

# Stability of monotone dynamical flow networks

Enrico Lovisari, Giacomo Como, and Ketan Savla

**Abstract**—We study stability properties of monotone dynamical flow networks. Demand and supply functions relate states and flows of the network, and the dynamics at junctions are subject to fixed turning rates. Our main result consists in the characterization of a stability region such that: If the inflow vector in the network lies strictly inside the stability region and a certain graph theoretical condition is satisfied, then a globally asymptotically stable equilibrium exists. In contrast, if the inflow vector lies strictly outside the region, then every trajectory grows unbounded in time. As a special case, our framework allows for the stability analysis of the Cell Transmission Model on networks with arbitrary topologies. These results extend and unify previous work by Gomes *et al.* on stability of the Cell Transmission Model on a line topology as well as that by the authors on throughput optimality in monotone dynamical flow networks.

**Index Terms**—Transportation systems; Monotone systems; Dynamical flow network; Stability.

## I. INTRODUCTION

Transportation systems have been receiving an increasing degree of attention from the control community due to their societal and economic impact, and because they are a prototype of cyber-physical networks amenable to be treated with the tools developed during the last decades [1], [2], [3]. Large part of the literature on macroscopic traffic models is based on PDEs models [4], [5]. Despite the fact that the latter have proved valuable to represent real world phenomena, solving the PDEs is often a very difficult problem to tackle even in the simplest settings. In this paper, we consider instead dynamical flow networks modeled as mass-conservation driven systems of ODEs on directed graphs. Each node of the graph represents a junction, while links correspond to physical links in the network. The ODEs model the dynamics of the densities, or occupancy levels, of particles flowing through the network. On each link, the desired outflow and the maximum inflow are given by the density dependent demand and supply functions, respectively. Certain links work as on-ramps and have fixed inflows corresponding to the constant rate at which particles enter the network from the external world. Conversely, the outflow from links that act as off-ramps is equal to their demand, and leaves the network.

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The model extends the celebrated Cell Transmission Model (CTM) [6], [7] to networks with arbitrary topology. Our model inherits from the original CTM for networks [7] the model for flow through merge nodes, but differs in treating the flow in diverge nodes. Indeed, both models employ *fixed turning rates* in freeflow, namely, when at a node the supply of downstream links is sufficient to accommodate the flow coming from upstream. However, while [7], [8] consider a FIFO policy in congestion, namely the total outflow from a link is bounded by the most congested link downstream, we assume that each turning is independent. This allows us to prove that the system is monotone [9], and to offer a strong characterization of the network equilibria structure: the space of possible inflow vectors is divided in regions. If the inflow vector is strictly inside one of these regions and a certain graph theoretical condition is satisfied, then the network admits a globally asymptotically stable equilibrium.

When the inflow vector crosses the boundary between two of such regions, the corresponding globally asymptotically stable equilibrium jumps to a strictly higher value, namely the network admits a sequence of phase transitions. When the inflow vector lies instead strictly outside the last region, all the trajectories grow unbounded in time. The contributions of this paper can be summarized as follows: 1) we propose a monotone model for cyclic, multi-origin multi-destination dynamical flow networks that exhibits enough structure to be fully characterized in terms of its stability properties, 2) we extend and unify existing results [10], [11] that obtained similar results for the line topology, 3) we extend our previous analysis of dynamical flow networks [12], [13] to the demand-and-supply setting. In particular, in [12], [13] the model for dynamical flow networks is substantially different since it is based on *dynamic routing* rather than on fixed turning rates and demand and supply functions. While it shares with the model in the present paper the monotonicity properties, it has stronger stability properties that do not hold in the present setting. The theoretical results are also different, as in the present paper we focus on analysing stability and properties of the equilibria, while in [12], [13] we prove maximal throughput and we discussed the resilience of the network to perturbations.

The paper is organized as follows: in the rest of this section we provide some basic notation. In Section II we describe the model, we study its monotonicity properties, and we state the main result. More details on the particular case of freeflow equilibrium are offered in Section III. Section IV provides a numerical example that illustrates the theoretical results. Finally, Section V draws the conclusions and provides future research directions, and we gather in

Appendix some technical results.

### A. Notation

The symbols  $\mathbb{R}$  and  $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$  denote the set of real and nonnegative real numbers, respectively. Let  $\mathcal{A}$  and  $\mathcal{B}$  be finite sets. Then  $|\mathcal{A}|$  denotes the cardinality of  $\mathcal{A}$ ,  $\mathbb{R}^{\mathcal{A}}$  (respectively,  $\mathbb{R}_+^{\mathcal{A}}$ ) the space of real-valued (nonnegative-real-valued) vectors whose components are indexed by elements of  $\mathcal{A}$ , and  $\mathbb{R}^{\mathcal{A} \times \mathcal{B}}$  the space of matrices whose real entries are indexed by pairs in  $\mathcal{A} \times \mathcal{B}$ . The transpose of a matrix  $M \in \mathbb{R}^{\mathcal{A} \times \mathcal{B}}$  is denoted by  $M' \in \mathbb{R}^{\mathcal{B} \times \mathcal{A}}$ , while  $\mathbf{0}$  and  $\mathbf{1}$  stand for an all-zero and all-one vectors of suitable dimension, respectively. The natural partial ordering of  $\mathbb{R}^{\mathcal{A}}$  will be denoted by  $x \preceq y$  for two vectors  $x, y \in \mathbb{R}^{\mathcal{A}}$  such that  $x_a \leq y_a$  for all  $a \in \mathcal{A}$ .

