respectively. From Lemma 3, one gets that

$$f_e^-(t) \leq f_e(t) \leq f_e^+(t) , \qquad \forall t \geq \tau , \qquad \forall e \in \mathcal{E}_v^+ . \tag{36}$$

On the other hand, Lemma 2 implies that $f^-(t)$ converges to $f^*(\lambda^-)$, and $f^+(t)$ converges to $f^*(\lambda^+)$, as t grows large. Hence, passing to the limit of large t in (36) yields

$$f_e^*(\lambda^-) \leq \liminf_{t \to +\infty} f_e(t) \leq \limsup_{t \to +\infty} f_e(t) \leq f_e^*(\lambda + \varepsilon) , \qquad \forall e \in \mathcal{E}_v^+ .$$

Form the arbitrariness of $\varepsilon > 0$, and the continuity of $f^*(\lambda)$ as a function of λ, it follows that $\lim_{t \to \infty} f(t) = f^*(\lambda)$, which proves the claim. ■

We are now ready to prove Theorem 1 by showing that, for any initial condition $\rho(0) \in \mathcal{R}$, the solution of the dynamical flow network (5) satisfies

$$\lim_{t \to +\infty} f_e(t) = f_e^* , \tag{37}$$

for all $e \in \mathcal{E}$. We shall prove this by showing via induction on $v = 0, 1, \ldots, n-1$ that, for all $e \in \mathcal{E}_v^+$, there exists $f_e^* \in [0, f_e^{\max}]$ such that (37) holds true. First, observe that, thanks to Lemma 2, this statement is true for $v = 0$, since the inflow at the origin is constant. Now, assume that the statement is true for all $0 \leq v < w$, where $w \in \{1, \ldots, n-2\}$ is some intermediate node. Then, since $\mathcal{E}_w^- \subseteq \cup_{v=0}^{w-1} \mathcal{E}_v^+$, one has that

$$\lim_{t \to +\infty} \lambda_w^-(t) = \lim_{t \to +\infty} \sum\nolimits_{e \in \mathcal{E}_w^-} f_e(t) = \sum\nolimits_{e \in \mathcal{E}_w^-} f_e^* = \lambda_w^* .$$

Then, Lemma 4 implies that, for all $e \in \mathcal{E}_w^+$, (37) holds true with $f_e^* = f_e^*(\lambda_w^*)$, thus proving the statement for $v = w$. This completes the proof of Theorem 1.

VII. PROOF OF THEOREM 2

In this section, we shall prove Theorem 2 on the strong resilience of dynamical flow networks. Throughout the section, we shall consider a given flow network $\mathcal{N}$, satisfying Assumptions 1 and 2, a distributed routing policy $\mathcal{G}$, and a constant inflow $\lambda_0 \geq 0$, and assume that there exists an equilibrium flow $f^* \in \mathcal{F}^*(\lambda_0)$ for the dynamical flow network. First, we shall show that $\gamma_1(f^*, \mathcal{G}) \leq R(\mathcal{N}, f^*)$. This will follow mainly from the assumption of acyclicity of the network topology, and the locality constraint on the information used by the distributed routing policy, as per Definition 2. Then, we shall prove that, if the distributed routing policy is locally responsive (as per Definition 8), then $\gamma_1(f^*, \mathcal{G}) = R(\mathcal{N}, f^*)$. The proof of this second result will heavily rely on Properties (a) and (c) of Definition 8.[3] In particular, these properties will allow us to prove some key diffusivity properties for the solution of the perturbed dynamical flow network.

A. Upper bound on the strong resilience

The second part of Theorem 2 is restated and proved below.

Lemma 5 (Upper bound on the strong resilience): Let $\mathcal{N}$ be a flow network satisfying Assumptions 1 and 2, $\lambda_0 \geq 0$ a constant inflow, and $\mathcal{G}$ any distributed routing policy. Assume that the associated dynamical flow network has an equilibrium flow $f^* \in \mathcal{F}^*(\lambda_0)$. Then, $\gamma_1(f^*, \mathcal{G}) \leq R(\mathcal{N}, f^*)$.

Proof: In order to prove the result it is sufficient to exhibit a family of admissible perturbations, with magnitude δ arbitrarily close to $R(\mathcal{N}, f^*)$, under which the network is not fully transferring. Let us fix some non-destination node $0 \leq v < n$ minimizing the right-hand side of (15), and put $\kappa := \sum_{e \in \mathcal{E}_v^+} f_e^{\max}$. For any $R(\mathcal{N}, f^*) < \delta < \kappa$, consider the admissible perturbation defined by

$$\tilde{\mu}_e(\rho_e) := \frac{\kappa - \delta}{\kappa} \mu_e(\rho_e)\,, \quad \forall e \in \mathcal{E}_v^+\,, \qquad \tilde{\mu}_e(\rho_e) := \mu_e(\rho_e)\,, \quad \forall e \in \mathcal{E} \setminus \mathcal{E}_v^+\,. \tag{38}$$

Clearly, the magnitude of such perturbation equals δ, while its stretching coefficient is 1.

