Robustness Analysis of Transportation Networks

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Abstract

In this paper, we study robustness properties of transportation networks with respect to its pre-disturbance equilibrium operating condition and the agents' response to disturbances under information constraints. We perform the analysis within a dynamical system framework over a directed acyclic graph between a single origin-destination pair. The dynamical system consists of a system of ordinary differential equations (ODEs), one for every edge of the graph. Every ODE is a mass balance equation for the corresponding edge, where the inflow term is a function of the agents' route choice behavior and the arrival rate at the base node of that edge, and the outflow term is related to the congestion properties of the edge. We consider disturbances that affect the congestion properties of the network by reducing the maximum flow carrying capacity of the edges and define the *margin of stability* of the network as the minimum capacity that needs to be removed from the network so that the traffic density on some of the edges grows unbounded in time. For a given pre-disturbance equilibrium operating condition, we derive upper bounds on the margin of stability under local information constraint on the agents' behavior and characterize the route choice behaviors that match this bound exactly. We also setup a simple optimization problem to find the most robust pre-disturbance equilibrium operating condition for the network and determine a set of edge-wise tolls that yield such a desired equilibrium operating condition.

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1 Introduction

Social planning for efficient usage of transportation networks (TNs) is attracting renewed research interest as transportation demand is fast approaching its infrastructure capacity. While there exists an abundant literature on socially optimal traffic assignments, e.g., see [2], robustness analysis of TNs has received very little attention. In this paper, we study the relationship of the robustness properties of a large-scale TN to its pre-disturbance equilibrium operating condition and the agents' response to the disturbance under information constraints.

We abstract the topology of the transportation network by a directed acyclic graph between a single origin-destination pair. For the analysis, we adopt a dynamical system framework that is composed of a system of ordinary differential equations (ODEs), one for every edge of the graph. Every ODE represents a mass balance equation for the corresponding edge, where the inflow term is a function of the agents' route choice behavior and the arrival rate at the base node of that edge, and the outflow term is function of the congestion properties of the edge. We consider a setup where, before the disturbance, the network is operating at an equilibrium operating condition and information about this equilibrium condition is shared by all the agents. Such an equilibrium condition might be thought of as the outcome of a slower time-scale learning process, e.g., see [3, 4, 5], in presence of incentive mechanisms such as tolls, e.g., see [6, 7]. After the disturbance, we assume that the global knowledge of the agents remains fixed and that the agents act by complementing the fixed global knowledge with real-time instantaneous local information. Such a setup is meant is give insight into the evolution of the network in the immediate aftermath of a disruption when the availability of accurate global information about the whole network is sparse or it is too time-consuming for the agents to incorporate the real-time information about the whole network because of the huge computational burden involved.

We consider disturbances that reduce the maximum flow carrying capacities of the edges by affecting their congestion properties. We define the margin of stability of the TN to be the maximum sum of capacity losses, under which the traffic densities on some of the edges grows unbounded in time. We then prove that, irrespective of the route choice behavior of the agents, the margin of stability is upper-bounded by the minimum of all the *node cuts* of the residual capacities of the TN. We then characterize the route choice behaviors that match this upper bound. Finally, we study the dependence of the margin of stability on the equilibrium, and formulate a simple optimization problem for finding the most robust equilibria. This is, in general, different from the classical socially optimal equilibrium, as well as from the user-optimal equilibrium. We also discuss the utility of tolls in yielding a desired equilibrium operating condition. Our results provide important guidelines for social planners in terms of determining robust equilibrium operating conditions and route choice behaviors for TNs. Alternate notions of robustness for networks have been proposed in [8, 9, 10].

The contributions of the paper are as follows: (i) we formulate a novel dynamical system framework for robustness analysis of transportation networks with respect to agents' response to disturbances under information constraints, (ii) we derive an upper bound on the margin of stability of the network under local information constraint and characterize the class of route choice behaviors under which this bound is tight, and (iii) we postulate the notion of *robustness price of anarchy* to quantify the loss in robustness due to sub-optimal equilibrium operating condition of a network and determine the set of edge-wise tolls that reduce this gap to zero.

The technical results of this paper rely on tools from several disciplines. The upper bounds on the margin of stability for a given equilibrium operating condition uses graph theory notions from flow networks, e.g., see [11]. The properties of the route choice functions that give maximum margin of stability are reminiscent of cooperative dynamics, e.g., see [12]. The problem of determining tolls for a desired equilibrium condition exploits the fact that the associated congestion game is a potential game and that the extremum of the potential function corresponds to the equilibrium.

The rest of the paper is organized as follows. In Section **??**, we describe basic notations and concepts useful for the paper and formulate the robustness analysis problem. In Sections **??** and **??**, we derive bounds on the margin of stability of the network. Section 4 discusses the problem of selection of the most robust equilibrium operating point of the network. In Section 5, we report simulation results. Finally, we conclude in Section 6 with remarks on future research directions.

Before proceeding, we introduce some basic notation to be used throughout the paper. .... DIRECTED GRAPH For each node $v \in \mathcal{V}$, $\mathcal{E}_v^+$ (respectively, $\mathcal{E}_-^v$) will denote the sets of its outgoing (incoming) edges, while $\mathcal{R}_v := \mathbb{R}_+^{\mathcal{E}_v^+}$, and $\mathcal{S}_v := \{p \in \mathcal{R} : \sum_e p_e = 1\}$, will stay for the set of non-negative vectors, and, respectively, of probability vectors over $\mathcal{E}_v^+$. VECTOR LABELING

2 Transportation networks and their margin of stability

In this section, we introduce our model of a dynamic transportation network. Then, we define its margin of stability, and present our main result characterizing it.

2.1 Dynamic transportation networks

We shall model a *transportation network* by:

(i) a topology, described as a finite directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} := \{0, 1, \dots, n\}$ is the node set, $\mathcal{E}$ is the link set, and 0 and n are respectively a source (i.e. a node without incoming edges) and a sink (i.e. a node without outgoing edges);

(ii) a family of flow functions $\mu_e : \mathbb{R}_+ \to \mathbb{R}_+$, describing the functional dependence $f_e = \mu_e(\rho_e)$ of the current flow f_e on the current density ρ_e on every link $e \in \mathcal{E}$;

(iii) a family of myopic route-choice functions $G^v : \mathcal{R}_v \to \mathcal{S}_v$ describing the relative frequency with which agents passing through some intermediate node $v \in \mathcal{V} \setminus \{d\}$, and observing a current local density $\rho^v := (\rho_e)_{e \in \mathcal{E}_v^+} \in \mathcal{R}_v$ choose the different outgoing links.

We shall assume that there is a constant incoming flow λ_0 at node 0. MORE EXPLANATION COMES HERE Then, an application of mass conservation laws both on the links and at the nodes of $\mathcal{G}$ leads one to consider the following dynamical system

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\rho_e(t) = \lambda_v(t) G^v_e(\rho^v(t)) - f_e(t)\,, \qquad \forall\, 0 \le v < n\,, \quad \forall\, e \in \mathcal{E}_v^+\,, \\ &f_e(t) := \mu_e(\rho_e(t))\,, \\ &\lambda_v(t) := \left\{ \begin{array}{lll} \sum_{e \in \mathcal{E}_v^-} f_e(t) & \text{if} & 0 < v < n \\ \lambda_0 & \text{if} & v = 0 \end{array} \right. \end{aligned} \tag{1}$$

to any transportation network as above.