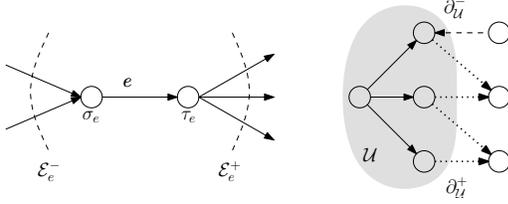


Fig. 1. Graphical illustration of some key notations. In the right, links comprising  $\partial_{\mathcal{U}}^-$  and  $\partial_{\mathcal{U}}^+$  are shown by dashed and dotted arrows, respectively; links comprising  $\mathcal{E}_{\mathcal{U}}^+ \setminus \partial_{\mathcal{U}}^+$  are shown in solid arrows.

A directed multi-graph is a couple  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ , where  $\mathcal{V}$  and  $\mathcal{E}$  stand for the node set and the link set, respectively, and are both finite. They are endowed with two vectors:  $\sigma, \tau \in \mathcal{V}^{\mathcal{E}}$ . For every  $e \in \mathcal{E}$ ,  $\sigma_e$  and  $\tau_e$  stand for the tail and head nodes respectively of link  $e$ . We shall always assume that there are no self-loops, i.e.,  $\tau_e \neq \sigma_e$  for all  $e \in \mathcal{E}$ . On the other hand, we allow for parallel links. For a node  $v \in \mathcal{V}$ , let  $\mathcal{E}_v^+ := \{e : \sigma_e = v\}$  and  $\mathcal{E}_v^- := \{e : \tau_e = v\}$ . For a link  $e \in \mathcal{E}$ , let  $\mathcal{E}_e^+ := \mathcal{E}_{\tau_e}^+$  be the set of links downstream to  $e$  and  $\mathcal{E}_e^- := \mathcal{E}_{\sigma_e}^-$  be the set of links upstream to  $e$ . For a node subset  $\mathcal{U} \subseteq \mathcal{V}$ , define  $\mathcal{E}_{\mathcal{U}}^+ := \cup_{u \in \mathcal{U}} \mathcal{E}_u^+$ . Let  $\partial_{\mathcal{U}}^+ := \{e \in \mathcal{E} : \sigma_e \in \mathcal{U}, \tau_e \notin \mathcal{U}\}$  and  $\partial_{\mathcal{U}}^- := \{e \in \mathcal{E} : \sigma_e \in \mathcal{V} \setminus \mathcal{U}, \tau_e \in \mathcal{U}\}$  be the set of links from  $\mathcal{U}$  to  $\mathcal{V} \setminus \mathcal{U}$  and from  $\mathcal{V} \setminus \mathcal{U}$  to  $\mathcal{U}$ , respectively. See Figure 1 for an illustration of some of these notations.

Given a graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ , a matrix  $J \in \mathbb{R}^{\mathcal{V} \times \mathcal{V}}$  is a weighted sublaplacian of  $\mathcal{G}$  if  $J_{vu} \geq 0$  for any  $v, u \in \mathcal{V}$ ,  $v \neq u$ , and  $\sum_u J_{uv} \leq 0$  for all  $v \in \mathcal{V}$ .

## II. MONOTONE DYNAMICAL FLOW NETWORKS

We study a transportation network modeled as a directed graph  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  in which the set of nodes  $\mathcal{V}$  represents junctions or the external world  $w$ , and the set of links represents physical links. A link  $e$  such that  $\sigma_e = w$  is called a on-ramp. The set of on-ramps is denoted by  $\mathcal{R} \subseteq \mathcal{E}$ .

Link  $e$ 's occupancy level, or density, is denoted by the symbol  $\rho_e \in [0, B_e]$ , where  $B_e$  is the maximum particles density allowed on the link. On on-ramps, we assume  $B_e = +\infty$ . Denote by  $R = \prod_{e \in \mathcal{E}} [0, B_e]$  the set of allowed

occupancy levels in the network. The densities are subject to the following mass-conservation driver dynamics

$$\dot{\rho}_e = f_e^{\text{in}}(\rho) - f_e^{\text{out}}(\rho) \quad (1)$$

where  $f_e^{\text{in}}(\rho)$  and  $f_e^{\text{out}}(\rho)$  denote the density dependent instantaneous inflow and the outflow on link  $e$ , defined below.

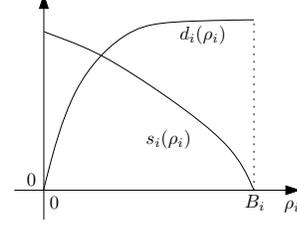


Fig. 2. Graphical illustration of demand and supply on a link  $i \in \mathcal{E} \setminus \mathcal{R}$ .