Let us consider the origin-destination cut-set $\mathcal{U} := \{0, 1, \ldots, v\}$, and put $\mathcal{E}_{\mathcal{U}}^+ := \{(u, w) \in \mathcal{E} : 0 \leq u \leq v, v < w \leq n\}$. Observe that the associated perturbed dynamical flow network satisfies, for every $0 \leq u < v$,

$$\tilde{\mu}_e(\tilde{\rho}_e(t)) = f_e^*\,, \qquad \forall t \geq 0\,, \quad \forall e \in \mathcal{E}_u^+\,.$$

In particular, this implies that $\tilde{\mu}_e(\tilde{\rho}_e(t)) = f_e^*$ for all $t \geq 0$, and for every link $e \in \mathcal{E}_{\mathcal{U}}^+ \setminus \mathcal{E}_v^+$. On the other hand, one has that

$$\tilde{\mu}_e(\tilde{\rho}_e(t)) < \tilde{f}_e^{\max} = \frac{\kappa - \delta}{\kappa} f_e^{\max}\,, \qquad \forall e \in \mathcal{E}_v^+\,,\ \forall t \geq 0\,.$$

Therefore, for all $t \geq 0$, one has that

$$\begin{aligned}
\sum_{e \in \mathcal{E}_{\mathcal{U}}^+} \tilde{\mu}_e(\tilde{\rho}_e(t)) &< \sum_{e \in \mathcal{E}_v^+} \tilde{f}_e^{\max} + \sum_{e \in \mathcal{E}_{\mathcal{U}}^+ \setminus \mathcal{E}_v^+} f_e^* \\
&= \frac{\kappa - \delta}{\kappa} \sum_{e \in \mathcal{E}_v^+} f_e^{\max} + \sum_{e \in \mathcal{E}_{\mathcal{U}}^+ \setminus \mathcal{E}_v^+} f_e^* \\
&= \sum_{e \in \mathcal{E}_v^+} f_e^{\max} - \delta - \sum_{e \in \mathcal{E}_v^+} f_e^* + \sum_{e \in \mathcal{E}_{\mathcal{U}}^+} f_e^* \\
&= R(\mathcal{N}, f^*) - \delta + \lambda_0\,.
\end{aligned} \tag{39}$$

Define the edge sets $\mathcal{A} := \bigcup_{w=v+1}^{n-1} \mathcal{E}_w^+$ and $\mathcal{B} := \bigcup_{w=v+1}^{n} \mathcal{E}_w^-$, and observe that $\mathcal{A} \cup \mathcal{E}_{\mathcal{U}}^+ = \mathcal{B}$. For $t \geq 0$, put $\zeta(t) := \sum_{e \in \mathcal{A}} \rho_e(t)$. Now, since

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\sum_{e \in \mathcal{E}_w^+} \tilde{\rho}_e(t)\right) &= \sum_{e \in \mathcal{E}_w^+} \left(\sum_{e \in \mathcal{E}_w^-} \tilde{f}_e(t)\right) G_e^v(\tilde{\rho}^w(t)) - \sum_{e \in \mathcal{E}_w^+} \tilde{f}_e(t) \\
&= \sum_{e \in \mathcal{E}_w^-} \tilde{f}_e(t) - \sum_{e \in \mathcal{E}_w^+} \tilde{f}_e(t)\,,
\end{aligned}$$

[3] Property (b) is in fact irrelevant for maximal robustness in the strong resilience sense, while it will be used in the next section to prove maximal robustness in the weak resilience sense of locally responsive distributed routing policies.

for every $v < w < n$, one gets, using (39), that

$$\begin{array}{rcl}\frac{\mathrm{d}}{\mathrm{d}t}\zeta(t) & = & \sum_{e\in\mathcal{B}}\tilde{f}_e(t)-\sum_{e\in\mathcal{E}_n^-}\tilde{f}_e(t)-\sum_{e\in\mathcal{A}}\tilde{f}_e(t)\\ & = & \sum_{e\in\mathcal{E}_{\mathcal{U}}^+}\tilde{f}_e(t)-\sum_{e\in\mathcal{E}_n^-}\tilde{f}_e(t)\\ & < & R(\mathcal{N},f^*)-\delta+\lambda_0-\tilde{\lambda}_n(t)\,.\end{array} \tag{40}$$

Now assume, by contradiction, that

$$\liminf_{t\to+\infty}\tilde{\lambda}_n(t) > R(\mathcal{N},f^*)-\delta+\lambda_0\,.$$

Then, there would exist some $\varepsilon > 0$ and $\tau \geq 0$ such that $\tilde{\lambda}_n(t) \geq R(\mathcal{N},f^*)-\delta+\lambda_0+\varepsilon$ for all $t \geq \tau$. It would then follow from (40) and Gronwall's inequality that $\zeta(t) \leq \zeta(\tau)-(t-\tau)\varepsilon$ for all $t \geq \tau$, so that $\zeta(t)$ would converge to $-\infty$ as t grows large, contradicting the fact that $\zeta(t) \geq 0$ for all $t \geq 0$. Hence, necessarily $\liminf_{t\uparrow\infty}\tilde{\lambda}_n(t) \leq R(\mathcal{N},f^*)-\delta+\lambda_0 < \lambda_0$, so that the perturbed dynamical flow network is not fully transferring. Then, from the arbitrariness of the perturbation's magnitude $\delta \in (R(\mathcal{N},f^*),\kappa)$, it follows that the network's strong resilience is upper bounded by $R(\mathcal{N},f^*)$. ■

B. Lower bound on the strong resilience

We shall now prove the second part of Theorem 2. Hence, throughout this subsection, $\mathcal{G}$ will be a locally responsive distributed routing policy.

First observe that, for any admissible perturbation, regardless of its magnitude, the perturbed dynamical flow network (11) satisfies all the assumptions of Theorem 1, which can therefore be applied to show the existence of a globally attractive perturbed limit flow $\tilde{f}^* \in \mathrm{cl}(\mathcal{F})$. This in particular implies that $\tilde{\lambda}_n(t) = \sum_{e\in\mathcal{E}_n^-}\tilde{f}_e(t)$ converges to $\tilde{\lambda}_n^* = \sum_{e\in\mathcal{E}_n^-}\tilde{f}_e^*$ as t grows large. However, this is not sufficient in order to prove strong resilience of the perturbed dynamical flow network (11), as it might be the case that $\tilde{\lambda}_n^* < \lambda_0$.

