# On the error exponent of Markov channels with ISI and feedback

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***Abstract*— We extend Burnashev's [5] classic result for the error exponent of discrete memoryless channels with feedback to the case of Markov channels with ISI and feedback. This is a nontrivial extension of our previous work [6] where we treated the case of Markov channels with feedback but without ISI. Tools from stochastic control theory are used to treat the case with ISI.**

## I. Introduction

It is well known that even perfect causal output feedback cannot increase the capacity of a discrete memoryless channel (DMC) [10]. Feedback, though, can help improving its reliability function. A classical result due to Burnashev [5] characterizes the reliability function of a DMC with a simple single-letter formula in the variable-length block-coding case. It is remarkable that in this framework, differently from when feedback is not available or when fixed-length coding is enforced, the reliability function is exactly known at any rate below capacity, and that it has nonzero slope approaching capacity. Recently, the problem of channel coding with feedback, and Burnashev's approach in particular, have attracted renewed interest from the researchers; see [12], [13], [8], [2], [6].

The present paper deals with a generalization of Burnashev's result to discrete Markov channels with perfect channel state information (CSI) both at the transmitter and at the receiver. In [6] we have presented some results for the special case when there is no intersymbol interference (ISI). Here we extend those results to deal with Markov channels with ISI. For this class of channels, under suitable ergodicity assumptions, we are able to exactly characterize the reliability function in the single-letter form

$$E_B(R) = D\left(1 - \frac{R}{C}\right). \qquad (1)$$

In (1), $R$ denotes the rate measured with respect to the average delay, while the capacity $C$ and the Burnashev coefficient $D$ are quantities defined as the solution of simple finite dimensional optimization problems involving the stochastic kernel describing the channel (see (3) and (6)). Our main result is contained in Theorem 1 stated at the end of Section II.

In order to prove the achievability of the exponent (1), we propose a simple two phase iterative transmission scheme based on the one first considered by Yamamoto and Itoh [14] for DMCs. The analysis of this scheme is presented in Section III and essentially relies on known results about the capacity of Markov channels with CSI [11], and the error exponent of binary hypothesis tests for irreducible Markov chains [9],[7].

Proving the converse result, instead, is more involved. This is because a lower bound on the error probability of any variable-length block-coding scheme has to be obtained. We follow Burnashev's original proof [5] and the ideas proposed in [2] (see also [13]). Specifically we provide two different bounds for the error probability, involving respectively the channel capacity $C$ and its Burnashev coefficient $D$, corresponding to two distinct phases which can be recognized in any sequential transmission scheme. Similarly to the memoryless case, martingale theory techniques, and in particular Doob's optional sampling theorem, are repeatedly used jointly with more standard information theoretic results as Fano's and log-sum inequalities.

The main issue when moving from the memoryless setting to the Markov setting consists in the necessity of considering the random dynamics of the state sequence. When there is no ISI, it has been shown in [6] that it is sufficient to track the evolution of empirical measures associated to the state sequence, a random process taking values in a finite-dimensional space, whose dynamics are independent of the transmitted message. When ISI is allowed, it turns out that both mutual information and maximal information divergence can no longer be optimized pointwise, since the dynamics of the state sequence now depend on the transmitted message through the channel input symbol. This implies that tracking the empirical measure of the state sequence only is no longer sufficient. To treat this issue we follow techniques developed in controlled Markov process theory. It is necessary to consider the empirical measure associated to the pair of the state sequence and the channel input distribution induced by the causal encoder. This empirical measure process takes values in an infinite dimensional space. However, due to the finiteness of both input and state sets, this space turns out to be compact so that many of the topological issues are simplified. In particular we follow Borkar's [4] convex analytical approach and derive, using the Hoeffding-Azuma inequality, a result generalizing the known theory of the average cost optimization problems to stopping time horizons. These arguments are presented in Section V.

## II. PROBLEM SETTING AND MAIN RESULT

### A. Stationary ergodic Markov channels

Throughout the paper $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{S}$ will respectively denote input, output and state set, all finite. For any finite set $\mathcal{A}$, $\mathcal{P}(\mathcal{A})$ will denote the space of probability measures over $\mathcal{A}$.

**Definition 1** *A stationary* Markov channel *is described by:*

- *a stochastic kernel consisting in a family* $\{P(\,\cdot\,,\cdot\,|\,s,x)\in\mathcal{P}(\mathcal{S}\times\mathcal{Y})|s\in\mathcal{S},x\in\mathcal{X}\}$ *of probability measures over* $\mathcal{S}\times\mathcal{Y}$*, indexed by elements of* $\mathcal{S}$ *and* $\mathcal{X}$*;*
- *an initial state distribution* $\boldsymbol{\mu}$ *in* $\mathcal{P}(\mathcal{S})$.

For a stationary Markov channel as in Definition 1, let $P_S(s_+|\,s,x):=\sum_y P(s_+,y|\,s,x)$ be the $\mathcal{S}$-marginals. We will consider the associated stochastic kernel $\{Q(\,\cdot\,|\,s,\boldsymbol{u})\in\mathcal{P}(\mathcal{S})|\,s\in\mathcal{S},\boldsymbol{u}\in\mathcal{P}(\mathcal{X})\}$, where for every channel input distribution $\boldsymbol{u}$ in $\mathcal{P}(\mathcal{X})$

$$Q(s_+\,|\,s,\boldsymbol{u}):=\sum_{x\in\mathcal{X}}P_S(s_+\,|\,s,x)\boldsymbol{u}(x)\,,\qquad s,s_+\in\mathcal{S}\,. \quad (2)$$

Given $\boldsymbol{\pi}:\mathcal{S}\to\mathcal{P}(\mathcal{X})$ (we refer to such a map as a deterministic stationary policy), denote by $Q_{\boldsymbol{\pi}}:=\big(Q\big(s_+\,|\,s,\boldsymbol{\pi}(s)\big)\big)_{s,s_+}$ the state transition stochastic matrix induced by $\boldsymbol{\pi}$. For $f:\mathcal{S}\to\mathcal{X}$ we shall write $Q_f$ in place of $Q_{\delta_{f(\cdot)}}$. Throughout the paper we will restrict ourselves to ergodic Markov channels, satisfying the following ergodicity assumption.