Each link has a demand function  $d_e$  that represents the ideal amount of flow on link  $e$  as a function of the density on  $e$ . We assume that  $d_e(\rho_e)$  is a continuous differentiable function with  $\frac{\partial d_e(\rho_e)}{\partial \rho_e} > 0$ ,  $d_e(0) = 0$  and  $\lim_{x \rightarrow \infty} d_e(x) = F_e > 0$ ,  $F_e$  being the ideal capacity, or maximum flow, of the link. Each non on-ramp link is also endowed with a supply function  $s_e$  that tells the maximum flow allowed into link  $e$ . We assume that  $s_e(\rho_e)$  is a continuous differentiable function such that  $\frac{\partial s_e(\rho_e)}{\partial \rho_e} < 0$ ,  $s_e(0) > 0$ , and  $s_e(B_e) = 0$ , for all  $e \in \mathcal{E} \setminus \mathcal{R}$ . Finally, for every link  $e \in \mathcal{E}$  we denote by the symbol  $C_e$  its capacity. For  $e \in \mathcal{R}$ , we set  $C_e = F_e$ . For  $e \in \mathcal{E} \setminus \mathcal{R}$ , by the monotonicity properties of demand and supply there exist a unique  $\hat{\rho}_e \in [0, B_e]$  such that  $d_e(\hat{\rho}_e) = s_e(\hat{\rho}_e)$ , namely, such that demand and supply are balanced. We then set  $C_e = d_e(\hat{\rho}_e)$ . We call this value capacity since it is the maximum flow on a link at equilibrium, as will be proven later. A graphical example of supply and demand functions for a link  $e \in \mathcal{E} \setminus \mathcal{R}$  is illustrated in Figure 2.

For each diverge node  $v \in \mathcal{V}$ , we assume that drivers have a set of fixed turning preferences  $\{R_{ej}\}_{j \in \mathcal{E}_v^+}$  such that  $R_{ej} \geq 0$  for all  $j \in \mathcal{E}_v^+$  and  $\sum_{j \in \mathcal{E}_v^+} R_{ej} = 1$ . These values tell how the flow from link  $e$  splits into the subsequent links.

Finally, we assume that there exists a nonempty set  $\mathcal{R}^o \subseteq \mathcal{E}$  of destination links, or off-ramps, such that  $\tau_e = w$ . If a particle leaves one of such links, it leaves the network.

The following minimal connectivity assumption ensures that particles are allowed to leave the network.

*Assumption 1:* For every link  $e \in \mathcal{E}$ , there exists at least one directed path from  $e$  to a link  $j \in \mathcal{R}^o$ .

Notice that, in particular, Assumption 1 implies that for any nonempty network there exists at least one off-ramp.

We consider the following routing policy:

$$f_{j \rightarrow e}(\rho) = R_{je} d_j(\rho_j) \min \left\{ 1, \frac{s_e(\rho_e)}{\sum_{k \in \mathcal{E}_{\sigma_e}^-} R_{ke} d_k(\rho_k)} \right\}$$

which can be interpreted as follows: If  $\sum_{k \in \mathcal{E}_{\sigma_e}^-} R_{ke} d_k(\rho_k) \leq s_e(\rho_e)$ , i.e., if the supply on  $e$  is sufficient to accommodate the whole ideal flow into  $e$ , then flow from  $j$  to  $e$  is equal to

its ideal value  $R_{je}d_j(\rho_j)$ . If instead  $\sum_{k \in \mathcal{E}_e^-} R_{ke}d_k(\rho_k) > s_e(\rho_e)$ , then the actual flow from  $j$  to  $e$  is a fraction of  $s_e(\rho_e)$  proportional to the ideal flow  $R_{je}d_j(\rho_j)$ .

We let finally

$$f_e^{\text{in}} = \begin{cases} \sum_{j \in \mathcal{E}_e^-} f_{j \rightarrow e}, & e \in \mathcal{E} \setminus \mathcal{R} \\ \lambda_e, & e \in \mathcal{R} \end{cases}$$

$$f_e^{\text{out}} = \begin{cases} \sum_{j \in \mathcal{E}_e^+} f_{e \rightarrow j}, & e \notin \mathcal{R}^\circ \\ d_e(\rho_e), & e \in \mathcal{R}^\circ \end{cases}.$$

We call the system (1) with the routing policies defined above a dynamical flow network with fixed preference rates. To make clear the dependence of the system on the inflow vector  $\lambda \in \mathbb{R}_+^{\mathcal{R}}$ , we write  $\dot{\rho} = \Phi(\rho, \lambda) = f^{\text{in}}(\rho) - f^{\text{out}}(\rho)$  and we denote by  $\phi^t(\rho^\circ, \lambda)$  the solution to  $\dot{\rho} = \Phi(\rho, \lambda)$  with initial condition  $\rho(0) = \rho^\circ$ .

*Remark 1:* The function  $\Phi(\rho, \lambda)$  is Lipschitz in  $\rho$  due to the properties of demand and supply and is smooth in  $\lambda$ .

*Remark 2:* The set  $R = \prod_{e \in \mathcal{E}} [0, B_e]$  is positively invariant for the system, since for  $\rho_e = 0$  one has  $f_e^{\text{out}}(\rho) \leq d_e(\rho_e) = 0$ , while  $\rho_e = B_e$  implies  $f_e^{\text{in}}(\rho) \leq s_e(\rho_e) = 0$ . From now on, we consider thus only trajectories that entirely belong to  $R$ .

*Remark 3:* In [10], [11] the authors consider the CTM on a directed line and assume that if a link, namely, a cell in the line, possesses an off-ramp, then the flow towards the off-ramp is a fraction of the total outflow from the link. As such, if the cell immediately downstream is completely congested, then the flow on the ramp is also stopped. In this paper we assume instead that off-ramps are subject to the balance of supply and demand. Consequently, the outflow from a link can be nonzero even if some of the subsequent links are congested. This is consistent with the idea that drivers whose destination can be reached from the off-ramp will leave the network even if downstream links are congested.