In fact, it turns out that, provided that the magnitude of the admissible perturbation is smaller than $R(\mathcal{N},f^*)$, the perturbed limit flow $\tilde{f}^*$ is an equilibrium flow for the perturbed dynamical flow network, so that $\tilde{\lambda}_n^* = \lambda_0$ and (11) is fully transferring. In order to show this, we need to study the *perturbed local system*

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\rho}_e(t) = \tilde{\lambda}(t)G_e^v(\tilde{\rho}^v(t))-\tilde{f}_e(t)\,,\qquad \tilde{f}_e(t)=\tilde{\mu}_e(\tilde{\rho}_e(t))\,,\qquad \forall e\in\mathcal{E}_v^+\,, \tag{41}$$

for every non-destination node $0 \leq v < n$, and nonnegative-real-valued, Lipschitz local input $\tilde{\lambda}(t)$. Indeed, Lemma 4 can be applied to the perturbed local system (41) establishing convergence of the perturbed local flows $\tilde{f}^v(t)$ to a local equilibrium flow $\tilde{f}^*(\lambda) \in \mathcal{F}_v$, provided that the input flow $\tilde{\lambda}(t)$ converges to a value λ which is strictly smaller than the sum of the perturbed flow capacities of the outgoing links. However, such local result is not sufficient to prove strong resilience of the entire perturbed dynamical flow network. The key property in order to prove such a global result is stated in Lemma 6, which describes how the flow redistributes upon the network perturbation. In particular, it ensures that the increase in flow on all the links downstream from a node whose outgoing links are affected by a given perturbation, is less than the magnitude of the disturbance itself. We shall refer to this property as to the *diffusivity* of the local perturbed system.

Lemma 6 (Diffusivity of the local perturbed system): Let $\mathcal{N}$ be a flow network satisfying Assumptions 1 and 2, $\mathcal{G}$ be a locally responsive distributed routing policy, $\lambda_0 \geq 0$ a constant inflow. Assume that $f^* \in \mathcal{F}^*(\lambda_0)$ is an equilibrium flow for the dynamical flow network (5). Let $\tilde{\mathcal{N}}$ be an admissible perturbation of $\mathcal{N}$, $0 \leq v < n$ be a nondestination node, $\lambda_v^* := \sum_{e\in\mathcal{E}_v^+}f_e^*$, and $\lambda \in [0,\sum_{e\in\mathcal{E}_v^+}\tilde{f}_e^{\max})$. Then, for every $\mathcal{J} \subseteq \mathcal{E}_v^+$,

$$\sum_{e\in\mathcal{J}}\left(\tilde{f}_e^*(\lambda)-f_e^*\right) \leq [\lambda-\lambda_v^*]_+ + \sum_{e\in\mathcal{E}_v^+}\delta_e\,, \tag{42}$$

where $\tilde{f}^*(\lambda)$ is the local equilibrium flow of the perturbed local system (11) with constant local input $\tilde{\lambda}(t) \equiv \lambda$, and $\delta_e := ||\mu_e(\,\cdot\,)-\tilde{\mu}_e(\,\cdot\,)||_\infty$.

Proof: Define $\lambda_v^* := \sum_{e\in\mathcal{E}_v^+}f_e^*$, and $\hat{\lambda} := \max\{\lambda,\lambda_v^*\}$. Let $\hat{\rho}^v(t)$ be the solution of the perturbed local system (41) with constant input $\tilde{\lambda}(t) \equiv \hat{\lambda}$, and initial condition $\hat{\rho}_e(0) = \rho_e^* := \mu_e^{-1}(f_e^*)$, for all $e \in \mathcal{E}_v^+$, and let $\hat{f}_e(e) := \tilde{\mu}_e(\hat{\rho}_e(t))$. We shall first prove that

$$\hat{f}_e(t) \geq f_e^*\,,\quad \forall\, t \geq 0 \quad \forall\, e \in \mathcal{E}_v^+\,. \tag{43}$$

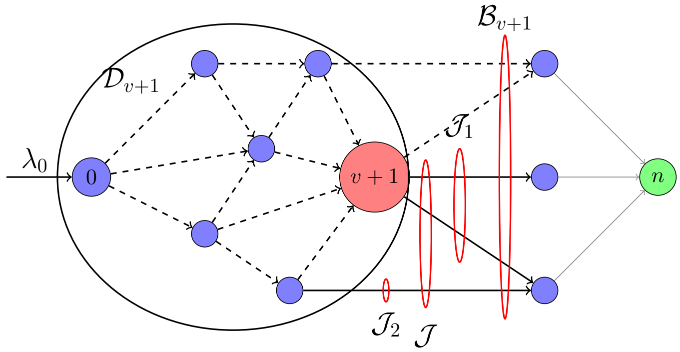

Fig. 4. Illustration of the sets used in proving the induction step.

For this, consider a point $\hat{\rho}^v \in \mathcal{R}_v$, such that $\hat{\rho}^v \neq \rho^*$, and there exists some $i \in \mathcal{E}_v^+$ such that $\hat{\rho}_i = \rho_i^*$ and $\hat{\rho}_e \geq \rho_e^*$ for all $e \neq i \in \mathcal{E}_v^+$. For such a $\hat{\rho}^v$ and i, Lemma 1 implies that $G_i^v(\hat{\rho}^v) \geq G_i^v(\rho^*)$. This, combined with the fact that $\hat{\lambda} \geq \lambda_v^*$ and $\tilde{\mu}_i(\hat{\rho}_i) \leq \mu_i(\hat{\rho}_i) = \mu_i(\rho_i^*)$, yields

$$\hat{\lambda}_v G_i^v(\hat{\rho}^v) - \tilde{\mu}_i(\hat{\rho}_i) \geq \lambda_v^* G_i^v(\rho^*) - \mu_i(\rho_i^*) = 0\,. \tag{44}$$