**Assumption 2** *For every* $f:\mathcal{S}\to\mathcal{X}$*,* $Q_f$ *is irreducible.*

Assumption 2 can be relaxed or replaced by other equivalent assumptions. Here we limit ourselves to observe that it involves only on the $S$-marginals $\{P_S\}$ of the Markov channel, and it is easily testable since it only requires a finite number of finite directed graphs to be strongly connected.

Since taking a convex combination does not reduce the support, Assumption 2 guarantees that for every $\boldsymbol{\pi}:\mathcal{S}\to\mathcal{P}(\mathcal{X})$ the stochastic matrix $Q_{\boldsymbol{\pi}}$ is irreducible. Then, by Perron-Frobenius theorem we have that $Q_{\boldsymbol{\pi}}$ has a unique invariant measure in $\mathcal{P}(\mathcal{S})$ which will be denoted by $\boldsymbol{\eta}_{\boldsymbol{\pi}}$.

Let us consider the cost function $c:\mathcal{S}\times\mathcal{P}(\mathcal{X})\to\mathbb{R}$,

$$c(s,\boldsymbol{u})=\sum_{x,y,s_+}\boldsymbol{u}(x)P(s_+,y|\,s,x)\log\frac{P(s_+,y|\,s,x)}{\sum_{x'}\boldsymbol{u}(x')P(s_+,y|\,s,x')}$$

It is easy to check that the term $c(s,\boldsymbol{u})$ equals the mutual information between an $\mathcal{X}$-valued random variable $X$ and an $\mathcal{S}\times\mathcal{Y}$-valued random variable $(S_+,Y)$ with marginal distribution of $X$ given by $\boldsymbol{u}$, and conditioned distribution of $(S_+,Y)$ given $X=x$ given by $P(\cdot,\cdot|\,s,x)$. This in particular implies that the function $c$ is continuous over $\mathcal{S}\times\mathcal{P}(\mathcal{X})$ and takes values in the bounded interval $[0,\log|\mathcal{X}|]$ . Define

$$C:=\max_{\boldsymbol{\pi}:\mathcal{S}\to\mathcal{P}(\mathcal{X})}\sum_{s\in\mathcal{S}}\boldsymbol{\eta}_{\boldsymbol{\pi}}(s)c(s,\boldsymbol{\pi}(s))\,. \quad (3)$$

The quantity $C$ defined above is known to be the capacity of the ergodic Markov channel we are considering when perfect CSI is available, with or without output feedback [11]. Notice that in the absence of ISI the invariant measure $\boldsymbol{\eta}$ is independent of the policy $\boldsymbol{\pi}$ so that (3) reduces to

$$C=\sum_{s\in\mathcal{S}}\boldsymbol{\eta}(s)C_s\,,\qquad C_s:=\max_{\boldsymbol{u}\in\mathcal{P}(\mathcal{X})}c(s,\boldsymbol{u})\,,$$

while when the state space is trivial (i.e. $|\mathcal{S}|=1$) it further simplifies to the usual definition of the capacity of a DMC.

Consider now the cost function $d:\mathcal{S}\times\mathcal{P}(\mathcal{X})\to[0,+\infty]$

$$d(s,\boldsymbol{u}):=\sup_{\boldsymbol{u}'}\sum_{x,y,s_+}\boldsymbol{u}(x)P(s_+,y|s,x)\log\frac{\sum_z\boldsymbol{u}(z)P(s_+,y|s,z)}{\sum_z\boldsymbol{u}'(z)P(s_+,y|s,z)} \quad (4)$$

The optimization in the righthand side of (4) is intended over all input distributions $\boldsymbol{u}'$ in $\mathcal{P}(\mathcal{X})$. Notice that the term to be optimized equals the Kullback-Leibler information divergence between the probability measures $P(s_+,y|s,x)\boldsymbol{u}(x)$ and $P(s_+,y|s,x)\boldsymbol{u}'(x)$ in $\mathcal{P}(\mathcal{X}\times\mathcal{S}\times\mathcal{Y})$. This implies that, if we introduce the quantity $\lambda:=\inf\{\lambda_s|\,s\in\mathcal{S}\}$, where

$$\lambda_s:=\inf\Big\{\inf_{x\in\mathcal{X}}P(s_+,y|\,s,x)\big|s_+,y:\exists z:P(s_+,y|\,s,z)>0\Big\}$$

$d$ is bounded and continuous over $\mathcal{S}\times\mathcal{P}(\mathcal{X})$ if and only if

$$\lambda>0\,. \quad (5)$$

Throughout the paper we will assume that (5) holds true, and define the Burnashev coefficient of a Markov channel as

$$D:=\max_{\boldsymbol{\pi}:\mathcal{S}\to\mathcal{P}(\mathcal{X})}\sum_{s\in\mathcal{S}}\boldsymbol{\eta}_{\boldsymbol{\pi}}(s)d(s,\boldsymbol{\pi}(s))\,. \quad (6)$$

We shall call the quantity $D$ defined in (6) the Burnashev coefficient of the channel. Notice that a simple convexity argument allows to conclude that both the maxima in (4) and in (6) are achieved in some corner points, which will be respectively denoted by $\boldsymbol{u}'=\delta_{x_s^1}$ and $\boldsymbol{\pi}(s)=\delta_{x_s^0}$. Notice that in the memoryless case $D$ coincides with the coefficient originally introduced by Burnashev.

### B. Causal feedback encoders and sequential decoders

**Definition 3** *A* causal feedback encoder *is the pair of a finite message set and a sequence of maps*

$$\Phi=\Big(\mathcal{W},\big\{\phi_t:\mathcal{W}\times\mathcal{Y}^{t-1}\times\mathcal{S}^t\to\mathcal{X}\big\}_{t\in\mathbb{N}}\Big)\,. \quad (7)$$

With Def.3, we are implicitly assuming that perfect state knowledge as well as perfect output feedback are available at the encoder side.