*Remark 4:* Our model employs a proportional rule for merge that correspond to that proposed in [7], [8]. As it is straightforward to see, it corresponds to set, for all  $j \in \mathcal{E}_e^-$ ,  $f_{j \rightarrow e}(\rho) = \alpha(\rho)d_j(\rho_j)$  where  $\alpha(\rho) \in [0, 1]$  is the maximum value for which  $\sum_{j \in \mathcal{E}_e^-} f_{j \rightarrow e}(\rho) \leq s_e(\rho_e)$ .

Concerning diverge, previous models employ the following FIFO rule

$$f_{e \rightarrow j} = \alpha(\rho)R_{ej}d_e(\rho_e)$$

where  $\alpha(\rho) \in [0, 1]$  is the maximum value such that  $f_{e \rightarrow j} = \min \{ \alpha(\rho)R_{ej}d_e(\rho_e), s_j(\rho_j) \}$  for all  $j \in \mathcal{E}_e^+$ . As explicitly pointed out in [7], this corresponds to assume that if vehicles that want to turn into  $j$  are blocked by the scarce supply in  $j$ , then the whole flow out from  $e$  (and hence, also vehicles that want to turn into  $k \neq j$ ) is slowed down. In the present paper we simply assume that turnings are independent one each other. Notice that this implies that part of the vehicles that would like to turn into  $j$  stay in  $e$ , increasing the demand and thus also the flow towards  $k \neq j$ . This is consistent with the idea that vehicles stopped in a jam might choose different paths instead of blindly follow their preference.

The rest of this section is devoted to establish some important properties of the proposed model. First of all, we show that the maximum flow on an edge at equilibrium is given by its capacity, hence the name.

*Lemma 1:* Consider the dynamical flow network with fixed preference rates (1) and let  $\rho^*$  be an equilibrium for the system. Then  $f_e(\rho^*) = f_e^{\text{in}}(\rho^*) = f_e^{\text{out}}(\rho^*) \leq C_e$  for any  $e \in \mathcal{E} \setminus \mathcal{R}$ .

We study now the monotonicity of the system. A system of the type

$$\dot{\rho} = \Phi(\rho)$$

is said to be a *monotone*, or *cooperative*, system [9] if  $\rho(0) \preceq \tilde{\rho}(0)$  implies  $\phi^t(\rho(0)) \preceq \phi^t(\tilde{\rho}(0))$  for any  $t \geq 0$ , where  $x \preceq y$  means  $x_i \leq y_i$  for all  $i$  and  $\phi^t(\rho^\circ)$  is the solution to  $\dot{\rho} = \Phi(\rho)$  with initial condition  $\rho(0) = \rho^\circ$ . By Kamke's theorem [9, Theorem 1.2], [14], monotonicity is equivalent to the property that for almost all  $\rho$ ,

$$\frac{\partial \Phi_e(\rho)}{\partial \rho_k} \geq 0, \quad \forall e \neq k.$$

The following lemma establishes that the system under consideration is indeed monotone.

*Lemma 2:* The dynamical flow network with fixed preference rates (1) is a monotone system for every  $\lambda \in \mathbb{R}_+^{\mathcal{R}}$ .

Monotonicity gives the system a high degree of structure that can be used to study its stability properties. In particular, let us call nominal trajectory the evolution of the state with initial condition  $\rho(0) = \mathbf{0}$ . As an immediate consequence of the monotonicity properties of the system, we obtain  $\phi_e^t(\mathbf{0}, \lambda) \succeq \phi_e^s(\mathbf{0}, \lambda)$  for all  $0 \leq s \leq t$  and for all  $e \in \mathcal{E}$ . Consequently,  $\lim_{t \rightarrow \infty} \phi_e^t(\mathbf{0}, \lambda)$  exists for every  $\lambda$ , and we can adopt the notation  $\rho^*(\lambda) := \lim_{t \rightarrow \infty} \phi_e^t(\mathbf{0}, \lambda)$ . The following lemma shows that such a limit is always strictly smaller than the maximum allowed density on any non on-ramp link.

*Lemma 3:* Consider the dynamical flow network with fixed preference rates (1). Then  $\rho_e^*(\lambda) < B_e$  for all  $e \in \mathcal{E} \setminus \mathcal{R}$ .

Monotonicity also allows us to prove the following lemma, which gathers several useful properties of the system under analysis that will be used in the rest of the paper. The result can be proved employing an  $\ell_1$  contraction principle for monotone system with mass conservation that was proved in [13] and is stated for completeness in Appendix.

*Lemma 4:* Consider the dynamical flow network with fixed preference rates (1). Then

- i) if  $\rho_e^*(\lambda) < +\infty$  for all  $e \in \mathcal{R}$ , then it holds true  $\limsup_{t \rightarrow \infty} \|\phi^t(\rho^\circ, \lambda)\| < +\infty$  for any initial condition  $\rho^\circ \in R$ . If there exists  $e \in \mathcal{R}$  such that  $\rho_e^* = +\infty$ , then  $\lim_{t \rightarrow \infty} \phi_e^t(\rho^\circ, \lambda) = +\infty$  for any initial condition  $\rho^\circ \in R$ .
- ii)  $\rho^*(\lambda)$  is a monotone function of the inflow vector

$$\lambda \preceq \tilde{\lambda} \implies \rho^*(\lambda) \preceq \rho^*(\tilde{\lambda}).$$

- iii) Assume that  $\tilde{\rho}$  is a globally asymptotically stable equilibrium for (1) with inflow vector  $\tilde{\lambda} \succeq \lambda$ . Then  $\tilde{\rho} \succeq \rho^*(\lambda)$ .