Considering the region $\Omega := \{\hat{\rho}^v \in \mathcal{R}_v : \hat{\rho}_j \geq \rho_j^*,\ \forall j \in \mathcal{E}_v^+\}$, and denoting by $\omega \in \mathbb{R}^{\mathcal{E}_v^+}$ the unit outward-pointing normal vector to the boundary of Ω at $\hat{\rho}^v$, (44) shows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}^v \cdot \omega = \left(\hat{\lambda}_v G^v(\hat{\rho}^v) - \tilde{\mu}_v(\hat{\rho}^v)\right) \cdot \omega \leq 0\,, \quad \forall \hat{\rho}^v \in \partial\Omega\,,\ t \geq 0\,.$$

Therefore, Ω is invariant under (41). Since $\hat{\rho}^v(0) = \rho^* \in \Omega$, this proves (43).

Lemma 2 implies that there exists a unique local equilibrium flow $\hat{f}^* := \tilde{f}^*(\hat{\lambda})$. Then, for any $\mathcal{J} \subseteq \mathcal{E}_v^+$, (43) implies that

$$\begin{array}{rcl} \sum_j \hat{f}_j^* & = & \hat{\lambda}_v^* - \sum_k \hat{f}_k^* \\ & \leq & \hat{\lambda}_v^* - \sum_k \tilde{\mu}_k(\rho_k^*) \\ & = & \hat{\lambda}_v^* - \lambda_v^* + \sum_j f_j^* + \sum_k \mu_k(\rho_k^*) - \sum_k \tilde{\mu}_k(\rho_k^*) \\ & \leq & [\hat{\lambda}_v^* - \lambda_v^*]_+ + \sum_j f_j^* + \sum_k \delta_k \\ & \leq & [\hat{\lambda}_v^* - \lambda_v^*]_+ + \sum_j f_j^* + \sum_{e \in \mathcal{E}_v^+} \delta_e\,, \end{array} \tag{45}$$

where the summation indices j and k run over $\mathcal{J}$, and $\mathcal{E}_v^+ \setminus \mathcal{J}$, respectively. Moreover, since $\lambda \leq \hat{\lambda}$ from Lemma 3, one gets that $\tilde{f}_e^*(\lambda) \leq \tilde{f}_e^*(\hat{\lambda}) = \hat{f}_e^*$ for all $e \in \mathcal{E}_v^+$. In particular, this implies that $\sum_{j \in \mathcal{J}} \tilde{f}_j^*(\lambda) \leq \sum_{j \in \mathcal{J}} \hat{f}_j^*$, for all $\mathcal{J} \subset \mathcal{E}_v^+$. This, combined with (45), proves (42). ■

The following lemma exploits the diffusivity property from Lemma 6 along with an induction argument on the topological ordering of the node set to prove that $R(\mathcal{N}, f^*)$ is indeed a lower bound on the strong resilience of the network under the locally responsive distributed routing policies.

Lemma 7: Consider a flow network $\mathcal{N}$ satisfying Assumptions 1 and 2, a locally responsive distributed routing policy $\mathcal{G}$, and a constant inflow $\lambda_0 \geq 0$. Assume that $f^* \in \mathcal{F}^*(\lambda_0)$ is an equilibrium flow for the associated dynamical flow network. Let $\tilde{\mathcal{N}}$ be an admissible perturbation of $\mathcal{N}$, of magnitude $\delta < R(\mathcal{N}, f^*)$. Then, the perturbed dynamical flow network (11) has a globally attractive equilibrium flow and hence it is fully transferring.

Proof: First recall that Theorem 1 can be applied to the perturbed dynamical network (11) in order to prove existence of a globally attractive limit flow $\tilde{f}^* \in \mathrm{cl}(\mathcal{F})$ for the perturbed dynamical network flow (11). For brevity in notation, for every $1 \leq v < n$, put $\lambda_v^* := \sum_{e \in \mathcal{E}_v^+} f_e^*$, $\tilde{\lambda}_v^* := \sum_{e \in \mathcal{E}_v^-} \tilde{f}_e$, and $\lambda_v^{\max} := \sum_{e \in \mathcal{E}_v^+} f_e^{\max}$. Also, for every node $v \in \mathcal{V}$, let $\mathcal{D}_v := \bigcup_{u=0}^{v} \mathcal{E}_u^+$ and $\mathcal{B}_v := \{(u, w) \in \mathcal{E} : 0 \leq u \leq v,\ v < w \leq n\}$ be, respectively, the set of all outgoing links, and the link-boundary of the node set $\{0, 1, \dots, v\}$.

We shall prove the following through induction on $u = 0, 1, \dots, n-1$:

$$\sum_{e \in \mathcal{J}} \left(\tilde{f}_e^* - f_e^*\right) \leq \sum_{e \in \mathcal{D}_u} \delta_e\,, \qquad \forall \mathcal{J} \subseteq \mathcal{B}_u\,. \tag{46}$$

First, notice that $\mathcal{B}_0 = \mathcal{D}_0 = \mathcal{E}_0^+$. Since $\sum_{e\in\mathcal{E}_0^+} \delta_e \le \delta < R(\mathcal{N}, f^*) \le \sum_{e\in\mathcal{E}_0^+}(f_\mathrm{e}^{\max} - f_e^*)$, we also have that $\lambda_0 < \tilde{\lambda}_v^{\max}$. Therefore, by using (42) of Lemma 6, one can verify that (46) holds true for $v = 0$.