Given a stationary Markov channel and a causal feedback encoder as in Def.3, we will consider an $\mathcal{W}$-valued random variable $W$ describing the message to be transmitted, a sequence $\boldsymbol{X}=(X_t)_{t\in\mathbb{N}}$ of $\mathcal{X}$-valued r.v.s (the channel input sequence), a sequence $\boldsymbol{Y}=(Y_t)_{t\in\mathbb{N}}$ of $\mathcal{Y}$-valued r.v.s (the channel output sequence), and a sequence $\boldsymbol{S}=(S_t)_{t\in\mathbb{N}}$ $\mathcal{S}$-valued r.v.s (the state sequence).

Consider the time ordering $W,S_1,X_1,Y_1,S_2,X_2,Y_2,\dots$ , and assume that the joint distribution of $W$, $\boldsymbol{X}$, $\boldsymbol{Y}$ and $\boldsymbol{S}$ is described by

$$\mathbb{P}_\Phi(W=w)=\frac{1}{|\mathcal{W}|}\,,\qquad\mathbb{P}_\Phi(S_1=s\big|W)=\boldsymbol{\mu}(s)\,,$$

$$\mathbb{P}_\Phi(X_t = x \big| W, \boldsymbol{S}_1^t, \boldsymbol{X}_1^{t-1}, \boldsymbol{Y}_1^{t-1}) = \delta_{\left\{\phi_t(W, \boldsymbol{Y}_1^{t-1}, \boldsymbol{S}_1^t)\right\}}(x)\,,$$

$$\mathbb{P}_\Phi(S_{t+1} = s, Y_t = y \big| W, \boldsymbol{S}_1^t, \boldsymbol{Y}_1^{t-1}, \boldsymbol{X}_1^t) = P\left(s, y \big| S_t, X_t\right)\,.$$

$\mathbb{E}_\Phi$ will denote the corresponding expectation operator.

It is convenient to introduce the following notation for the information patterns available at the encoder and decoder side. For every $t$ we define the sigma-fields $\mathcal{E}_t := \sigma\left(\boldsymbol{S}_1^t, \boldsymbol{Y}_1^{t-1}\right)$, describing the feedback information available at the encoder side, and $\mathcal{F}_t := \sigma\left(\boldsymbol{S}_1^t, \boldsymbol{Y}_1^t\right)$, describing the information available at the decoder. Clearly $\{\emptyset, \Omega\} = \mathcal{E}_0 = \mathcal{F}_0 \subseteq \mathcal{E}_1 \subseteq \mathcal{F}_1 \subseteq \ldots$. In particular we end up with two nested filtrations: $\mathcal{F} := (\mathcal{F}_t)_{t \in \mathbb{Z}^+}$ and $\mathcal{E} := (\mathcal{E}_t)_{t \in \mathbb{Z}_+}$.

**Definition 4** *A sequential decoder for a causal feedback encoder $\Phi$ as in (7) is a pair $\Psi = (T, \psi)$, where $T$ is a stopping time for the decoder filtration $\mathcal{F}$ and $\psi$ an $\mathcal{W}$-valued $\mathcal{F}_T$-measurable random variable.*

Given a causal feedback encoder $\Phi$ as in Def. 3 and a sequential decoder $\Psi$ as in Def. 4, their error probability is

$$p_e(\Phi, \Psi) := \mathbb{P}_\Phi\left(\psi \neq W\right)\,.$$

Following Burnashev's approach we shall take the expected decoding time $\mathbb{E}_\Phi[T]$ as a measure of the delay and accordingly define the rate of the coding scheme by

$$R := \log|\mathcal{W}| / \mathbb{E}_\Phi[T]\,.$$

### C. Main result

We are now ready to state our main result. It is formulated in an asymptotic setting, considering countable families of causal encoders and sequential decoders with asymptotic average rate below capacity and vanishing error probability.

**Theorem 1** *For any $R$ in $(0, C)$*

1) *any family $(\Phi^n, \Psi^n)_{n\in\mathbb{N}}$ of causal encoder and sequential decoder pairs such that*

$$\lim_{n\in\mathbb{N}} p_e(\Phi^n, \Psi^n) = 0\,, \qquad \limsup_{n\in\mathbb{N}} \frac{\log|\mathcal{W}_n|}{\mathbb{E}_{\Phi^n}[T^n]} \geq R\,, \quad (8)$$

*satisfies*

$$\limsup_{n\in\mathbb{N}} -\frac{1}{\mathbb{E}_{\Phi^n}[T^n]} \log p_e\left(\Phi^n, \Psi^n\right) \leq E_B(R)\,. \quad (9)$$

2) *there exists a family $(\Phi^n, \Psi^n)_{n\in\mathbb{N}}$ of causal encoder and sequential decoder pairs satisfying (8) and such that*

$$\lim_{n\in\mathbb{N}} -\frac{1}{\mathbb{E}_{\Phi^n}[T^n]} \log p_e\left(\Phi^n, \Psi^n\right) = E_B(R)\,. \quad (10)$$

We observe that Burnashev's original result [5] for memoryless channels can be recovered as a particular case of Theorem 1 when the state space is trivial, i.e. $|\mathcal{S}| = 1$.

## III. An asymptotically optimal scheme

In order to prove Part 2 of Theorem 1, thus showing the achievability of the Burnashev's exponent $E_B(R)$, we propose an iterative transmission scheme consisting in a generalization of the one introduced by Yamamoto and Itoh [14] for memoryless channels. This scheme consists of a sequence of epochs. Each epoch is made up of two distinct transmission phases, respectively named communication and confirmation phase. In the communication phase the message to be sent is encoded in a block code and transmitted over the channel. At the end of this phase the decoder makes a tentative decision about the message sent based on the observation of the channel outputs and of the state sequence. As perfect feedback is available, the result of this decision is known at the encoder. In the confirmation phase a binary acknowledge message, confirming the decoder's estimation if it is correct, or denying it when it is wrong, is sent by the encoder through a fixed-length repetition code. The decoder performs a binary hypothesis test in order to decide whether a deny or a confirmation message has been sent. If a confirmation is detected the transmission halts, while if a deny is detected the system restarts transmitting the same message with the same protocol. Again because of perfect feedback availability at the encoder, there are no synchronization problems.