Point i) of Lemma 4 states a dichotomy: Either the system is bounded for any initial condition, and the trajectory starting from zero initial condition converges to an equilibrium. Or the occupancy levels grow unbounded, for any initial condition. Consequently, the set

$$\Lambda := \{\lambda \in \mathbb{R}_+^{\mathcal{R}} : \max_{e \in \mathcal{E}} \rho_e^*(\lambda) < +\infty\},$$

is the largest set of inflow vectors for which the system (1) is stable. The rest of the paper aims at characterizing  $\Lambda$ . A first property is straightforward from point ii) of Lemma 4, which implies that as  $\lambda$  decreases in each component, the corresponding limit point of the nominal trajectory cannot increase. Therefore,  $\Lambda$  is a connected set that includes  $\mathbf{0}$ .

To state our main result, we first define  $\mathcal{B} \subset \Lambda$  to be the set of  $\lambda$  for which there exists  $k \in \mathcal{E} \setminus \mathcal{R}$  such that

$$\sum_{j \in \mathcal{E}_{\sigma_k}^-} R_{jk} d_j(\rho_j^*(\lambda)) = s_k(\rho_k^*(\lambda)).$$

For every  $\lambda \notin \mathcal{B}$ , we define the dual graph  $\mathcal{G}^d$  associated with  $\rho^*(\lambda)$  as follows: Let

$$J_\lambda = \nabla \Phi(\rho)|_{\rho=\rho^*(\lambda)}.$$

Then the dual graph  $\mathcal{G}^d = (\mathcal{V}^d, \mathcal{E}^d)$  has set of nodes  $\mathcal{V}^d = \mathcal{E}$ , and has an edge  $(e, j) \in \mathcal{E}^d$  if  $[J_\lambda]_{je} > 0$ .

Finally, we shall say that the dual graph  $\mathcal{G}^d$  is rooted if for every  $e \in \mathcal{V}^d$  there a directed path from  $e$  to an offramp  $j \in \mathcal{R}^o \subseteq \mathcal{V}^d$ .

The following theorem is the main result of this paper. To prove it, one uses Lemma 7 and the fact that  $\mathcal{G}^d$  is rooted to show that the system linearized around the equilibrium is stable, and then proves that stability is global by Lemma 6.

*Theorem 1:* Consider the dynamical flow network with fixed preference rates (1) with inflow vector  $\lambda \in \Lambda \setminus \mathcal{B}$ . Assume that in the dual graph  $\mathcal{G}^d$  is rooted. Then  $\rho^*(\lambda) = \lim_{t \rightarrow \infty} \phi^t(\mathbf{0}, \lambda)$  is a globally asymptotically stable equilibrium.

As illustrated in Figure 3, the set  $\Lambda$  is a connected set that includes  $\mathbf{0}$ , and is divided into regions by the surfaces described by  $\mathcal{B}$ . Whenever the inflow vector lies strictly inside one of the regions that do not compose  $\mathcal{B}$  (in shades of grey in Figure 3), the system admits a globally asymptotically stable equilibrium if the dual graph is rooted, and whenever the inflow vector lies strictly outside  $\Lambda$ , the system is unstable for any initial condition. Finally, the trajectory of  $\rho^*(\lambda)$  is not continuous in  $\lambda$ , but rather exhibits phase transitions when the inflow vector crosses one of the surfaces defined by  $\mathcal{B}$ .

*Remark 5:* The main theorem shows that when  $\lambda \notin \mathcal{B}$  and the corresponding limit point  $\rho^*(\lambda)$  is an equilibrium, then it is unique and globally asymptotically stable. We also know that  $\rho^*(\lambda)$  is a monotone function of  $\lambda$  itself by point (ii) of Lemma 4. This set of equilibria can exhibit phase transitions in  $\lambda$ . Indeed, consider two regions  $\Lambda_1$  and  $\Lambda_2$  that are separated by a surface in  $\mathcal{B}$ . Let  $\{\lambda_1^k\}_{k \in \mathbb{N}} \subseteq \Lambda_1$  and  $\{\lambda_2^k\}_{k \in \mathbb{N}} \subseteq \Lambda_2$  be two sequences of elements of  $\Lambda_1$  and  $\Lambda_2$  that converge to the same value  $\tilde{\lambda} \in \mathcal{B}$ , i.e.,

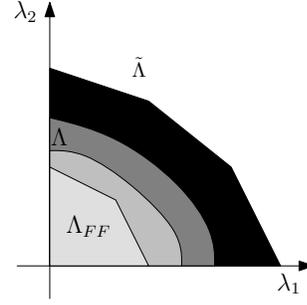


Fig. 3. Graphical illustration of the partition of the set  $\Lambda$  for a network with two on-ramps and  $\lambda = (\lambda_1, \lambda_2)$ . The smallest region  $\Lambda_{FF}$  for which the equilibrium admits a globally asymptotically stable equilibrium is called  $\Lambda_{FF}$  and is studied in Section III. Other stability regions are given in shades of grey. If  $\lambda$  lies strictly inside any of them and the corresponding dual graph is rooted, the system admits a globally asymptotically stable equilibrium. The black region  $\tilde{\Lambda}$  is an upper bound on  $\Lambda$ . If  $\lambda$  lies strictly outside  $\Lambda$ , then any trajectory grows unbounded, as shown in Remark 6.