Now, for some $v \le n-2$, assume that (46) holds true for every $u \le v$. Consider a subset $\mathcal{J} \subseteq \mathcal{B}_{v+1}$ and let $\mathcal{J}_1 := \mathcal{J} \cap \mathcal{E}_{v+1}^+$ and $\mathcal{J}_2 := \mathcal{J} \setminus \mathcal{J}_1$ (e.g., see Figure 4). By applying Lemma 6 to the set $\mathcal{J}_1$, one gets that

$$\sum\nolimits_{e\in\mathcal{J}_1} \left(\tilde{f}_e^* - f_e^*\right) \le \left[\tilde{\lambda}_{v+1}^* - \lambda_{v+1}^*\right]_+ + \sum\nolimits_{e\in\mathcal{E}_{v+1}^+} \delta_e, \quad \forall\, t \ge 0. \tag{47}$$

It is easy to check that $\mathcal{J}_2 \subseteq \mathcal{B}_v$ and $\mathcal{E}_{v+1}^- \subseteq \mathcal{B}_v$. Therefore, using (46) for the sets $\mathcal{J}_2$ and $\mathcal{J}_2 \cup \mathcal{E}_{v+1}^-$, one gets the following inequalities respectively:

$$\sum\nolimits_{e\in\mathcal{J}_2} \left(\tilde{f}_e^* - f_e^*\right) \le \sum\nolimits_{e\in\mathcal{D}_v} \delta_e, \tag{48}$$

$$\sum\nolimits_{e\in\mathcal{J}_2} \left(\tilde{f}_e^* - f_e^*\right) + \sum\nolimits_{e\in\mathcal{E}_{v+1}^-} \left(\tilde{f}_e^* - f_e^*\right) \le \sum\nolimits_{e\in\mathcal{D}_v} \delta_e. \tag{49}$$

Consider the two cases: $\tilde{\lambda}_{v+1}^* \le \lambda_{v+1}^*$, or $\tilde{\lambda}_{v+1}^* > \lambda_{v+1}^*$. By adding up (47) and (48), in the first case, or (47) and (49) in the second case, one gets that

$$\sum_{e\in\mathcal{J}} \left(\tilde{f}_e^* - f_e^*\right) = \sum_{e\in\mathcal{J}_1} \left(\tilde{f}_e^* - f_e^*\right) + \sum_{e\in\mathcal{J}_2} \left(\tilde{f}_e^* - f_e^*\right) \le \sum_{e\in\mathcal{E}_{v+1}^+} \delta_e + \sum_{e\in\mathcal{D}_v} \delta_e \le \sum_{e\in\mathcal{D}_{v+1}} \delta_e\,.$$

This proves (46) for node $v+1$ and hence the induction step.

Fix $1 \le v < n$. Since $\mathcal{E}_v^- \subseteq \mathcal{B}_{v-1}$, (46) with $u = v-1$ implies that

$$\tilde{\lambda}_v^* = \sum_{e\in\mathcal{E}_v^-} \tilde{f}_e^* \le \sum_{e\in\mathcal{E}_v^-} f_e^* + \sum_{e\in\mathcal{D}_{v-1}} \delta_e = \sum_{e\in\mathcal{E}_v^+} f_e^* + \sum_{e\in\mathcal{E}} \delta_e - \sum_{e\in\mathcal{E}\setminus\mathcal{D}_{v-1}} \delta_e\,,$$

where the third step follows from the fact that $\sum_{e\in\mathcal{E}_v^-} f_e^* = \sum_{e\in\mathcal{E}_v^+} f_e^*$ by conservation of mass. Then, since $\mathcal{E}_v^+ \subseteq \mathcal{E} \setminus \mathcal{D}_{v-1}$, one gets that

$$\begin{aligned}
\tilde{\lambda}_v^* &\le \sum\nolimits_{e\in\mathcal{E}_v^+} f_e^* + \delta - \sum\nolimits_{e\in\mathcal{E}_v^+} \delta_e \\
&< \sum\nolimits_{e\in\mathcal{E}_v^+} f_e^* + R(\mathcal{N}, f^*) - \sum\nolimits_{e\in\mathcal{E}_v^+} \delta_e \\
&\le \sum\nolimits_{e\in\mathcal{E}_v^+} f_e^* + \sum\nolimits_{e\in\mathcal{E}_v^+} \left(f_\mathrm{e}^{\max} - f_e^*\right) - \sum\nolimits_{e\in\mathcal{E}_v^+} \delta_e \\
&= \sum\nolimits_{e\in\mathcal{E}_v^+} \left(f_\mathrm{e}^{\max} - \delta_e\right) \\
&= \sum\nolimits_{e\in\mathcal{E}_v^+} \tilde{f}_\mathrm{e}^{\max}\,.
\end{aligned}$$

Hence, it follows from Lemma 2 applied to the perturbed local system (41) that

$$\tilde{f}_e^* = \tilde{f}_e^*(\tilde{\lambda}_v^*) < \tilde{f}_e^{\max}\,, \qquad \forall e \in \mathcal{E}_v^+\,, \tag{50}$$

for all $1 \le v < n-1$. Moreover, since $\lambda_0 = \sum_{e\in\mathcal{E}_v^+} f_e^* < \sum_{e\in\mathcal{E}_v^+} f_e^{\max}$, applying Lemma 2 again to the perturbed local system (41) shows that (50) holds true for $v = 0$ as well. Hence $\tilde{f}_e^* < f_e^{\max}$ for all $e \in \mathcal{E}$, so that the limit flow $\tilde{f}^* \in \mathcal{F}$, and hence it is necessarily an equilibrium flow of the perturbed dynamical flow network (11), as argued in Remark 1. Therefore, (11) is fully transferring. ■

The second part of Theorem 2 now immediately follows from 7, and the arbitrariness of the admissible perturbation of magnitude smaller than $R(\mathcal{N}, f^*)$.