More precisely we design our scheme as follows. Given a design rate $R$ in $(0, C)$, let us fix an arbitrary value $\gamma$ in $(\frac{R}{C}, 1)$. For every $n$ in $\mathbb{N}$, consider a message set $\mathcal{W}_n$ of cardinality $|\mathcal{W}_n| = \exp(\lfloor nR \rfloor)$ and two blocklengths $\hat{n}$ and $\tilde{n}$ respectively defined as $\hat{n} = \lceil n\gamma \rceil$, $\tilde{n} := n - \hat{n}$.

### A. Fixed-length coding for the transmission phase

It is known that $C$ defined in (3) equals the capacity of the stationary Markov channel we are considering [11]. This implies that, since $\lim \frac{\log|\mathcal{W}_n|}{\hat{n}} = \frac{R}{\gamma} < C$, there exists a sequence of causal encoders with no output feedback $\{\hat{\phi}_t^n : \mathcal{W}_n \times \mathcal{S}^t \to \mathcal{X}\}$, and a sequence of decoders of fixed length $\hat{n}$ $\{\hat{\psi}^n : \mathcal{S}^{\hat{n}} \times \mathcal{Y}^{\hat{n}} \to \mathcal{W}_n\}$ with asymptotically vanishing error probability. More precisely, the pair $(\hat{\Phi}^n, \hat{\Psi}^n)$ can be chosen with error probability going to zero uniformly with respect to initial state distribution and the transmitted message. Thus, denoting by $p(n)$ the maximum over all initial state distributions $\boldsymbol{\mu}$ and messages $w$ in $\mathcal{W}_n$ of the error probability of the pair $(\hat{\phi}^n, \hat{\Psi}^n)$ conditioned on the transmission of $W = w$, we have that $\lim p(n) = 0$. The pair $(\hat{\Phi}^n, \hat{\Psi}^n)$ will be used in the first phase of each epoch of our iterative transmission scheme.

### B. Binary hypothesis test for the confirmation phase

For the second phase we consider a causal repetition encoder $\tilde{\Phi}^n$ using the stationary policy optimizing the average cost $d$ in (6). More specifically, using the notation introduced at the end of Sec.II-A, we define $\tilde{\phi}_t^n : \{0, 1\} \times \mathcal{S}^t \to \mathcal{X}$, $\tilde{\phi}_t^n(m, \boldsymbol{s}) = x_{s_t}^m$. Suppose that an acknowledge message $m = 0$ is sent. Then it is easy to verify that the pair sequence $(S_{t+1}, Y_t)_{t=1}^{\tilde{n}}$ is distributed like a Markov chain with state space $\{(s,y) \in \mathcal{S}\times\mathcal{Y} : \exists s,x : P(v,y|s,x) > 0\}$ and transition probabilities $P_0(v,y|s,z) := P(v,y|s,x_s^0)$. Analogously, if a deny message $m = 1$ has been sent, with transition probabilities $P_1(v,y|s,z) := P(v,y|s,x_s^1)$. It follows that a decoder for $\tilde{\Phi}^n$ is simply a binary hypothesis test between two Markov chain hypotheses. Using binary hypothesis test results for irreducible Markov chains [9], [7], it is possible to show that the decoder $\tilde{\Psi}^n$ can be chosen in such a way that, asymptotically in $n$, its type-1 error probability achieves the exponent $D$ while its type-0 error probability is vanishing. More specifically, since the state space is finite, we have that, defining $p_0(n)$ (respectively $p_1(n)$) as the maximum over all possible initial state distributions $\boldsymbol{\mu}$ of the error probability of the pair $(\tilde{\Phi}^n, \tilde{\Psi}^n)$ conditioned on the transmission of a $'0'$ ($'1'$) message, we have $\lim_n p_0(n) = 0$ and $\lim_n \frac{-\log p_1(n)}{\tilde{n}} = D$.

### C. Performances of the proposed scheme

Once fixed $\hat{\Phi}^n$, $\hat{\Psi}^n$, $\tilde{\Phi}^n$ and $\tilde{\Psi}^n$, the iterative protocol described at the beginning of this section defines a causal encoder $\Phi^n = (\mathcal{W}_n, (\phi_t^n))$ and a sequential decoder $\Psi^n = (T^n, \psi^n)$. It can be verified that $p_e(\Phi^n, \Psi^n) \le p_1(n)$, and

$$\mathbb{P}_{\Phi^n}(T^n > kn) \le (p_0(n) + p(n))^k\,, \qquad k \ge 0\,, \tag{11}$$

i.e. $T^n$ is dominated by a scaled geometric r.v. It follows that

$$\lim_{n\in\mathbb{N}} \frac{\log|\mathcal{W}_n|}{\mathbb{E}_{\Phi^n}[T^n]} = R\,, \qquad \lim_{n\in\mathbb{N}} \frac{-\log p_e(\Phi^n, \Psi^n)}{\mathbb{E}_{\Phi^n}[T^n]} = D(1-\gamma)\,,$$

and (10) can be deduced from the arbitrariness of $\gamma$ in $(\frac{R}{C}, 1)$.

We emphasize the fact that the two phases in each epoch of the scheme are of fixed length, while the number of epochs can be variable. Hence, the overall transmission length of the scheme is variable. However, (11) guarantees that with high probability the transmission halts after the first epoch, a property making this scheme appealing for practical implementation, as already noticed in [8]. Moreover the first transmission phase only requires an asymptotically vanishing error probability, not necessarily at an exponential rate.

## IV. An upper bound on the achievable error exponent

In his original proof for memoryless channels [5], Burnashev suggested to look, given an arbitrary causal encoder, at the evolution of the stochastic dynamical system describing the entropy of the a posteriori distribution of the transmitted message. He proved two bounds for the conditional expected values of the decrements of the a posteriori entropy and of its logarithm respectively based on the capacity $C$ and the coefficient $D$. Then he combined these results using standard martingale arguments and a generalized Fano's inequality, obtaining a lower bound for the error probability of a generic sequential coding scheme.