$\lim_{k \rightarrow \infty} \lambda_1^k = \lim_{k \rightarrow \infty} \lambda_2^k = \tilde{\lambda}$ . Then it is possible that  $\lim_{k \rightarrow \infty} \rho^*(\lambda_1^k) = \rho_1^* \prec \rho_2^* = \lim_{k \rightarrow \infty} \rho^*(\lambda_2^k)$ , namely, the system exhibits a phase transition from  $\rho_1^*$  to  $\rho_2^*$ . Therefore, on a path of inflow vectors that starts from  $\lambda = \mathbf{0}$  and increases in every component, the system exhibits a sequence of globally asymptotically stable equilibria that is continuous except for a series of jumps, until it becomes a point at infinity for  $\lambda$  large enough.

*Remark 6:* For generic networks it is difficult to characterize the set  $\Lambda$  in an explicit way, because its boundaries depend on the shape of demands and supplies in the links of the network. A simple upper bound is given by the max-flow min-cut theorem. To this aim, call a subset  $\mathcal{U} \subseteq \mathcal{V}$  a cut, and define  $\lambda_{\mathcal{U}} = \sum_{r \in \mathcal{R}: \sigma_r \in \mathcal{U}} \lambda_r$  where the sum is on the ramps that stem from nodes in  $\mathcal{U}$ . Define the polytope  $\tilde{\Lambda} := \{\lambda \in \mathbb{R}_+^{\mathcal{R}} : \lambda_{\mathcal{U}} \leq \sum_{e \in \partial_{\mathcal{U}}^+} C_e, \forall \mathcal{U}\}$  - shown in black in Figure 3. Then a sufficient condition for instability of the system is  $\lambda \notin \tilde{\Lambda}$ . Indeed, if by contradiction  $\lambda_{\bar{\mathcal{U}}} > \sum_{e \in \partial_{\bar{\mathcal{U}}}^-} C_e$  on a cut  $\bar{\mathcal{U}}$  and  $\rho^*(\lambda)$  were an equilibrium, then

$$\lambda_{\bar{\mathcal{U}}} + \sum_{e \in \partial_{\bar{\mathcal{U}}}^-} f_e^{\text{out}}(\rho_e^*) = \sum_{e \in \partial_{\bar{\mathcal{U}}}^+} f_e^{\text{out}}(\rho^*) \Rightarrow \lambda_{\bar{\mathcal{U}}} \leq \sum_{e \in \partial_{\bar{\mathcal{U}}}^-} C_e.$$

where we used Lemma 1. This bound is in general not tight, as shown for example in Section IV. However, it also shows that for dynamical networks with supply and demand the classical static theory is insufficient to explain the complex interplay between the flows in the network.

A precise characterization of the equilibrium can be moreover offered for the freeflow region  $\Lambda_{FF}$ , which is the region that contains the origin, as shown in the next section.

### III. STABILITY IN THE FREEFLOW REGION

The fixed turning rates  $\{R_{ej}\}$ , and the inflows  $\lambda_e$  imply the existence of a unique equilibrium flow  $f^*$  that is the solution of the system of linear equations

$$R^T f^* + \lambda^a = 0 \quad (2)$$

where

$$[R]_{ej} = \begin{cases} R_{ej}, & \text{if } \tau_e = \sigma_j \\ -1, & \text{if } e = j \\ 0, & \text{otherwise} \end{cases}$$

and  $\lambda^a \in \mathbb{R}_+^{\mathcal{E}}$  is such that  $\lambda_e^a = \lambda_e$  if  $e \in \mathcal{R}$ , and  $\lambda_e^a = 0$  otherwise.

The matrix  $R$  is Metzler and under Assumption 1 it is invertible [2], so (2) has indeed a unique solution  $f^* = -R^{-T}\lambda^a$ . To this equilibrium flow we can associate a candidate equilibrium  $\rho^*$  such that  $\rho_e^* = d_e^{-1}(f_e^*)$  under the very mild assumption  $f_e^* < F_e$  for all  $e \in \mathcal{E}$ . We can now define the polytope, or freeflow region,

$$\Lambda_{FF} = \{\lambda \in \Lambda : f_e^* < C_e, \forall e \in \mathcal{E}, f^* = -R^{-T}\lambda^a\},$$

which is the set of vector inflows whose associated equilibrium flows  $f^*$ , solutions of (2), are component-wise strictly smaller than the capacities. The next proposition states that for this set of vector inflows the system admits a globally asymptotically stable equilibrium. As such,  $\rho^*$  is also the limit of the nominal trajectory. The next proposition is a particular case of Theorem 1, as it can be shown that the dual graph associated with  $\lambda \in \Lambda_{FF}$  is always rooted.

*Proposition 1:* Consider the dynamical flow network with fixed preference rates (1), let  $\lambda \in \Lambda_{FF}$ , and set  $\rho_e^* = d_e^{-1}(f_e^*)$  for all  $e \in \mathcal{E}$ . Then  $\rho^*$  is a globally asymptotically stable equilibrium.