The main feature of the above-proposed model of a dynamic transportation network resides in the fact that the myopic route choice G^v at the intermediate nodes $v = 0, 1, \dots, n-1$ is a function only of the local information on the current traffic density available to the agents passing through node v. On the other hand, the structural form of such a dependence may depend on some global information on the traffic network which has been accumulated through a slower time-scale process. While such a two time-scale process has been analyzed in our related work [] through a singular perturbation approach, the focus of this paper is on the fast scale dynamics described by the dynamic system (1), and on its behavior in the presence of perturbations of the system. For this reason, the route choice is approximated by a static function of the local information.

2.2 Main structural assumptions

Throughout the paper, we shall make the following assumption on the graph topology:

Assumption 1 *The graph $\mathcal{G}$ is acyclic.*

It follows from the Assumption 1 (see Appendix ??? OR DO WE HAVE AN EXPLICIT REFERENCE?) that the nodes of $\mathcal{G}$ can be labeled in such a way that

$$\mathcal{E}^-_{v+1} \subseteq \bigcup\nolimits_{0 \le u \le v} \mathcal{E}^+_u \,, \qquad \forall 0 \le v < n \,. \tag{2}$$

We shall assume such an ordering to have been chosen once for all, and stick to it throughout the paper.

We shall make the following assumption on the flow functions:

Assumption 2 *For every link $e \in \mathcal{E}$, the map $\mu_e : \mathbb{R}_+ \to \mathbb{R}_+$ is Lipschitz continuous, strictly increasing, and such that $\mu_e(0) = 0$, and the flow capacity*

$$f^{\max}_e := \sup\{\mu_e(\rho_e) : \rho_e \in \mathbb{R}_+\} \tag{3}$$

is finite.

It is important to observe that the typical flow functions analyzed in the transportation literature are not strictly increasing on all $\mathbb{R}^+$, but rather have a $\cap$-shaped graph. While we shall perform our analysis under the simplifying Assumption 2 for the clarity of exposition, as we shall clarify later on (see Sect. ???), our results will allow us to analyze these more realistic models as well.

We now proceed to describe the structural assumptions on the myopic route-choice function. The first one is:

Assumption 3 *For all intermediate node $v = 0, 1, \dots, n-1$, the map $G^v : \mathcal{R}_v \to \mathcal{S}_v$ is differentiable and such that*

$$\frac{\partial}{\partial \rho_j} G_e(\rho^v) > 0 \,, \qquad \forall j \neq e \in \mathcal{E}^+_v \,, \quad \forall \rho^v \in \mathcal{R}_v \,. \tag{4}$$

Assumption 3 captures a fundamental feature of the myopic behavior of the agents in response to their current available local information: as the current density on some link is increased, the probability of choosing each of the other links increases. This assumption is reminiscent of the notion of cooperative dynamic system [**?**, **?**]. As we shall see, it will prove fundamental for the validity of our main results. We shall further assume that:

Assumption 4 *For all $0 \leq v < n$, and every sequence $\{\rho^v(m) : m \in \mathbb{N}\}$ such that*

$$\lim_{m \to +\infty} \rho_j(m) = +\infty \,, \qquad \limsup_{m \to +\infty} \rho_e(m) < +\infty \,,$$

for some $j, e \in \mathcal{E}_v^+$, one has that

$$\lim_{m \to +\infty} G_j^v(\rho^v(m)) = 0 \,. \tag{5}$$

Assumption 4 guarantees that, if the density is exploding on one but not on all the outgoing links from a given node, then the frequency with which that link is chosen drops down to zero..... ADD SOME EXPLANATION: THAT SHOULD NOT BE HARD TO JUSTIFY Observe that Assumptions 3 and 4 are completely local, for they do not involve any global knowledge of the network topology or flow functions.

Finally, we shall need the following structural assumption of global nature:

Assumption 5 *There exists an equilibrium density $\rho^{eq} \in \mathbb{R}_+^{\mathcal{E}}$ for the dynamical system (1).*

NEED TO PUT JUSTIFICATION HERE
The following result guarantees uniqueness of the equilibrium, and is proved in the Appendix.

Lemma 2.1 *Let Assumptions 1, 2, 3, and 5 hold. Then, ρ^{eq} is the unique, globally attractive equilibrium of the dynamical system (1).*

2.3 Perturbed systems and margin of stability

We shall consider perturbations of the dynamical system (1), described as a reduction of the flow functions on the links. Specifically, for each link $e \in \mathcal{E}$, we shall consider a perturbed flow function $\tilde{\mu}_e(\,\cdot\,)$, satisfying Assumption 2, and such that

$$\tilde{\mu}_e(\rho_e) \leq \mu_e(\rho_e) \,, \qquad \forall \rho_e \geq 0 \,.$$

Such a modification of the flow functions will be referred to as an *admissible perturbation*. We shall consider the *perturbed dynamical system*

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\rho}_e(t) = \tilde{\lambda}_v(t) G_e^v(\tilde{\rho}^v(t)) - \tilde{f}_e(t)\,, \qquad \forall\, 0 \leq v < n\,, \quad \forall\, e \in \mathcal{E}_v^+\,,\\
&\tilde{f}_e(t) := \tilde{\mu}_e(\tilde{\rho}_e(t))\,,\\
&\tilde{\lambda}_v(t) := \left\{ \begin{array}{lll} \sum_{e\in\mathcal{E}_v^-} \tilde{f}_e(t) & \text{if} & 0 < v < n \\ \lambda_0 & \text{if} & v = 0\,, \end{array} \right.
\end{aligned}
\tag{6}
$$

and call it stable if its solution starting from the equilibrium ρ^* of the unperturbed system remains bounded in time.

We shall measure the magnitude of an admissible perturbation by the total reduction of the maximum flow capacity:

$$
\delta := \sum_{e\in\mathcal{E}} \delta_e\,, \qquad \delta_e := \sup\left\{\tilde{\mu}_e(\rho_e) - \mu_e(\rho_e) :\ \rho_e \geq 0\right\}\,. \tag{7}
$$

Finally, we shall define the margin of stability of the transportation network as the infimum of the magnitudes of the admissible perturbations which make the perturbed system (6) unstable. The following is our main result, characterizing the margin of stability of a transportation network:

Theorem 2.2 *The margin of stability of a transportation network satisfying Assumptions 1-5 is given by*

$$
\gamma := \min\left\{ \sum\nolimits_{e\in\mathcal{E}_v^+} \left(f_{\mathrm{e}}^{\max} - f_e^* \right) :\ v = 0, 1, \ldots, n-1 \right\}\,. \tag{8}
$$

Remark 2.3 *It is worth comparing the quantity γ defined in (8) with*

$$
\Gamma := \min\left\{ \sum\nolimits_{e\in\mathcal{E}_\mathcal{S}^+} f_{\mathrm{e}}^{\max} :\ \mathcal{S} \subseteq \mathcal{V},\, 0 \in \mathcal{S},\, n \notin \mathcal{S} \right\} - \lambda_0\,, \tag{9}
$$

where $\mathcal{E}_\mathcal{S}^+ := \{(u,v) \in \mathcal{E} :\ u \in \mathcal{S},\, v \notin \mathcal{S}\}$ is set of outgoing edges from a node subset $\mathcal{S} \subseteq \mathcal{V}$. The quantity Γ defined in (9) is the difference between the minimum capacity of all cut-set of the transportation network separating the origin node 0 from the destination node n, and the incoming flow λ_0. It is not hard to verify that, in general, $\gamma \leq \Gamma$: to see this, it is sufficient to consider, for every $\mathcal{S} \subseteq \mathcal{V}$ with $0 \in \mathcal{S}$ and $n \notin \mathcal{S}$, the node $v_\mathcal{S} := \max \mathcal{S}$, and observe that, thanks to (2), $\mathcal{E}_{v_\mathcal{S}}^+ \subseteq \mathcal{E}_\mathcal{S}^+$, and then

$$
\sum\nolimits_{e\in\mathcal{E}_\mathcal{S}^+} f_{\mathrm{e}}^{\max} - \lambda_0 = \sum\nolimits_{e\in\mathcal{E}_\mathcal{S}^+} (f_{\mathrm{e}}^{\max} - f_e^*) \geq \sum\nolimits_{e\in\mathcal{E}_{v_\mathcal{S}}^+} (f_{\mathrm{e}}^{\max} - f_e^*)\,.
$$