Here we follow the approach proposed in [8] (see also [12]) and look at the evolution of the maximum a posteriori error probability associated to any causal encoder. We first obtain a lower bound to the error probability involving the sum of the mutual information terms: this is given in Lemma 3 whose proof is based on an application of Fano's inequality and Doob's optional sampling theorem. Then we obtain an upper bound to the error exponent achievable by a binary hypothesis test: see Lemma 4 which can be proved by using the log-sum inequality and the optional sampling theorem again. Combining these two Lemmas we obtain Theorem 5. These results are generalizations of those presented in [5], [2], [6], the main difference consisting in the fact that they are not optimized with respect to the channel input distribution. This optimization instead is taken successively in an asymptotic setting, using the results presented in Sec.V.

### A. A first bound on the error probability

It will be convenient to define for every $t \ge 0$ the $\sigma$-algebra $\mathcal{G}_t := \mathcal{E}_{t+1}$ describing the encoder's feedback information at time $t+1$; $\mathcal{G} := (\mathcal{G}_t)_{t\in\mathbb{Z}_+}$ will denote the corresponding filtration and

$$\tilde{P}^{\Phi}_{MAP}(t) := 1 - \max_{w\in\mathcal{W}} \{\mathbb{P}_\Phi(W = w|\mathcal{G}_t)\}\,.$$

will denote the a posteriori error probability given the encoder's feedback information at time $t+1$. It is known that the decoder minimizing the error probability over the class of fixed-length decoders is the maximum a posteriori one. Thus, given a stopping time $\tau$ for the decoder filtration $\mathcal{F}$, the error probability of $\Psi = (\tau, \psi)$, where $\psi$ is an arbitrary $\mathcal{F}_\tau$-measurable $\mathcal{W}$-valued r.v., is lower bounded by

$$p_e(\Phi, (\tau, \psi)) \ge \mathbb{E}_\Phi\left[\tilde{P}^{\Phi}_{MAP}(\tau)\right]\,. \tag{12}$$

It is thus sufficient to lower bound the righthand side of (12). In particular, since the random variable $W$ is uniformly distributed over the message set $\mathcal{W}$, and since $S_1$ is independent of $W$, we have that $\tilde{P}^{\Phi}_{MAP}(0) = (|\mathcal{W}|-1)/|\mathcal{W}|$. Moreover it is not difficult to prove the following recursive lower bound.

**Lemma 2** *Given any causal feedback encoder $\Phi$, for any $t \ge 0$*

$$\tilde{P}^{\Phi}_{MAP}(t+1) \ge \lambda \tilde{P}^{\Phi}_{MAP}(t) \qquad \mathbb{P}_\Phi - a.s.$$

We associate to every causal encoder $\Phi$ the sequence of $\mathcal{P}(\mathcal{X})$-valued random variables $(\boldsymbol{\Upsilon}_{\Phi,t})$ defined by

$$\boldsymbol{\Upsilon}_{\Phi,t}(x) := \mathbb{P}_\Phi(X_t = x|\,\mathcal{E}_t)\,, \qquad t\in\mathbb{N},\, x\in\mathcal{X}\,. \tag{13}$$

$\boldsymbol{\Upsilon}_{\Phi,t}$ is $\mathcal{E}_t$-measurable and represents the channel input distribution induced by the encoder $\Phi$ at time $t$.

For every $\delta$ in $(0, \frac{1}{2})$, we now consider the random variable

$$\tau_\delta := \min\left\{T, \inf\left\{t\in\mathbb{N} : \tilde{P}^{\Phi}_{MAP}(t) \le \delta\right\}\right\}\,. \tag{14}$$

The following result relates three relevant quantities characterizing the performances of any causal encoder sequential decoder pair: cardinality of the message set, error probability, and the sum of the mutual information costs up to time $\tau_\delta$.

**Lemma 3** *For any causal encoder $\Phi = (\mathcal{W}, (\phi_t))$, any sequential decoder $\Psi = (T, \psi)$ and any $0 < \delta < \frac{1}{2}$,*

$$\mathbb{E}_\Phi\left[\sum_{t=1}^{\tau_\delta} c(S_t, \boldsymbol{\Upsilon}_{\Phi,t})\right] \ge \left(1 - \delta - \frac{p_e(\Phi,\Psi)}{\delta}\right)\log|\mathcal{W}| - \mathrm{H}(\delta)\,. \tag{15}$$

*B. A lower bound to the error probability of a composite binary hypothesis test*

We now consider a particular binary hypothesis testing problem which will arise while proving the main result. Consider a nontrivial binary partition of the message set

$$\mathcal{W} = \mathcal{W}_0 \cup \mathcal{W}_1\,, \qquad \mathcal{W}_0 \cap \mathcal{W}_1 = \emptyset\,, \qquad \mathcal{W}_0, \mathcal{W}_1 \neq \emptyset\,, \quad (16)$$

and a sequential binary hypothesis test $\tilde{\Psi} = (T, \tilde{\psi})$ (where $T$ is a stopping time with respect to $\mathcal{F}$, and $\tilde{\psi}$ is an $\mathcal{F}_T$-measurable $\{0,1\}$-valued random variable) between the hypotheses $\{W \in \mathcal{W}_0\}$ and $\{W \in \mathcal{W}_1\}$. For any $t$, we define the $\mathcal{P}(\mathcal{X})$-valued r.v.s $\boldsymbol{\Upsilon}^0_{\Phi,t}$ and $\boldsymbol{\Upsilon}^1_{\Phi,t}$ by

$$\boldsymbol{\Upsilon}^i_{\Phi,t}(x) = \mathbb{P}_\Phi\left(X_t = x \middle|\, W \in \mathcal{W}_i, \mathcal{E}_t\right)\,, \qquad x \in \mathcal{X},\ i = 0, 1\,.$$

The random variable $\boldsymbol{\Upsilon}^i_{\Phi,t}$ represents the channel input distribution at time $t$ induced by the encoder $\Phi$ when restricted to the message subset $\mathcal{W}_i$.