VIII. Proof of Theorem 3

This section is devoted to the proof of Theorem 3 on the weak resilience of dynamical flow networks. Throughout this section, we shall consider a given flow network $\mathcal{N}$ satisfying Assumptions 1 and 2, a distributed routing policy $\mathcal{G}$, a constant inflow $\lambda_0 \ge 0$. First, we shall prove that, if $\mathcal{G}$ is an arbitrary distributed routing policy such that the associated dynamical flow network has an equilibrium flow $f^* \in \mathcal{F}^*(\lambda_0)$, then the min-cut capacity of the network, $C(\mathcal{N})$, provides an upper bound on the weak resilience $\gamma_0(f^*, \mathcal{G})$. This will follow from a basic conservation of mass argument. Then, we shall show that, when the distributed routing policy $\mathcal{G}$ is locally responsive, as per Definition 8, then the weak resilience of the associated dynamical flow network coincides with $C(\mathcal{N})$. This will follow from some arguments in part close to those developed in Section VII-B, and in part exploiting the additional Property (b) of Definition 8, which was not necessary for showing maximal robustness of locally responsive distributed routing policies in the strong resilience sense.

A. Upper bound on the weak resilience

We start by proving that $C(\mathcal{N})$ is indeed an upper bound on the weak resilience. Recall that, if $f^* \in \mathcal{F}^*(\lambda_0)$ is an equilibrium flow of the associated dynamical flow network, then its weak resilience is defined as the double limit $\lim_{\theta\uparrow\infty}\lim_{\alpha\downarrow 0}\gamma_{\alpha,\theta}(f^*)$, where $\gamma_{\alpha,\theta}(f^*)$ is the infimum magnitude of all the admissible perturbations of stretching coefficient less than or equal to θ for which the associated perturbed dynamical flow network is not α-transferring with respect to f^*. The following result will readily imply the first part of Theorem 3.

Lemma 8 (Upper bound on the weak resilience): Let $\mathcal{N}$ be a flow network satisfying Assumptions 1 and 2, $\mathcal{G}$ a distributed routing policy, and $\lambda_0 \geq 0$ a constant inflow. Assume that $f^* \in \mathcal{F}(\lambda_0)$ is an equilibrium for the dynamical flow network (5). Then, for every $\alpha \in (0, 1]$, and every $\theta \geq 1$,

$$\gamma_{\alpha,\theta}(f^*, \mathcal{G}) \leq C(\mathcal{N}) - \frac{\alpha}{2}\lambda_0\,.$$

Proof: Consider a minimal origin-destination cut, i.e., some $\mathcal{U} \subseteq \mathcal{V}$ such that $0 \in \mathcal{U}$, $n \notin \mathcal{U}$, and $\sum_{e\in\mathcal{E}^+_{\mathcal{U}}} f^{\max}_e = C(\mathcal{N})$. Define $\varepsilon := \alpha\lambda_0/(2C(\mathcal{N}))$, and consider an admissible perturbation such that $\tilde{\mu}_e(\rho_e) = \varepsilon\mu_e(\rho_e)$ for every $e \in \mathcal{E}^+_{\mathcal{U}}$, and $\tilde{\mu}_e(\rho_e) = \mu_e(\rho_e)$ for all $e \in \mathcal{E} \setminus \mathcal{E}^+_{\mathcal{U}}$. It is readily verified that the magnitude of such perturbation satisfies

$$\delta = (1-\varepsilon)\sum\nolimits_{e\in\mathcal{E}^+_{\mathcal{U}}} f^{\max}_e = (1-\varepsilon)C(\mathcal{N}) = C(\mathcal{N}) - \frac{\alpha}{2}\lambda_0\,,$$

while its stretching coefficient is 1. Moreover, $\sum_{e\in\mathcal{E}^+_{\mathcal{U}}} \tilde{f}^{\max}_e = \varepsilon\sum_{e\in\mathcal{E}^+_{\mathcal{U}}} f^{\max}_e = \alpha\lambda_0/2$. Then, arguing in the same way as in the proof of Lemma 5, one shows that $\liminf_{t\uparrow\infty}\tilde{\lambda}_n(t) < \alpha\lambda_0$, so that the perturbed dynamical network is not α-transferring. This implies the claim. ■

Observe now that it immediately follows from Lemma 8 that

$$\gamma_0(f^*, \mathcal{G}) = \lim_{\theta\uparrow\infty}\lim_{\alpha\downarrow 0}\gamma_{\alpha,\theta}(f^*, \mathcal{G}) \leq \lim_{\theta\uparrow\infty}\lim_{\alpha\downarrow 0}\left(C(\mathcal{N}) - \alpha\lambda_0/2\right) = C(\mathcal{N})\,,$$

which proves the first part of Theorem 3.

B. Lower bound on the weak resilience

We now prove the lower bound on the weak resilience of a dynamical flow network when the distributed routing policy $\mathcal{G}$ is locally responsive. To start with, let us recall that in this case Theorem 1 implies the existence of a globally attractive limit flow $\tilde{f}^* \in \mathrm{cl}(\mathcal{F})$ for the perturbed dynamical flow network associated to any admissible perturbation $\tilde{\mathcal{N}}$. Define $\tilde{\lambda}^*_0 = \lambda_0$, and $\tilde{\lambda}^*_v = \sum_{e\in\mathcal{E}^-_v}\tilde{f}^*_e$, for $0 < v \leq n$.

Lemma 9: Consider a dynamical flow network $\mathcal{N}$ satisfying Assumptions 1 and 2, with locally responsive distributed routing policy $\mathcal{G}$. For every $\theta \geq 1$, there exists $\beta_\theta \in (0, 1)$ such that, if $\tilde{\mathcal{N}}$ is an admissible perturbation of $\mathcal{N}$ with stretching coefficient less than or equal to θ, and $\tilde{f}^* \in \mathrm{cl}(\tilde{\mathcal{F}})$ is the limit flow vector of the corresponding perturbed dynamical flow network (11), then

$$\tilde{f}^*_e \geq \beta_\theta\tilde{\lambda}^*_v\,,$$

for every non-destination node $0 \leq v < n$, and every link $e \in \mathcal{E}^+_v$ for which $\tilde{f}^*_e \leq \tilde{f}^{\max}_e/2$.