(DOES THIS DESERVE TO BE CLARIFIED FURTHER?)
While in some particular cases, such as the parallel link topology of Example ???, one may have $\Gamma = \gamma$, the gap $\Gamma - \gamma$ can be arbitrarily large, as the example proposed in Fig.??? shows.
SHOULD WE PUT AN EXAMPLE HERE OR IN A SEPARATE ENVIRONMENT SHOWING THE GAP CAN BE ARBITRARILY LARGE??
*Using arguments along the lines of those employed in Sect. 3.1, is not hard to show that Γ provides an upper bound on the margin of stability even if the locality constraint on the information used for the agents' myopic route choice is removed. In fact, one may exhibit agents' route choice which are functions of the global current traffic density, for which the margin of stability is exactly Γ [**?**]. Hence, one may interpret the gap $\Gamma - \gamma$ as the margin of stability loss due to the locality constraint on the information available to the agents.*

In fact, it is possible to consider intermediate levels of information available to the agents, which interpolate between the one-hop information of our current modeling of the transportation network, and the global information described above. EXPAND ON THIS??

3 Proof of the main result

In this section, we shall prove Theorem 2.2. First, we shall show that γ is indeed an upper bound on the margin of stability of the transportation network. This will follow only from the assumption of acyclicity of the network topology, and locality of the information available to the agents, and will be independent from Assumptions 3 and 4 on the route choice function. In contrast, these assumptions will prove fundamental when showing that γ is also a lower bound on the margin of stability of the transportation network. In particular, Assumption 3 on the cooperative nature of the local route choice function will allow us to prove some key monotonicity properties for the solution of the perturbed dynamical system. Our arguments will also lead to a proof of Lemma 2.1.

3.1 Upper bound on the margin of stability

We start by proving that γ is indeed an upper bound on the margin of stability. To see this, it is sufficient to exhibit a family of perturbations, with magnitude δ arbitrarily close to γ, which make the system unstable. Let us fix some $v \in \{0, 1, \ldots, n-1\}$ minimizing the right-hand side of (8), put $\beta := \sum_{e \in \mathcal{E}_v^+} f_e^{\max}$ and for any $\delta \in (\gamma, \beta)$, consider the perturbed flow functions

$$\begin{aligned} \tilde{\mu}_e(\rho_e) &:= \tfrac{\beta-\delta}{\beta}\mu_e(\rho_e)\,, \qquad && \forall e \in \mathcal{E}_v^+\,, \\ \tilde{\mu}_e(\rho_e) &:= \mu_e(\rho_e)\,, && \forall e \in \mathcal{E} \setminus \mathcal{E}_v^+\,. \end{aligned} \tag{10}$$

Clearly this is an admissible pertubation, and has magnitude δ. Thanks to (2), an inductive argument easily shows that the solution of the perturbed system (6) satisfies $\tilde{\rho}_e(t) = \rho_e^*$, for all $t \geq 0$, $u \leq v$, and $e \in \mathcal{E}_u^+$. Hence, in particular $\lambda_v(t) = \beta - \gamma$, so that

$$\frac{\mathrm{d}}{\mathrm{d}t} \sum_{e \in \mathcal{E}_v^+} \tilde{\rho}_e(t) = \lambda_v(t) - \sum_{e \in \mathcal{E}_v^+} \tilde{f}_e(t) \geq \beta - \gamma - \sum_{e \in \mathcal{E}_v^+} \tilde{f}_e^{\max} = \delta - \gamma > 0\,.$$

It then follows from Gronwall's inequality that

$$\sum_{e \in \mathcal{E}_v^+} \tilde{\rho}_e(t) \geq \sum_{e \in \mathcal{E}_v^+} \rho_e^* + (\delta - \gamma)t\,, \tag{11}$$

which shows that the traffic network is unstable. Then, from the arbitrariness of the perturbation's magnitude $\delta \in (\gamma, \beta)$, it follows that the traffic network's margin of stability is upper-bounded by γ. As remarked at the beginning of this section, notice that this upper bound on the margin of stability does not depend on Assumptions 3, and 4, but only on the acyclicity of the network, and locality of the route choice.

3.2 Lower bound on the margin of stability

We shall now prove that γ is also a lower bound on the margin of stability of the transportation network. We shall prove this through a series of intermediate results.

We start with the following result ensuring existence of a local equilibrium for the perturbed system. (IN THE FOLLOWING LEMMA POSSIBLY SWITCH TILDE TO HAT)

Lemma 3.1 *For every intermediate node $v \in \{0, 1, \ldots, n-1\}$, and every constant local input flow $\tilde{\lambda}_v \in (0, \sum_{e \in \mathcal{E}_v^+} \tilde{f}_e^{\max})$, there exists a unique local equilibrium density vector $\rho^* \in \mathcal{R}_v$, and corresponding local equilibrium flow $\tilde{f}^* := \tilde{\mu}(\tilde{\rho}^*)$ such that*

$$\tilde{\lambda}_v G_e^v(\tilde{\rho}^*) = \tilde{f}_e^*\,, \qquad \forall e \in \mathcal{E}_v^+\,.$$

Proof Consider the set $\mathcal{F} := \{f \in \mathcal{R}_v : \sum_j f_j = \tilde{\lambda}_v\,,\ f_j < \tilde{f}_j^{\max}, \forall j \in \mathcal{E}_v^+\}$ of feasible equilibrium flows, and let $\overline{\mathcal{F}}$ be its closure in $\mathcal{R}_v$. Let $\tilde{\mu}^{-1} : \mathcal{F} \to$

$\mathcal{R}_v$ be the component-wise inverse of the perturbed flow function on the outgoing edges. For $j \in \mathcal{E}_v^+$, consider the real-valued function

$$h_j(f) := \tilde{\lambda}_v G_j^v \left(\tilde{\mu}^{-1}(f)\right) - f_j \,, \qquad \forall f \in \mathcal{F} \,,$$

and extend it by continuity to $\overline{\mathcal{F}}$. First observe that, for all $f \in \overline{\mathcal{F}}$, $\sum_j h_j(f) = \tilde{\lambda}_v - \sum_j f_j = 0$. Moreover, $h_j(f) = \tilde{\lambda}_v G_j^v(f) \geq 0$ for all $f \in \mathcal{F}$ is such that $f_j = 0$. Finally, thanks to Assumption 4, one has that, if $f \in \overline{\mathcal{F}}$ is such that $f_j = \tilde{f}_j^{\max}$, then $h_j(f) = -\tilde{f}_j^{\max} \leq 0$. It follows that, if $\Phi_j^t(f) = f_j(t)$ for $t \geq 0$, where $f(t)$ is the solution of the initial value problem

$$\frac{\mathrm{d}}{\mathrm{d}t} f_j(t) = h_j(f(t)) \,, \qquad f_j(0) = f_j \,, \qquad j \in \mathcal{E}_v^+ \,, \tag{12}$$

then $\Phi^t(f) \in \overline{\mathcal{F}}$ for all $f \in \overline{\mathcal{F}}$. Now, fix some $f, g \in \overline{\mathcal{F}}$, and, for $t \geq 0$, put $\delta(t) := ||\Phi^t(f) - \Phi^t(g)||_1$, and let $\mathcal{J}, \mathcal{K} \subseteq \mathcal{E}_v^+$ be such that $\Phi_j^t(f) > \Phi_j^t(g)$ iff $j \in \mathcal{J}$, and $\Phi_k^t(f) < \Phi_k^t(g)$ iff $k \in \mathcal{K}$. Thanks to Assumption 3, one has that

$$\sum\nolimits_{j \in \mathcal{J}} G_j^v(\mu^{-1}(\Phi^t(f))) \leq \sum\nolimits_{j \in \mathcal{J}} G_j^v(\mu^{-1}(\Phi^t(g))) \,,$$

and, similarly,

$$\sum\nolimits_{k \in \mathcal{K}} G_k^v(\mu^{-1}(\Phi^t(f))) \geq \sum\nolimits_{k \in \mathcal{K}} G^k(\mu^{-1}(\Phi^t(g)) \,.$$