Consider now a stopping time $\tau$ for the filtration $\mathcal{G}$, such that $\tau \leq T$. Suppose that $\mathcal{W}_1$ is a $\mathcal{G}_\tau$-measurable random subset of the message set $\mathcal{W}$. The following lower bound to the error probability of the binary test $\tilde{\Psi}$ conditioned on $\mathcal{G}_\tau$ can be proved using the log-sum inequality and Doob's optional sampling theorem.

**Lemma 4** *Let $\Phi$ be any causal encoder, $\tau \leq T$ stopping times for $\mathcal{G}$. Then, for every $\mathcal{G}_\tau$-measurable random message subset $\mathcal{W}_1$*

$$\mathbb{E}_\Phi\left[\sum_{t=\tau+1}^{T} d\left(S_t, \boldsymbol{\Upsilon}^{\mathbb{1}_{\{W \in \mathcal{W}_1\}}}_{\Phi,t}\right)\middle|\, \mathcal{G}_\tau\right] \geq -\log \frac{\mathbb{P}_\Phi\left(\tilde{\psi} \neq \mathbb{1}_{\{W \in \mathcal{W}_1\}} \middle|\, \mathcal{G}_\tau\right)}{Z/4} \quad (17)$$

*$\mathbb{P}_\Phi$-a.s., where $Z := \min_{i=0,1}\left\{\mathbb{P}_\Phi\left(W \in \mathcal{W}_i | \mathcal{G}_\tau\right)\right\}$.*

*C. Burnashev bound for Markov channels*

From Lemmas 2, 3 and 4, it is possible to prove the following.

**Theorem 5** *Given a causal feedback encoder $\Phi = (\mathcal{W}, (\phi_t))$ and a sequential decoder $\Psi = (T, \psi)$, for every $\delta > 0$, there exists a $\mathcal{G}_{\tau_\delta}$-measurable random subset $\mathcal{W}_1$ of $\mathcal{W}$ such that*

$$\begin{aligned}\frac{D}{C}\mathbb{E}_\Phi\left[\sum_{t=1}^{\tau_\delta} c(S_t, \boldsymbol{\Upsilon}_{\Phi,t})\right] + \mathbb{E}_\Phi\left[\sum_{t=\tau_\delta+1}^{T} d\left(S_t, \boldsymbol{\Upsilon}^{\mathbb{1}_{\{W \in \mathcal{W}_1\}}}_{\Phi,t}\right)\right] \\ \geq -\log p_e\left(\Phi, \Psi\right) + \frac{D}{C}\log|\mathcal{W}|\left(1 - \alpha\right) + \beta\,,\end{aligned} \quad (18)$$

*where $\mathcal{W}_0 = \mathcal{W} \setminus \mathcal{W}_1$, $\tau_\delta$ is defined by (14), and*

$$\alpha := \delta + \frac{p_e(\Phi, \Psi)}{\delta}\,, \qquad \beta := \log\frac{\lambda\delta}{4} - \frac{D}{C}\,\mathrm{H}(\delta)\,.$$

Inequality (18) constitutes a generalization of Burnashev's (4.1) in [5] (see also (12) in [2]). Indeed, in the memoryless case Burnashev's result can be recovered from (18) by optimizing its lefthand side with respect to the channel input distribution. In the general case of Markov channels with ISI it is more convenient to take this optimization in an asymptotic setting as explained below.

In order to obtain the single-letter characterization of Theorem 1, it is necessary to consider a countable family of causal encoders $(\Phi^k)$ and a corresponding family of sequential decoders $(\Psi^k)$ satisfying (8). The idea is to consider a positive real sequence $(\delta_k)$ and to show that both

$$\tau^k := \inf\left\{t \in \mathbb{N} \middle|\, t \geq T^k \text{ or } P^{\Phi^k}_{MAP}(t) \leq \delta_k\right\}\,,$$

and $T^k - \tau^k$ 'diverge' in the sense of satisfying (20) below. The sequence $(\delta_k)$ needs to be properly chosen: we want it to be asymptotically vanishing in order to guarantee that $\tau^k$ diverges, but not too fast since otherwise $T^k - \tau^k$ would not diverge. It turns out that one possible good choice is $\delta_k := \frac{-1}{\log p_e(\Phi^k, \Psi^k)}$. From (2) it follows that

$$\lim_{k \in \mathbb{N}} \delta_k = 0\,, \qquad \lim_{k \in \mathbb{N}} \frac{p_e\left(\Phi^k, \Psi^k\right)}{\delta_k} = 0\,. \quad (19)$$

**Lemma 6** *In the previous setting, for every $M$ in $\mathbb{N}$, we have*

$$\lim_{k \in \mathbb{N}} \mathbb{P}^{\Phi^k}\left(\tau^k \leq M\right) = 0\,, \qquad \lim_{k \in \mathbb{N}} \mathbb{P}^{\Phi^k}\left(T^k - \tau^k \leq M\right) = 0\,. \quad (20)$$

Lemma 6 allows us to use the results in the appendix in order to conclude that

$$\limsup_{k \in \mathbb{N}} \frac{1}{\mathbb{E}_{\Phi^k}[\tau^k]}\mathbb{E}_{\Phi^k}\left[\sum_{t=1}^{\tau^k} c(S_t, \boldsymbol{\Upsilon}_{\Phi^k,t})\right] \leq C\,, \quad (21)$$

$$\limsup_{k \in \mathbb{N}} \frac{1}{\mathbb{E}_{\Phi^k}[T^k - \tau^k]}\mathbb{E}_{\Phi^k}\left[\sum_{t=\tau^k+1}^{T^k} d\left(S_t, \boldsymbol{\Upsilon}^{\mathbb{1}_{\{W \in \mathcal{W}_1\}}}_{\Phi,t}\right)\right] \leq D\,. \quad (22)$$

By taking the limit of both sides of (18) and substituting (21) and (22), we get (9).