Proof: Fix some link $e \in \mathcal{E}$ for which $\tilde{f}^*_e \leq \tilde{f}^{\max}_e/2$. Define $\rho^\theta \in \mathcal{R}_v$ by $\rho^\theta_j = 0$ for all $j \in \mathcal{E}^+_v$, $j \neq e$, and $\rho^\theta_e = \theta\rho^\mu_e$, where recall that ρ^μ_e is the median density of the flow function μ_e. Since the stretching coefficient of $\tilde{\mathcal{N}}$ is less than or equal to θ, one has that the median densities of the perturbed and the unperturbed flow functions satisfy $\tilde{\rho}^\mu_e \leq \theta\rho^\mu_e$. This and the fact that $\tilde{f}^*_e \leq \tilde{f}^{\max}_e/2$ imply that $\tilde{\rho}^*_e \leq \tilde{\rho}^\mu_e \leq \rho^\theta_e$, while clearly $\tilde{\rho}^*_j \geq 0 = \rho^\theta_j$ for all $j \in \mathcal{E}^+_v$, $j \neq e$. Now, let $\beta_\theta := G^v_e(\rho^\theta)$, and observe that, thanks to Property (b) of Definition 8, one has $\beta_\theta > 0$. Then, from Lemma 1 one gets that

$$G^v_e(\tilde{\rho}^*) = \frac{1}{2}\left(G^v_e(\tilde{\rho}^*) + 1 - \sum\nolimits_{j\neq e} G^v_j(\tilde{\rho}^*)\right) \geq \frac{1}{2}\left(G^v_e(\rho^\theta) + 1 - \sum\nolimits_{j\neq e} G^v_j(\rho^\theta)\right) = G^v_e(\rho^\theta) = \beta_\theta\,. \qquad (51)$$

On the other hand, since $\tilde{f}^*_e \leq \tilde{f}^{\max}_e/2 < \tilde{f}^{\max}_e$, Lemma 2 implies that necessarily $\tilde{\lambda}^*_v G^v_e(\tilde{\rho}^*) = \tilde{f}^*_e$. The claim now follows by combining this and (51). ■

We are now ready to prove the following lower bound on the resilience.

Lemma 10: Let $\mathcal{N}$ be a flow network satisfying Assumptions 1 and 2, $\lambda_0 \in [0, C(\mathcal{N}))$ a constant inflow, and $\mathcal{G}$ a locally responsive distributed routing policy. Let $f^* \in \mathcal{F}(\lambda_0)$ be an equilibrium for the dynamical flow network (5). Then, for every $\theta \geq 1$, and every $\alpha \in (0, \beta_\theta^n]$, the resilience of the associated dynamical flow network satisfies

$$\gamma_{\alpha,\theta}(f^*, \mathcal{G}) \geq C(\mathcal{N}) - 2|\mathcal{E}|\lambda_0 \beta_\theta^{1-n}\alpha\,,$$

where $\beta_\theta \in (0,1)$ is as in Lemma 9.

Proof: Consider an arbitrary admissible perturbation $\tilde{\mathcal{N}}$ of stretching coefficient less than or equal to θ, and of magnitude

$$\delta \leq C(\mathcal{N}) - 2|\mathcal{E}|\lambda_0 \beta_\theta^{1-n}\alpha\,. \tag{52}$$

We shall iteratively select a sequence of nodes $0 =: v_0, v_1, \ldots, v_k := n \in \mathcal{V}$ such that, for every $1 \leq j \leq k$,

$$\exists i \in \{0, \ldots, j-1\} \qquad \text{such that} \qquad (v_i, v_j) \in \mathcal{E}\,, \quad \tilde{f}^*_{(v_i,v_j)} \geq \lambda_0 \alpha \beta_\theta^{j-n}\,. \tag{53}$$

Since $v_k = n$, and $\beta_\theta^{k-n} \geq 1$, the above with $j = k \leq n$ will immediately imply that

$$\lim_{t\to+\infty} \tilde{\lambda}_n(t) = \tilde{\lambda}^*_n = \sum\nolimits_{e\in\mathcal{E}_n^-} \tilde{f}^*_e \geq \alpha\lambda_0\beta_\theta^{k-n} \geq \alpha\lambda_0\,,$$

so that the perturbed dynamical flow network is α-transferring. The claim will then readily follow from the arbitrariness of the considered admissible perturbation.

First, let us consider the case $j = 1$. Assume by contradiction that $\tilde{f}^*_e < \lambda_0\alpha\beta_\theta^{1-n}$, for every link $e \in \mathcal{E}_0^+$. Since $\alpha \leq \beta_\theta^n$, this would imply that $\tilde{f}^*_e < \beta_\theta\lambda_0$ and hence, by Lemma 9, that $\tilde{f}^{\max}_e \leq 2\tilde{f}^*_e$ for all $e \in \mathcal{E}_0^+$, so that

$$\sum\nolimits_{e\in\mathcal{E}_0^+} \tilde{f}^{\max}_e \leq 2\sum\nolimits_{e\in\mathcal{E}_0^+} \tilde{f}^*_e < 2\alpha|\mathcal{E}_0^+|\beta_\theta^{1-n}\lambda_0 \leq 2\alpha|\mathcal{E}|\beta_\theta^{1-n}\lambda_0\,.$$

Combining the above with the inequality $C(\mathcal{N}) \leq \sum_{e\in\mathcal{E}_0^+} f^{\max}_e$, one would get

$$\delta \geq \sum\nolimits_{e\in\mathcal{E}_0^+}\left(f^{\max}_e - \tilde{f}^{\max}_e\right) > C(\mathcal{N}) - 2\alpha|\mathcal{E}|\beta_\theta^{1-n}\lambda_0\,,$$

thus contradicting the assumption (52). Hence, necessarily there exists $e \in \mathcal{E}_0^+$ such that $\tilde{f}^*_e \geq \lambda_0\alpha\beta_\theta^{1-n}$, and choosing v_1 to be the unique node in $\mathcal{V}$ such that $e \in \mathcal{E}_{v_1}^-$, one sees that (53) holds true with $j = 1$.