As a consequence

$$\sum_{e \in \mathcal{E}_v^+} \operatorname{sgn}(\Phi_e^t(f) - \Phi_e^t(g)) \left(G_e^v(\Phi^t(f)) - G_e^v(\Phi^t(g))\right) \leq 0 \,, \quad t \geq 0 \,.$$

It follows that

$$\begin{aligned}
\chi(t) &:= ||\Phi^t(f) - \Phi^t(g)||_1 \\
&= \int_0^t \sum_e \operatorname{sgn}(\Phi_e^s(f) - \Phi_e^s(g)) \left(h_e(\Phi^s(f)) - h_e(\Phi^s(g))\right) \mathrm{d}s \\
&\leq \int_0^t \sum_e \operatorname{sgn}(\Phi_e^s(f) - \Phi_e^s(g)) \left(\Phi_e^s(g) - \Phi_e^s(f)\right) \mathrm{d}s \\
&= \int_0^t -\chi(s) \mathrm{d}s \,,
\end{aligned}$$

and then Gronwall's inequality implies that

$$||\Phi^t(f) - \Phi^t(g)||_1 = \chi(t) \leq \chi(0) e^{-t} = ||f - g||_1 e^{-t} \,, \qquad \forall f, g \in \overline{\mathcal{F}} \,. \tag{13}$$

Therefore, for all $t > 0$, $\Phi^t : \overline{\mathcal{F}} \to \overline{\mathcal{F}}$ is a contraction, and it admits a unique fixed point $\tilde{f}^* \in \overline{\mathcal{F}}$ by Banach's contraction mapping principle. Applying (13) with $g = \tilde{f}^*$ and arbitrary $f(0) = f \in \overline{\mathcal{F}}$ shows that $\tilde{f}^*$ is a (globally attractive) fixed point for the system (12), so that $h_e(\tilde{f}^*) = 0$ for all $e \in \mathcal{E}_v^+$. In particular, this implies that necessarily $\tilde{f}_e^* < \tilde{f}_e^{\max}$ for all e such that $\tilde{f}_e^{\max} > 0$. Therefore, for all $e \in \mathcal{E}_v^+$, one has that $\tilde{\rho}_e^* := \tilde{\mu}_e^{-1}(\tilde{f}_e^*) < +\infty$, and satisfies

$$\tilde{\lambda}_v G_e^v(\rho^*) = h_e(\tilde{f}^*) + \tilde{f}_e^* = \tilde{\mu}_e(\rho_e^*)\,, \qquad e \in \mathcal{E}_v^+\,,$$

which concludes the proof.

The following result establishes some local stability and diffusivity properties of the perturbed system, which mainly rely on Assumption 3 on the route choice function. More specifically, it shows that the increase in flow on all the edges downstream from a node whose outgoing edges are affected by a given perturbation, is less than the magnitude of the disturbance itself.

Lemma 3.2 *Consider some admissible perturbation, of magnitude as in (7), and some intermediate node $v \in \{0, 1, , \ldots, n-1\}$. Let $\tilde{\lambda}_v(t)$ be a continuous local input flow satisfying*

$$\sup\{\tilde{\lambda}_v(t) : t \geq 0\} = \tilde{\lambda}_v^* \leq \sum\nolimits_{e \in \mathcal{E}_v^+} \tilde{f}_e^{\max}\,, \qquad t \geq 0\,, \tag{14}$$

and consider the local perturbed system

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\tilde{\rho}_e = \tilde{\lambda}_v(t) G_e^v(\tilde{\rho}^v) - \tilde{\mu}_e(\tilde{\rho}_e)\,, \\ \tilde{\rho}_e(0) = \rho_e^*\,, \qquad\qquad e \in \mathcal{E}_v^+\,. \end{cases}$$

Then, for all $t \geq 0$, and every subset of outgoing links $\mathcal{J} \subseteq \mathcal{E}_v^+$,

$$\sum\nolimits_{e \in \mathcal{J}} (\tilde{\mu}_e(\tilde{\rho}_e(t)) - f_e^*) \leq \left[\tilde{\lambda}_v^* - \lambda_v^*\right]_+ + \sum\nolimits_{e \in \mathcal{E}_v^+} \delta_e\,. \tag{15}$$

Proof Let $\hat{\lambda}_v^* := \max\{\tilde{\lambda}_v^*, \lambda_v^*\}$, and $\hat{\rho}^v(t)$ be the solution of the initial value problem

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_j = \hat{\lambda}_v^* G_j^v(\hat{\rho}^v) - \tilde{\mu}_j(\hat{\rho}_j) \\ \hat{\rho}_j = \rho_j^*\,. \end{cases}$$

Observe that, thanks to Assumption 3, for every $\rho^v \neq \rho^* \in \mathcal{R}_v$ such that

$$\rho_j = \rho_j^* \quad \text{for some } j \in \mathcal{E}_v^+\,, \qquad \rho_e \geq \rho_e^*, \quad \forall e \neq j \in \mathcal{E}_v^+\,,$$

one has $G_j^v(\rho^v) > G_j^v(\rho^*)$, and then,

$$\hat{\lambda}^* G_j^v(\rho^v) - \tilde{\mu}_j(\rho_j) > (\sum\nolimits_e f_e^*) G_j^v(\rho^*) - \mu_j(\rho_j^*) = 0\,.$$

Therefore, if we consider the region $\mathcal{R}^* := \{\rho^v : \ \rho_j \geq \rho_j^*, \ \forall j \in \mathcal{E}\}$, and denote by n the outward-pointing normal vector to its boundary, one has that

$$\left(\hat{\lambda}^* G_j^v(\rho^v) - \tilde{\mu}_j(\rho_j)\right) \cdot n < 0 \,, \quad \forall \rho^v \in \partial\mathcal{R}^* \,, \ t \geq 0 \,.$$

It follows that

$$\hat{\rho}_e(t) \geq \rho_e^* \,, \qquad \forall e \in \mathcal{E}, \ \forall t \geq 0 \,. \tag{16}$$

On the other hand, thanks to Lemma 3.1, there exists a perturbed equilibrium point $\hat{\rho}^*$ such that

$$\hat{\lambda}_v^* G_j^v(\hat{\rho}^*) = \tilde{\mu}_j(\hat{\rho}_j^*) \,, \qquad \forall j \in \mathcal{E}_v^+ \,.$$

Then, for all $\rho^v \neq \hat{\rho}^*$ such that

$$\rho_j = \hat{\rho}_j^* \quad \text{for some } j \in \mathcal{E}, \qquad\qquad \rho_e \leq \hat{\rho}_e^*, \quad \forall e \neq j \in \mathcal{E} \,.$$