## V. A RESULT ON CONTROLLED MARKOV CHAINS

This final section deals with some considerations about controlled Markov chains which are needed in order to prove (21) and (22). In particular we will follow the convex-analytical approach introduced by Borkar [4] for average cost problems.

We consider a discrete time, stationary controlled Markov process with state space $\mathcal{S}$, control space $\mathcal{U} = \mathcal{P}(\mathcal{X})$, transition kernel $\{Q(\,\cdot\,|\, s, \boldsymbol{u}) \in \mathcal{P}(\mathcal{S}) |\, s \in \mathcal{S}, \boldsymbol{u} \in \mathcal{U}\}$ defined by (2) and initial state distribution $\boldsymbol{\mu}$, and let $g : \mathcal{S} \times \mathcal{U} \to \mathbb{R}$ be a continuous cost function. The evolution of the system is described by a state sequence $\boldsymbol{S} = (S_t)_{t \in \mathbb{N}}$ and a control sequence $\boldsymbol{U} = (U_t)_{t \in \mathbb{N}}$. If at time $t$ the system is in state $S_t = s$ in $\mathcal{S}$, and a control $U_t = u$ in $\mathcal{U}$ is chosen according to some policy, then a cost $g(s, u)$ is incurred and the system moves to next state $S_{t+1}$ in $\mathcal{S}$ according to the probability distribution $Q(\,\cdot\,|\, s, u)$. Once the transition into next state has occurred, a new action is chosen and the process is repeated. The control at time $t$, $U_t$, can be chosen as a (randomized) function of the past history, i.e. the state values $\boldsymbol{S}_1^t$ up to time $t$ and the past control values $\boldsymbol{U}_1^{t-1}$ up to time $t - 1$. More precisely we consider a filtration $\{\emptyset,\Omega\}=\mathcal{E}_0\subseteq\mathcal{E}_1\subseteq\ldots$ such that $S_t$ is $\mathcal{E}_t$-measurable and define an admissible policy $\boldsymbol{\pi}$ as a sequence $(\pi_t)_{t\in\mathbb{N}}$ of $\mathcal{P}(\mathcal{U})$-valued random variables, each $\pi_t$ being $\mathcal{E}_t$-measurable; the set of all admissible policies will be denoted by $\Pi$. The joint distribution of the state sequence $\boldsymbol{S}$ and the control sequence $\boldsymbol{U}$ induced by an admissible policy $\boldsymbol{\pi}$ will be denoted by $\mathbb{P}_{\boldsymbol{\pi}}$, the expectation operator by $\mathbb{E}_{\boldsymbol{\pi}}$.

We have already noticed that, under the irreducibility assumption we made, given any stationary policy $\boldsymbol{\pi}:\mathcal{S}\to\mathcal{U}$ a unique invariant measure $\boldsymbol{\eta_\pi}$ exists for the matrix $Q_{\boldsymbol{\pi}}$. We will associate to such an invariant measure the so called occupation measure $\hat{\boldsymbol{\eta}}_{\boldsymbol{\pi}}$ in $\mathcal{P}(\mathcal{S}\times\mathcal{U})$ defined by

$$\langle\hat{\boldsymbol{\eta}}_{\boldsymbol{\pi}},g\rangle:=\sum_{s\in\mathcal{S}}\boldsymbol{\eta_\pi}(s)g(s,\boldsymbol{\pi}(s))\,,$$

for every $g\in\mathcal{C}_b(\mathcal{S}\times\mathcal{U})$, the space of continuous functions on $\mathcal{S}\times\mathcal{U}$. It is a well-known fact in controlled Markov chains theory [1], [4] that the set $K_e:=\{\hat{\boldsymbol{\eta}}_{\boldsymbol{\pi}}\,|\,\boldsymbol{\pi}:\mathcal{S}\to\mathcal{U}\}$ is closed in $\mathcal{P}(\mathcal{S}\times\mathcal{U})$. Moreover it is possible to show that $K_e$ actually coincides with the the set of extreme points of the convex set $K:=\{\boldsymbol{\eta}:\boldsymbol{F}(\boldsymbol{\eta})=\boldsymbol{0}\}$ of the zeros of the map $\boldsymbol{F}:\mathcal{P}(\mathcal{S}\times\mathcal{U})\to[0,1]^{\mathcal{S}}$,

$$F_s(\boldsymbol{\eta}):=\boldsymbol{\eta}\left(\{s\}\times\mathcal{U}\right)-\langle\eta(\cdot,\cdot),Q(\,s\,|\cdot,\cdot)\rangle\,.$$

Due to the finiteness of $\mathcal{X}$, the control space $\mathcal{U}=\mathcal{P}(\mathcal{X})$ is compact; since also the state space $\mathcal{S}$ is finite, the simplex $\mathcal{P}(\mathcal{S}\times\mathcal{U})$ turns out to be compact as well. Thus, the continuous map

$$g^*:\mathcal{P}(\mathcal{S}\times\mathcal{U})\to\mathbb{R}\,,\qquad g^*:\boldsymbol{\eta}\mapsto\langle\boldsymbol{\eta},g\rangle$$

achieves its maximum over both the compacts $K$ and $K_e$. Moreover, since $g^*$ is linear, $K$ is convex and $K_e$ is the set of its extreme points, those maxima do coincide. Therefore,

$$G:=\max_{\boldsymbol{\eta}\in K}\langle\boldsymbol{\eta},g\rangle=\langle\hat{\boldsymbol{\eta}}_{\boldsymbol{\pi}^*},g\rangle\,,$$

for some stationary policy $\boldsymbol{\pi}^*:\mathcal{S}\to\mathcal{U}$. It is known that, for every diverging sequence of positive integers $(n_k)$, and any sequence of admissible policies $(\boldsymbol{\pi}^k)$,

$$\limsup_{k\in\mathbb{N}}\frac{1}{n_k}\mathbb{E}_{\boldsymbol{\pi}^k}\left[\sum_{t=1}^{n_k}g(S_t,U_t)\right]\leq G\,. \tag{23}$$

Here we will generalize (23) considering stopping times.