*Remark 7:* When the network consists in a direct chain of links with one origin and one destination, the result recovers contributions already appeared in [10], [11]. However, not only our result is concerned with global stability of a generic, possibly multi-origin multi-destination and cyclic network. More importantly, in [10], [11] the freeflow equilibrium is the unique possible globally asymptotically stable equilibrium. In a network, instead, more complex behaviors arise.

#### IV. A NUMERICAL EXAMPLE

We consider the cyclic network with  $N = 10$  links, two on-ramps and two off-ramps illustrated in Figure 4. We consider on all links  $e \in \mathcal{E}$  demand and supply functions given by  $d_e(\rho_e) = \min\{F_e, \rho_e\}$ , where  $F_e$  is a large value and  $s_e(\rho_e) = \max\{2C_e - \rho_e, 0\}$ . With this choice  $C_e$  is indeed the capacity on each link. We set  $C_e = 2$  for  $e \neq 6, 9$ , and  $C_6 = C_9 = 0.5$ . We set  $\lambda_1 = 0.5$  and we let  $\lambda_2$  vary. We also set  $R_{34} = R_{38} = 0.5$ , and  $R_{45} = .75$  and  $R_{410} = .25$ .

As illustrated in Figure 5, for  $\lambda_2 \in [0, 1/8) \cup (1/8, 1/6) \cup (1/6, 4/5)$  the vector  $(\lambda_1, \lambda_2) = (0.5, \lambda_2)$  is not in  $\mathcal{B}$  and the system admits a unique globally asymptotically stable equilibrium, that in particular is a freeflow equilibrium for  $\lambda_2 \in [0, 1/8)$ . Instead, the values  $(0.5, \lambda_2)$  for  $\lambda_2 = 1/8$  and  $\lambda_2 = 1/6$  lie on two surfaces defined by  $\mathcal{B}$ . When they are crossed the equilibrium exhibits jumps in the component  $\rho_8^*(\lambda)$  and  $\rho_5^*(\lambda)$  of the equilibria, for  $\lambda_2 = 1/8$  and  $\lambda_2 = 1/6$ , respectively. More interestingly, when  $\lambda_2 = 1/8$ ,  $\rho_e^*(\lambda)$  is uniquely determined for  $e \neq 8$ , but there is a whole segment in which  $\rho_8^*(\lambda)$  can take values (represented in thick black in Figure 5). In other terms, equilibria manifold

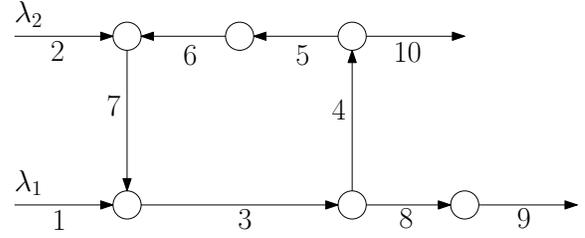


Fig. 4. The network used in the simulations.

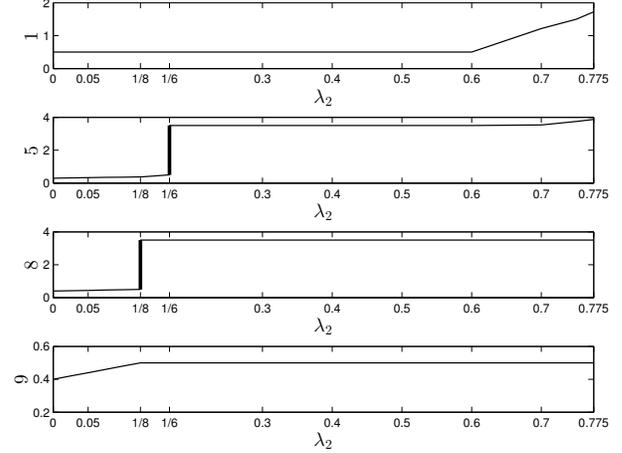


Fig. 5. The equilibria for the simulated system for  $\lambda_2 \in [0, 0.775]$  and  $\lambda_1 = 0.5$ . In thick line, for  $\lambda_2 = 1/8$  and  $\lambda_2 = 1/6$ , the equilibria interval that appear on links 8 and 5, respectively. For any other  $\lambda_2$ , the equilibrium in the system is unique and globally asymptotically stable. As  $\lambda_2 \rightarrow 0.8$ , the component  $\rho_5^*(\lambda)$  increases and when  $\lambda_2 = 0.8$  (not shown) the system becomes unstable.

appear, similarly to what is shown in [10]. Analogously, for  $\lambda_2 = 1/6$  the component  $\rho_5^*(\lambda)$  of the equilibria is allowed to take values in a segment. Finally, when  $\lambda_2 = 0.8$  any trajectory grows unbounded in the second component. It can be noticed that this corresponds to an equilibrium condition that would require link 5 to get completely congested and have zero supply  $s_5 = 0$ . Obviously, the trajectory also grows unbounded for any  $\lambda_2 > 0.8$ . This situation is illustrated in Figure 5.

#### V. CONCLUSIONS

This paper studies a macroscopic traffic model based on dynamical flow networks driven by mass-conservation laws. We extend the Cell Transmission Model to the network setting in such a way that the system is monotone. This allows us to characterize the stability properties and the structure of the equilibria of the system as a function of the inflow vectors in the network. Future research directions include the generalization of these results beyond the demand and supply setting and distributed optimal control for traffic networks.