Assumption 3 implies that $G_j^v(\rho^v) < G_j^v(\hat{\rho}^*)$, and then

$$\hat{\lambda}^* G_j^v(\rho^v) - \tilde{\mu}_j(\rho_j) < \hat{\lambda}^* G_j^v(\hat{\rho}^*) - \tilde{\mu}_j(\hat{\rho}_j^*) = 0 \,.$$

Therefore,

$$\left(\hat{\lambda}^* G_j^v(\rho^v) - \tilde{\mu}_j(\rho_j)\right) \cdot \hat{n} < 0 \,, \quad \forall \rho^v \in \partial\mathcal{R}^* \,,$$

where

$$\hat{\mathcal{R}}^* := \{\rho^v : \ \rho_j \leq \hat{\rho}_j^* \,, \ \forall j \in \mathcal{E}_v^+\} \,,$$

and $\hat{n}$ is the outward-pointing normal vector to its boundary. This implies that

$$\hat{\rho}_e(t) \leq \hat{\rho}_e^* \,, \qquad \forall e \in \mathcal{E}, \ \forall t \geq 0 \,,$$

from which, in particular, it follows that

$$\sum_{e \in \mathcal{E}} \tilde{\mu}_e(\hat{\rho}_e(t)) \leq \sum_{e \in \mathcal{E}} \tilde{\mu}_e(\hat{\rho}_e^*) = \hat{\lambda}^* \,. \tag{17}$$

Now, combining (17) with (16), one gets that, for all $t \geq 0$

$$\begin{aligned}
\textstyle\sum_{j \in \mathcal{J}} \tilde{\mu}_j(\hat{\rho}_j(t)) \ &\leq \ \hat{\lambda}^* - \sum_{j \notin \mathcal{J}} \tilde{\mu}_j(\hat{\rho}_j(t)) \\
&\leq \ \hat{\lambda}^* - \sum_{j \notin \mathcal{J}} \tilde{\mu}_j(\rho_j^*) \\
&= \ \left[\tilde{\lambda}_v^* - \lambda_v^*\right]_+ + \sum_{j \in \mathcal{J}} f_j^* + \sum_{j \notin \mathcal{J}} \mu_j(\rho_j^*) - \sum_{j \notin \mathcal{J}} \tilde{\mu}_j(\rho_j^*) \\
&\leq \ \left[\tilde{\lambda}_v^* - \lambda_v^*\right]_+ + \sum_{j \in \mathcal{J}} f_j^* + \textstyle\sum_{e \in \mathcal{E}_v^+} \delta_e \,.
\end{aligned}$$

To complete the proof, it remains to show that

$$\hat{\rho}_j(t) \geq \rho_j(t)\,, \qquad t \geq 0\,. \tag{18}$$

for all $j \in \mathcal{E}_v^+$. In order to see this, first observe that $\hat{\rho}_j(0) = \rho_j(0)$. Moreover, if $\hat{\rho}_j(t) = \rho_j(t)$ for some j, and $\hat{\rho}_e(t) \geq \rho_e(t)$ for all other e, then Assumption (A4) guarantees that $G_j^v(\hat{\rho}^v) \geq G_j^v(\rho^v)$, and, as a consequence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho}_j = \hat{\lambda}^* G_j^v(\hat{\rho}^v) - \tilde{\mu}_j(\hat{\rho}_j) \geq \lambda(t) G_j^v(\rho^v) - \tilde{\mu}_j(\rho_j) = \frac{\mathrm{d}}{\mathrm{d}t}\rho_j\,,$$

which in turn can be shown to imply (18). ■

One can exploit the local stability and diffusivity properties from Lemma 3.2 along with an induction argument on the depth of an acyclic graph to prove that γ is indeed a lower bound to the margin of stability of the traffic network. For $v = 0, 1, \ldots, n$, define

$$D_v := \bigcup\nolimits_{u=0}^{v} \mathcal{E}_u^+\,, \qquad B_v := D_v \setminus \{(u,w) \in \mathcal{E} \mid u, w \in \{0, \ldots, v\}\}\,.$$

Lemma 3.3 *Consider an admissible perturbation of magnitude* $\delta = \sum_e \delta_e$ *as in (7). Then, for any* $v = 0, \ldots, n-1$, *and for every* $\mathcal{J} \subseteq \mathcal{B}_v$

$$\sum_{e \in \mathcal{J}} \left(\tilde{\mu}_e(t) - f_e^*\right) \leq \sum_{e \in D_v} \delta_e, \quad \forall t \geq 0\,. \tag{19}$$

Proof We shall proceed by induction on $v = 0, 1, \ldots, n$.

First, notice that $B_0 = D_0 = \mathcal{E}_o^+$. Therefore, by applying Lemma 3.2, with $\tilde{\lambda}(t) = \lambda_0$, one can verify that Equation (19) is true for $v = 0$.

Now, assume that (19) be true for some $v \leq n-2$. Consider a set $\mathcal{J} \subseteq B_{v+1}$ and let $\mathcal{J}_1 := \mathcal{J} \cap \mathcal{E}_{v+1}^+$ and $\mathcal{J}_2 := \mathcal{J} \setminus \mathcal{J}_1$. It is easy to check that $J_2 \subseteq \mathcal{B}_v$. By applying Lemma 3.2 to the set $\mathcal{J}_1$, one gets that

$$\sum_{e \in J_1} \left(\tilde{\mu}_e(\tilde{\rho}_e(t)) - f_e^*\right) \leq \left[\tilde{\lambda}_{v+1}^* - \lambda_{v+1}^*\right]_+ + \sum_{e \in \mathcal{E}_{v+1}^+} \delta_e, \tag{20}$$

where $\tilde{\lambda}_{v+1}^* = \sup\{\sum_{e \in \mathcal{E}_{v+1}^-} \tilde{\mu}_e(\rho_e(t)) : t \geq 0\}$. By applying the induction step on $\mathcal{J}_2$ and $\mathcal{J}_2 \cup \mathcal{E}_{w+1}^-$, one gets the following inequalities:

$$\sum_{e \in \mathcal{J}_2} \left(\tilde{\mu}_e(\tilde{\rho}_e(t)) - f_e^*\right) \leq \sum_{e \in D_v} \delta_e, \tag{21}$$

$$\sum_{e \in \mathcal{J}_2} \left(\tilde{\mu}_e(\tilde{\rho}_e(t)) - f_e^*\right) + \sum_{e \in \mathcal{E}_{v+1}^-} \left(\tilde{\mu}_e(\tilde{\rho}_e(t)) - f_e^*\right) \leq \sum_{e \in D_v} \delta_e. \tag{22}$$

Consider the two cases: $\tilde{\lambda}^*_{v+1} \leq \lambda^*_{v+1}$, or $\tilde{\lambda}^*_{v+1} > \lambda^*_{v+1}$. By adding up equations (20) and (21), in the first case, or (20) and (22) in the second case, one gets that

$$\begin{aligned}\textstyle\sum_{e\in\mathcal{J}} (\tilde{\mu}_e(\rho_e(t)) - f^*_e) &= \textstyle\sum_{e\in\mathcal{J}_1} (\tilde{\mu}_e(\rho_e(t)) - f^*_e) + \sum_{e\in\mathcal{J}_2} (\tilde{\mu}_e(\rho_e(t)) - f^*_e) \\ &\leq \textstyle\sum_{e\in\mathcal{E}^+_{v+1}} \delta_e + \sum_{e\in D_v} \delta_e \\ &\leq \textstyle\sum_{e\in D_{v+1}} \delta_e \,.\end{aligned}$$

Since $\mathcal{J} \subseteq B_{v+1}$, this proves Equation (19). ■

In particular, Lemma 3.3 implies that

$$\tilde{\lambda}^*_v \leq \lambda^*_v + \sum_{e\in D_{v-1}} \delta_e, \qquad \forall v = 1, \ldots, n-1\,. \tag{23}$$