Now, fix some $1 < j^* \leq k$, and assume that (53) holds true for every $1 \leq j < j^*$. Then, by choosing i as in (53), one gets that

$$\tilde{\lambda}^*_{v_j} = \sum\nolimits_{e\in\mathcal{E}_{v_j}^+} \tilde{f}^*_e \geq \tilde{f}^*_{(v_i,v_j)} \geq \lambda_0\alpha\beta_\theta^{j-n} \geq \lambda_0\alpha\beta_\theta^{j^*-1-n}\,, \qquad \forall 1 \leq j < j^*\,. \tag{54}$$

Moreover,

$$\tilde{\lambda}^*_{v_0} = \lambda_0 > \lambda_0\alpha\beta_\theta^{-n} \geq \lambda_0\alpha\beta_\theta^{j^*-1-n}\,. \tag{55}$$

Let $\mathcal{U} := \{v_0, v_1, \ldots, v_{j^*-1}\}$ and $\mathcal{E}_\mathcal{U}^+ \subseteq \mathcal{E}$ be the set of links with tail node in $\mathcal{U}$ and head node in $\mathcal{V}\setminus\mathcal{U}$. Assume by contradiction that $\tilde{f}^*_e < \lambda_0\alpha\beta_\theta^{j^*-n}$ for all $e \in \mathcal{E}_\mathcal{U}^+$. Thanks to (54) and (55), this would imply that, $\tilde{f}^*_e < \beta_\theta\tilde{\lambda}^*_j$, for every $0 \leq j < j^*$ and $e \in \mathcal{E}_{v_j}^+ \cap \mathcal{E}_\mathcal{U}^+$. Then, Lemma 9 would imply that $\tilde{f}^{\max}_e \leq 2\tilde{f}^*_e$ for all $e \in \mathcal{E}_\mathcal{U}^+ = \cup_{j=0}^{j^*-1}(\mathcal{E}_{v_j}^+ \cap \mathcal{E}_\mathcal{U}^+)$. This would yield

$$\sum\nolimits_{e\in\mathcal{E}_\mathcal{U}^+} \tilde{f}^{\max}_e \leq \sum\nolimits_{e\in\mathcal{E}_\mathcal{U}^+} 2\tilde{f}^*_e < 2\sum\nolimits_{e\in\mathcal{E}_\mathcal{U}^+} \lambda_0\alpha\beta_\theta^{j^*-n} \leq 2|\mathcal{E}|\lambda_0\alpha\beta_\theta^{1-n}\,.$$

From the above and the inequality $C(\mathcal{N}) \leq \sum_{e\in\mathcal{E}_\mathcal{U}^+} f^{\max}_e$, one would get

$$\delta \geq \sum\nolimits_{e\in\mathcal{E}_\mathcal{U}^+}\left(f^{\max}_e - \tilde{f}^{\max}_e\right) > C(\mathcal{N}) - 2\alpha|\mathcal{E}|\beta_\theta^{1-n}\lambda_0\,,$$

thus contradicting the assumption (52). Hence, necessarily there exists $e \in \mathcal{E}_\mathcal{U}^+$ such that $\tilde{f}^*_e \geq \lambda_0\alpha\beta_\theta^{1-n}$, and choosing v_{j^*} to be the unique node in $\mathcal{V}$ such that $e \in \mathcal{E}_{v_{j^*}}^-$ one sees that (53) holds true with $j = j^*$. Iterating this argument until $v_{j^*} = n$ proves the claim. ∎

It is now easy to see that Lemma 10 implies that $\lim_{\alpha\downarrow 0}\gamma_{\alpha,\theta} \geq C(\mathcal{N})$ for every $\theta \geq 1$, thus showing that $\gamma_0 \geq C(\mathcal{N})$.

IX. Conclusion

In this paper, we studied robustness properties of dynamical flow networks, where the dynamics on every link is driven by the difference between the inflow, which depends on the upstream routing decisions, and the outflow, which depends on the particle density, on that link. We considered distributed routing policies that depend only on the local information about the particle densities in the network. We proposed a class of locally responsive distributed routing policies that yield the maximum resilience under local information constraint, with respect to malicious disturbances that reduce the flow functions of the links of the network. We also established the relationship between the resilience and the topology as well as the initial equilibrium flow of the network. The findings of this paper stand to provide important guidelines for management of several large scale critical infrastructures both from planning as well as real-time operation point of view.

In future, we plan to extend the research in several directions. We plan to rigorously study the robustness properties of the network with finite link-wise capacity for the densities, and formally establish the results on the resilience as suggested in Section V. We plan to study the scaling of the resilience with respect to the amount of information, e.g., multi-hop as opposed to just single-hop, available to the routing policies. We also plan to perform robustness analysis in a probabilistic framework to complement the adversarial framework of this paper, possibly considering other general models for disturbances. In particular, it would be interesting to study robustness with respect to sequential disturbances than just one-shot disturbance considered in this paper. We plan to consider a setting with buffer capacities on the nodes and study the scaling of the resilience with such buffer capacities. We also plan to consider more general graph topologies, e.g., graphs having cycles and multiple origin-destination pairs.

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