In order to do that, for every $n$ we introduce the empirical measure $\boldsymbol{\upsilon}_n$ in $\mathcal{P}(\mathcal{S}\times\mathcal{U})$ defined by

$$\langle\boldsymbol{\upsilon}_n,h\rangle:=\frac{1}{n}\sum_{t=1}^{n}h(S_t,U_t)\,,\qquad\forall\;h\in\mathcal{C}_b(\mathcal{S}\times\mathcal{U})\,.$$

Clearly under the ergodicity assumption made we have that $\boldsymbol{\upsilon}_n$ converges to $\hat{\boldsymbol{\eta}}_{\boldsymbol{\pi}}$, $\mathbb{P}_{\boldsymbol{\pi}}$-a.s. for every stationary policy $\boldsymbol{\pi}$. Moreover it is known that for every possibly non stationary policy $\boldsymbol{\pi}$ the set of limit points of the empirical measure sequence $(\boldsymbol{\upsilon}_n)$ is contained in $K$, $\mathbb{P}_{\boldsymbol{\pi}}$-a.s.. This result is usually proved using a martingale central limit theorem [1], [4] to show that $\boldsymbol{F}(\boldsymbol{\upsilon}_n)$ converges to 0 a.s.. Lemma 7 below provides a bound on the tails of the distribution of $\boldsymbol{F}(\boldsymbol{\upsilon}_n)$ which is uniform with respect to the chosen policy. Its proof relies on an application of Hoeffding-Azuma inequality [7].

**Lemma 7** *For every $n$ in $\mathbb{N}$ there exists two nonnegative valued random variables $A_n$ and $B_n$ such that*

$$||\boldsymbol{T}(\boldsymbol{\upsilon}_n)||_1=A_n+B_n\,,\qquad A_n\leq 2/n\,,$$

*while for every $\varepsilon>0$,*

$$\mathbb{P}_{\boldsymbol{\pi}}\left(B_n\geq\varepsilon\right)\leq 2|\mathcal{S}|\exp\left(-n\varepsilon^2/4|\mathcal{S}|^2\right)\,,\quad\forall\,\boldsymbol{\pi}\in\Pi\,. \tag{24}$$

We emphasize that the bound provided in (24) is uniform with respect to the admissible policy $\boldsymbol{\pi}$. This fact, together with the observation that the map $\gamma:\mathbb{R}^+\to\mathbb{R}$

$$\gamma(x):=\sup\left\{\langle\boldsymbol{\eta},c\rangle\,\middle|\,\boldsymbol{\eta}\in\mathcal{P}(\mathcal{S}\times\mathcal{U}):\,||\boldsymbol{T}(\boldsymbol{\eta})||\leq x\right\}$$

is upper semicontinuous allows to prove the following.

**Lemma 8** *For every sequence $(\boldsymbol{\pi}^k)$ of admissible policies and any sequence $(\tau^k)$ of stopping times such that*

$$\lim_{k\in\mathbb{N}}\mathbb{P}_{\boldsymbol{\pi}^k}(\tau^k\leq M)=0\,,\qquad\forall M\in\mathbb{N}\,, \tag{25}$$

*we have*

$$\limsup_{k\in\mathbb{N}}\frac{1}{\mathbb{E}_{\boldsymbol{\pi}^k}\left[\tau^k\right]}\mathbb{E}_{\boldsymbol{\pi}^k}\left[\sum_{t=1}^{\tau^k}g(S_t,U_t)\right]\leq G\,. \tag{26}$$

## REFERENCES


[1] A. Arapostathis, V. S. Borkar, E. Fernandez-Gaucherand, M.K. Ghosh and S.I. Marcus, "Discrete-time controlled Markov processes with average cost criterion: A survey", *SIAM J. Control Optim.*, vol. 31, pp. 282-344, 1993.

[2] P. Berlin, B. Nakiboglu, B. Rimoldi, E. Telatar, "A Simple Derivation of Burnashev's Reliability function", preprint, http://arxiv.org/abs/cs/0610145, 2006.

[3] V. S. Borkar, *Probability Theory: An Advanced Course*, Springer, 1995.

[4] V. S. Borkar, "Convex analytic methods in Markov decision processes", *Handbook of Markov Decision Processes*, E. A. Feinberg and A. Shwartz eds., Kluwer, 2001.

[5] M. V. Burnashev, "Data Transmission over a Discrete Channel with Feedback. Random Transmission Time", *Prob. of Inform. Transm.*, vol. 12 (4), pp. 1030, 1976.

[6] G. Como, S. Yuksel, S. Tatikonda, "On the error exponent of Markov channels with feedback", *proc. of ISIT 2007*, 2007.

[7] A. Dembo, O. Zeitouni, *Large Deviations Techniques and Applications*, 2nd ed., Springer, 1998.

[8] B. Nakiboglu, R. G. Gallager, "Error exponents for variable-length block codes with feedback and cost constraints", preprint, http://arxiv.org/abs/cs.IT/0612097, 2006.

[9] S. Natarajan, "Large deviations, hypotheses testing, and source coding for finite Markov chains", *IEEE Trans. Inform. Theory*, vol. 31, pp. 360-365, 1985.

[10] C. E. Shannon "The zero error capacity of a noisy channel", *IEEE Trans. Inform. Theory*, vol. 2, pp. 8-19, 1956.

[11] S. Tatikonda, S. Mitter, "The Capacity of Channels with Feedback", preprint, http://arxiv.org/abs/cs/0609139, 2006.

[12] A. Tchamkerten, I. E. Telatar, "On the Universality of Burnashev's Error Exponent", *IEEE Trans. Inform. Theory*, vol. 44, pp. 2940-2944, 2005.

[13] A. Tchamkerten, I. E. Telatar, "Variable Length Coding over an Unknown Channel", *IEEE Trans. Inform. Theory*, vol. 52, pp. 2126-2145, 2006.

[14] H. Yamamoto, K. Itoh, "Asymptotic performance of a modified Schalkwijk-Barron scheme for channels with noiseless feedback", *IEEE Trans. Inf. Theory*, vol. 25, pp. 729-733, 1979.