Now consider an admissible perturbation of magnitude $\delta < \gamma$. From (23), and the inequality

$$\sum_{e\in D_v} \delta_e \leq \delta < \gamma \leq \sum_{e\in\mathcal{E}^+_v} (f^{\max}_e - f^*_e)\,,$$

one finds that, for all $v = 0, 1, \ldots, n-1$,

$$\begin{aligned}\tilde{\lambda}^*_v &\leq \textstyle\lambda^*_v + \sum_{e\in D_{v-1}} \delta_e \\ &= \textstyle\lambda^*_v + \sum_{e\in D_v} \delta_e - \sum_{e\in\mathcal{E}^+_v} \delta_e \\ &< \textstyle\lambda^*_v + \sum_{e\in\mathcal{E}^+_v} \left(f^{\max}_{\mathrm{e}} - f^*_e - \delta_e\right) \\ &= \textstyle\sum_{e\in\mathcal{E}^+_k} (f^{\max}_{\mathrm{e}} - \delta_e) \\ &= \textstyle\sum_{e\in\mathcal{E}^+_k} \tilde{f}^{\max}_e\,.\end{aligned}$$

Then, an iterated application of Lemma 3.2, for all intermediate nodes $v = 0, 1, \ldots, n-1$, proves that γ is indeed a lower bound to the margin of stability of the transportation network, thus completing the proof of Theprem 2.2.

4 Robust equilibrium selection and optimal toll selection

In the previous sections, we studied robustness properties of a transportation network around a given equilibrium point. We now return to our secondary objective of identifying the most robust equilibrium operating point for the network. For this, we shall assume that initial equilibrium point of the transportation network is of Wardrop type. Such an assumption is justified by ..... EXPLAIN AND REFER TO OUR WORK ON STABILITY

4.1 Wardrop equilibria

We shall strengthen Assumption 2 to the following:

Assumption 6 *For every* $e \in \mathcal{E}$*,* $\mu_e : \mathbb{R}_+ \to \mathbb{R}_+$ *is continuously differentiable, strictly increasing, strictly concave, and is such that* $\mu_e(0) = 0$*,* $\mu'_e(0) < +\infty$ *and* $\limsup_{\rho_e \to +\infty} \mu_e(\rho_e) < +\infty$*, where* $\mu'_e(0) := \lim_{\rho_e \to 0^+} \mu'_e(\rho_e)$*.*

For a $\mu(\rho)$ satisfying (A2), there exists a continuous inverse $\mu_e^{-1}(\,\cdot\,)$ for all $e \in \mathcal{E}$. Therefore, with flow on an edge being the product of speed and density on that edge, one can define the edge-wise delay functions $T_e : \mathbb{R}_+ \to \mathbb{R}_{>0}$ representing the flow-dependent time taken to traverse an edge as

$$T_e(f_e) := \begin{cases} \mu_e^{-1}(f_e)/f_e & \text{if } f_e > 0, \\ \frac{1}{\mu'_e(0)} & \text{if } f_e = 0, \end{cases} \qquad \forall e \in \mathcal{E}. \tag{24}$$

For a $\mu(\rho)$ satisfying (A2), let $f_{\mathrm{e}}^{\max} := \limsup_{\rho_e \to +\infty} \mu_e(\rho_e)$ be the maximum flow carrying capacity of edge e. Let $f^{\max} \succ \mathbf{0}$ be the vector of maximum flow carrying capacities of the edges in $\mathcal{E}$. For a given $\mathcal{G}$ and $f^{\max} \succ \mathbf{0}$, define the set of admissible flows through $\mathcal{G}$ as

$$\mathbb{F}(\mathcal{G}, f^{\max}) := \{f \succeq \mathbf{0} \mid f \preceq f^{\max}, \sum_{e \in \mathcal{E}_u^+} f_e = \sum_{e \in \mathcal{E}_u^-} f_e + \mathbb{1}_o(u) \quad \forall\, u \in \mathcal{V} \backslash \{d\}\}.$$

Throughout this paper, we will assume that $\mathcal{G}$ and $f^{\max}$ are such that $\mathbb{F}(\mathcal{G}, f^{\max}) \neq \emptyset$.

Let $\Upsilon \succeq \mathbf{0}$ be the edge-wise vector of tolls, with Υ_e denoting the toll on edge e. We now describe the game-theoretic framework for the transportation network that is relevant to describe the appropriate notion of an equilibrium operating condition from the agents point of view. We briefly describe the standard congestion game setup for the transportation network under consideration in this paper. Assuming that one unit of toll corresponds to a unit amount of delay, the utility of a driver associated with edge e when the flow on it is f_e is $-\left(T_e(f_e) + \Upsilon_e\right)$ and hence the utility associated with a path $p \in \mathcal{P}$ is $-\sum_{e \in p} \left(T_e\left(f_e\right) + \Upsilon_e\right)$.

We shall assume that the network is initially operating at a *Wardrop* equilibrium condition. We refer the reader to [13] for sufficient conditions for the stability of Wardrop equilibria under settings similar to the one considered in this paper. We now recall the notion of a Wardrop equilibrium [2] that also includes the effect of tolls.

Definition 4.1 *A Wardrop equilibrium is a vector* $f^* \in \mathbb{F}(\mathcal{G}, f^{\max})$ *that satisfies the following for all* $p, q \in \mathcal{P}$*:*

$$f_e > 0 \quad \forall e \in p \cup q \Longrightarrow \sum_{e \in p} \left(T_e\left(f_e\right) + \Upsilon_e\right) = \sum_{e \in q} \left(T_e\left(f_e\right) + \Upsilon_e\right) \ ,$$

$$f_e > 0 \quad \forall e \in p, \quad \exists e' \in q \ s.t. \ f_{e'} = 0 \Longrightarrow \sum_{e \in p} \left(T_e\left(f_e\right) + \Upsilon_e\right) \leq \sum_{e \in q} \left(T_e\left(f_e\right) + \Upsilon_e\right) \ .$$

The following result guarantees the existence and uniqueness of Wardrop equilibrium in our setting.

Proposition 4.2 *Given a* $\mathcal{G}$ *satisfying (A1),* $\mu(\rho)$ *satisfying (A2) and* $\Upsilon \succeq \mathbf{0}$*, there exists a unique Wardrop equilibrium* $f^* \in \mathbb{F}(\mathcal{G}, f^{\max})$*.*

Proof It follows from assumption (A2) that, for all $e \in \mathcal{E}$, the delay function $T_e(f_e)$ is continuous, strictly increasing, and such that $T_e(0) > 0$. The proposition then follows by applying Theorems 2.4 and 2.5 from [2].

4.2 Robust equilibrium selection as an optimization problem

The robust equilibrium selection problem can be posed as an optimization problem as follows:

$$\begin{aligned} \text{maximize} \ & \Gamma\left(\mathcal{G}, \Pi_1, f^*\right), \\ \text{subj. to} \ & f^* \in \mathbb{F}(\mathcal{G}, f^{\max}). \end{aligned} \tag{25}$$

The solution to this optimization problem can help a system planner evaluate the distribution of traffic flow that is most robust to disruptions and can implement this distribution using, for example, using tolls Υ, e.g., see [6]. Similar optimization problems and their solution methodologies have been widely studied in context of traffic assignment in [2].

Equation (**??**) shows that, under these conditions, Γ^* is a minimum of a set of functions linear in f^* and hence is concave in f^*. Therefore the optimization problem stated in Equation (25) is equivalent to minimizing a convex function over a convex polytope. However, note that the objective function, $\Gamma\left(\mathcal{G}, \Pi_1, f^*\right)$ is non-smooth and one needs to use non-smooth convex optimization techniques, e.g., see [14], to solve this problem.

4.3 The robustness price of anarchy

Conventionally, transportation networks have been viewed as static flow networks, where a given equilibrium traffic flow is an outcome of driver's selfish behavior in response to the delays associated with various paths and the incentive mechanisms in place. The price of anarchy [15] has been suggested as a metric to measure how sub-optimal a given equilibrium is with respect to the societal optimal equilibrium, where the societal optimality is related to the average delay faced by the agents. In the context of robustness analysis of transportation networks, it is natural to consider societal optimality from the robustness point of view, thereby motivating a notion of the robustness price of anarchy. Formally, it can be defined as

$$P\left(\mathcal{G}, \Pi, f^*\right) = \Gamma^*\left(\mathcal{G}, \Pi\right) - \Gamma\left(\mathcal{G}, \Pi, f^*\right).$$

It is worth noting that, for a parallel topology, we have that $\Gamma^*\left(\mathcal{G}, \Pi_1, f^*\right) = \Gamma^*\left(\mathcal{G}, \Pi, f^*\right) = \sum_{e\in\mathcal{E}} f^{\max}_{\mathrm{e}} - 1$ for all f^*. That is, the margin of stability is independent of the equilibrium operating condition and hence, for a parallel topology, $P\left(\mathcal{G}, \Pi, f^*\right) = 0$ for all f^*. However, for a general topology and a general equilibrium, this quantity is non-zero. In the next section, we discuss the use of tolls to yield a robust equilibrium point for a given topology, i.e., the one for which the robustness price of anarchy is zero.

4.4 Tolls for the robust equilibrium point

In this section, we determine the set of edge-wise tolls Υ that yield a desired equilibrium operating condition for the network.

Proposition 4.3 *Given a graph $\mathcal{G}$ satisfying (A1), flow functions μ satisfying (A2), the set of tolls that yield a desired equilibrium operating condition $f^* \in \mathbb{F}(\mathcal{G}, f^{\max}) \cap \mathbb{R}^{|\mathcal{E}|}_{>0}$ is given by*

$$\Upsilon^{\mathrm{eq}} = \left(\max_{e\in\mathcal{E}} \frac{T_e(1)}{T_e(f^{\mathrm{eq},0}_e)}\right) T(f^{eq,0}) - T(f^*),$$

where $f^{eq,0} \in \mathbb{F}(\mathcal{G}, f^{\max})$ is the Wardrop equilibrium for tolls set to zero.

Proof Let S be a simplex of dimension $|\mathcal{P}|$, i.e., number of paths in $\mathcal{G}$ between o and d. Consider the function $V : S \to \mathbb{R}$ that serves as a potential function for the congestion game at hand [16]:

$$\begin{aligned} V(\pi) =& \sum_{e\in\mathcal{E}} \int_0^{f_e} \left(\Upsilon_e + T_e(z)\right) dz \\ =& \Upsilon^T f + \sum_{e\in\mathcal{E}} \int_0^{f_e} T_e(z) dz, \end{aligned} \tag{26}$$

where $f = A^T\pi$, with $A \in \{0,1\}^{|\mathcal{P}|\times|\mathcal{E}|}$ being the path-edge incidence matrix, i.e., for all $e \in \mathcal{E}$ and $p \in \mathcal{P}$, $A_{p,e} = 1$ if $e \in p$ and zero otherwise. Equation (26) can be rewritten as

$$V(\pi) = (A\Upsilon)^T\pi + \sum_{e\in\mathcal{E}} \int_0^{(A^T\pi)_e} T_e(z)dz,$$

Following assumption (A2) and the discussion thereafter, it is easy to see that $V(\pi)$ is convex in π. It is known, e.g., see Theorem 2.1 in [2], that the unique Wardrop equilibrium corresponding to a given set of tolls is equivalent to the first order optimality condition of the following optimization problem:

$$\begin{aligned} &\text{minimize } V(\pi), \\ &\text{subj. to } \pi \in S. \end{aligned} \tag{27}$$

Let $\zeta \in \mathbb{R}$ be the Lagrange multiplier corresponding to the constraint in (27). The Lagrangian function can then be written as $L(\pi,\zeta) := (A\Upsilon)^T\pi + \sum_{e\in\mathcal{E}} \int_0^{(A^T\pi)_e} T_e(z)dz + \zeta\left(1 - \mathbf{1}^T\pi\right)$.

Considering the first order optimality conditions, the necessary and sufficient condition for $f^* \in \mathbb{F}(\mathcal{G}, f^{\max}) \cap \mathbb{R}^{|\mathcal{E}|}_{>0}$ to be a Wardrop equilibrium is the existence of $\Upsilon^{\text{eq}} \succeq \mathbf{0}$ and $\zeta^* \in \mathbb{R}$ that satisfy the following condition:

$$A\left(\Upsilon^{\text{eq}} + T(f^*)\right) = \zeta^*\mathbf{1}. \tag{28}$$

Since $f^{eq,0}$ is a Wardrop equilibrium for $\tau = 0$, the first order optimality conditions imply that there exists $\hat{\zeta} \in \mathbb{R}$ such that

$$AT(f^{eq,0}) = \hat{\zeta}\mathbf{1}. \tag{29}$$

Using Equation (29) and simple algebra, one can verify that Equation (28) is satisfied for $\Upsilon^{\text{eq}} = \left(\max_{e\in\mathcal{E}} \frac{T_e(1)}{T_e(f_e^{\text{eq},0})}\right) T(f^{eq,0}) - T(f^*)$ and $\zeta^* = \left(\max_{e\in\mathcal{E}} \frac{T_e(1)}{T_e(f_e^{\text{eq},0})}\right)\hat{\zeta}$.

Remark 4.4 *The set of tolls that yield a desired equilibrium operating condition is not unique. In fact, any toll of the form* $\Upsilon^{\text{eq}} = \eta T(f^{eq,0}) - T(f^*)$, *with* $\eta \geq \max_{e\in\mathcal{E}} \frac{T_e(f_e^*)}{T_e(f_e^{\text{eq},0})}$ *would yield* f^* *as the equilibrium condition. Proposition 4.3 gives just one such set of tolls.*

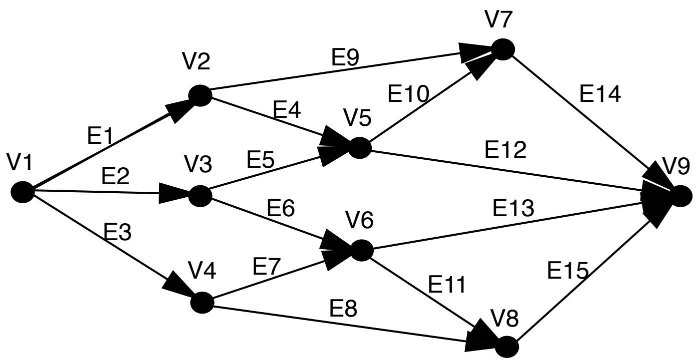

Figure 1: The graph topology used in simulations.

5 Simulations

In this section, through numerical experiments, we show the implications of the results when the flow functions are set to the ones commonly accepted in the transportation literature, e.g., see [17]. In transportation literature, the flow functions are defined over a final interval of the form $[0, \rho_e^{\max}]$, where $\rho_e^{\max}$ is the maximum traffic density that link e can handle. Additionally, μ_e is assumed to be strictly concave and achieves its maximum in $(0, \rho_e^{\max})$. For example, consider the following:

$$\mu_e(\rho_e) = \frac{4 f_{\mathrm{e}}^{\max} \rho_e (\rho_e^{\max} - \rho_e)}{(\rho_e^{\max})^2}, \quad \rho_e \in [0, \rho_e^{\max}]. \tag{30}$$

Note that the most important difference from (A2) is that in this case μ_e is not strictly increasing. However, we illustrate via simulations that one can exploit the flexibility in choosing β in the i-logit function to use the results on the margin of stability.

For the simulations, we selected the following parameters:

- The graph topology $\mathcal{G}$ is shown in Figure 1.
- The link-wise flow functions were selected to be a modified version of Equation (30):

$$\mu_e(\rho_e) = \frac{4 f_{\mathrm{e}}^{\max} \rho_e (\rho_e^{\max} - \rho_e)}{(\rho_e^{\max})^2} \mathbb{1}_{\rho_j \le \rho_j^{\max} \forall j \in \mathcal{E}_v^+}, \quad \rho_e \in [0, \rho_e^{\max}], \quad \forall e \in \mathcal{E},$$

  with the maximum flow carrying capacity and the maximum traffic density given by $f^{\max} = [\frac{2}{5}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{3}{10}\ \frac{3}{10}\ \frac{1}{2}\ \frac{1}{3}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}]^T$ and $\rho^{\max} = \mathbf{1}_{15}$ respectively.

- The equilibrium flow distribution was selected to be $f^* = [\frac{1}{3}\ \frac{1}{3}\ \frac{1}{3}\ \frac{2}{9}\ \frac{1}{6}\ \frac{1}{6}\ \frac{2}{9}\ \frac{1}{9}\ \frac{1}{9}\ \frac{7}{54}\ \frac{7}{54}\ \frac{7}{27}\ \frac{7}{27}\ \frac{13}{54}\ \frac{13}{54}]^T$.

[KS]: need to be a bit more careful in writing the indicator function in the equation for flow functions

- For the route choice function, a modified version of Equation (**??**) is used. The modification is done to respect the finite traffic density constraint on the links. The modified route choice function is as follows

$$G_e^v(\rho^v) = \frac{f_e^* \exp(-\beta(\rho_e - \rho_e^*))\mathbb{1}_{\rho_e \le \rho_e^*}}{\sum_{j \in \mathcal{E}_v^+} f_j^* \exp(-\beta(\rho_j - \rho_j^*))\mathbb{1}_{\rho_j \le \rho_j^*}},$$

  where β will be an independent parameter for the simulations.

5.1 Effect of β on the margin of stability

In this section, we study the effect of β on the margin of stability. One can verify using Theorem **??** that the margin of stability for the network with the given parameters is upper bounded by 11/45. Consider the disturbance vector δ such that $\delta_{12} = 0.1$ and $\delta_i = 0$ for all $i \in \{1, \ldots, 15\} \setminus \{12\}$. Note that the one-norm of this disturbance vector is strictly less than 11/45. Figures 2 and 3 illustrate the role of β in stability.

5.2 Cascaded instability

Figure 4 illustrates the cascading effect in the instability of the network.

6 Conclusion

In this paper, we studied robustness properties of transportation networks with respect to its pre-disturbance equilibrium operating condition and the agents' response to the disturbance. We considered disturbances that reduce the maximum flow carrying capacities of the edges by affecting their congestion properties. We define the margin of stability of the network to be the maximum sum of capacity losses, under which the traffic densities on all the edges remain bounded over time. We characterized the class of route choice functions that yield the maximum margin of stability for a given equilibrium operating condition and also the formulated an optimization problem to find the most robust equilibrium point. Finally, we discussed the use of tolls in yielding a desired equilibrium operating condition.

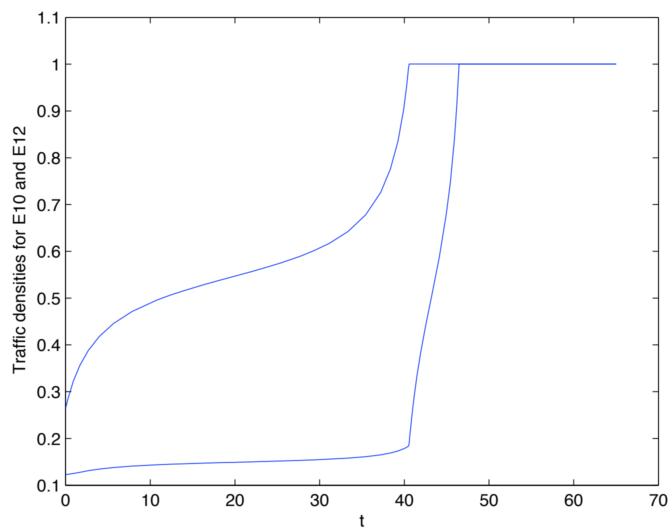

Figure 2: Plot of densities of links 10 and 12 for $\beta = 1$.

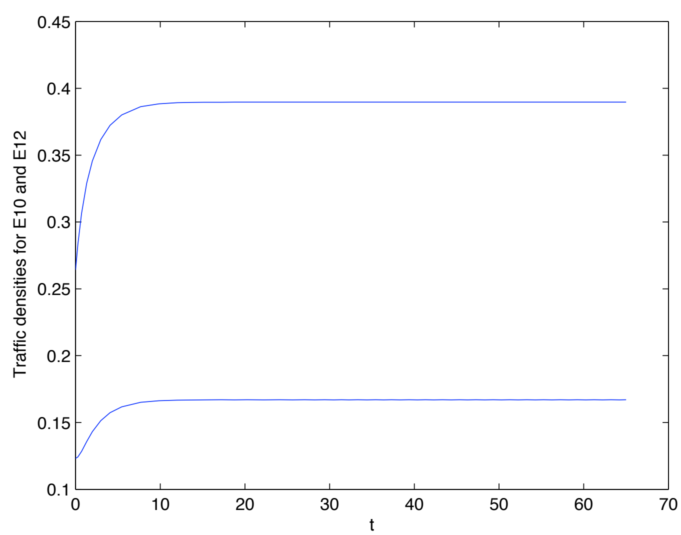

Figure 3: Plot of densities of links 10 and 12 for $\beta = 5$.

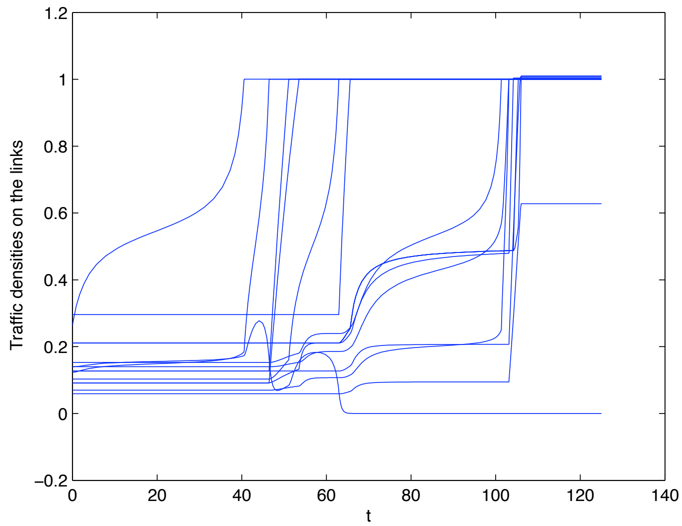

Figure 4: Plot of densities of all the links for $\beta = 1$.

In future, we plan to extend the research in several directions. First, we plan to study the dependence of the margin on stability on the amount of information available to the agents. We also plan to perform robustness analysis in a probabilistic framework versus the min-max framework of this paper, possibly considering other general models for disturbances. Finally, we also plan to consider more general graph topologies, e.g., graphs have cycles, multiple origin-destination pairs etc.

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