#### APPENDIX

The next result is a simple adaptation of the  $\ell_1$  contraction principle for monotone dynamical systems with mass conservation that is proven in [13].

*Lemma 5:* Let  $\Phi : \mathbb{R}_+^m \rightarrow \mathbb{R}^m$  be a Lipschitz map such that

$$\frac{\partial}{\partial x_j} \Phi_i(x) \geq 0, \quad \forall i \neq j \in \{1, \dots, m\} \quad (3)$$

and that

$$\sum_{1 \leq i \leq m} \frac{\partial}{\partial x_j} \Phi_i(x) \leq 0, \quad \forall j \in \{1, \dots, m\} \quad (4)$$

for every  $x \in \mathbb{R}_+^m$ . Then

$$\sum_{1 \leq i \leq n} \text{sgn}(x_i - y_i) (\Phi_i(x) - \Phi_i(y)) \leq 0, \quad \forall x, y \in \mathbb{R}_+^m. \quad (5)$$

The previous lemma has the following result as a corollary.

*Lemma 6:* Let  $\dot{x} = f(x)$  be a monotone system for which (4) holds true, and let  $x^*$  be an equilibrium for the system. Then  $x^*$  is locally asymptotically stable if and only if it is globally asymptotically stable.

*Proof:* Sufficiency is obvious. For necessity, assume  $x^*$  is locally asymptotically stable. Then there exists a  $\mathcal{KL}$  function  $\beta(\cdot, \cdot)$  such that  $\|\phi^t(x) - x^*\| \leq \beta(x - x^*, t)$  for all  $x \in \mathcal{B}_\varepsilon(x^*)$ , a sufficiently small closed ball around the equilibrium in the  $\ell_1$  topology [15, Lemma 4.5], and so if  $x^\circ \in \mathcal{B}_\varepsilon(x^*)$  then  $\phi^t(x) \rightarrow x^*$ . Therefore, to prove global stability we need to show that for any  $x^\circ \notin \mathcal{B}_\varepsilon(x^*)$ , i.e.,  $\|x^\circ - x^*\|_1 > \varepsilon$ , there exists a finite time  $T \geq 0$  such that  $\phi^T(x^\circ) \in \mathcal{B}_\varepsilon(x^*)$ . Let  $\tilde{x} = x^* + \frac{\varepsilon}{\|x^\circ - x^*\|_1} (x^\circ - x^*)$ , for which it is easily seen that  $\|\tilde{x} - x^*\|_1 = \varepsilon$ , i.e.,  $\tilde{x} \in \mathcal{B}_\varepsilon(x^*)$ , and  $\|x^\circ - x^*\|_1 = \|x^\circ - \tilde{x}\|_1 + \|\tilde{x} - x^*\|_1 = \|x^\circ - \tilde{x}\|_1 + \varepsilon$ , and consider the trajectories of the system starting from  $x^\circ$  and  $\tilde{x}$ . By Lemma 5,  $\frac{d}{dt} \|\phi^t(x^\circ) - \phi^t(\tilde{x})\|_1 \leq 0$ , namely  $\|\phi^t(x^\circ) - \phi^t(\tilde{x})\|_1 \leq \|x^\circ - \tilde{x}\|_1$ . By the triangle inequality,

$$\begin{aligned} \|\phi^t(x^\circ) - x^*\|_1 &\leq \|\phi^t(x^\circ) - \phi^t(\tilde{x})\|_1 + \|\phi^t(\tilde{x}) - x^*\|_1 \\ &= \|x^\circ - \tilde{x}\|_1 + \|\phi^t(\tilde{x}) - x^*\|_1 \\ &= \|x^\circ - x^*\|_1 - \varepsilon + \|\phi^t(\tilde{x}) - x^*\|_1. \end{aligned}$$

Due to the properties of the  $\mathcal{KL}$  functions, there exists  $T_{\frac{\varepsilon}{2}} \geq 0$  such that  $\beta(x - y, t) \leq \frac{\varepsilon}{2}$  for all  $y \in \mathcal{B}_\varepsilon(x^*)$  and for all  $t \geq T_{\frac{\varepsilon}{2}}$ . Thus, we have

$$\begin{aligned} \|\phi^t(x^\circ) - x^*\|_1 &\leq \|x^\circ - x^*\|_1 - \varepsilon + \|\phi^t(\tilde{x}) - x^*\|_1 \\ &\leq \|x^\circ - x^*\|_1 - \frac{\varepsilon}{2} \end{aligned}$$

for all  $t \geq T_{\frac{\varepsilon}{2}}$ . If  $\phi^{T_{\frac{\varepsilon}{2}}}(x^\circ) \in \mathcal{B}_\varepsilon(x^*)$ , the proof is complete with  $T = T_{\frac{\varepsilon}{2}}$ . Otherwise, the same argument can be reiterated. Since each step the  $\ell_1$  distance between  $\phi^t(x)$  and  $x^*$  decreases by at least  $\frac{\varepsilon}{2} > 0$ , in no more than  $\lceil \frac{2\|x^\circ - x^*\|_1}{\varepsilon} \rceil$  steps, i.e., for  $T \leq \lceil \frac{2\|x^\circ - x^*\|_1}{\varepsilon} \rceil T_{\frac{\varepsilon}{2}}$ , it holds  $\phi^T(x^\circ) \in \mathcal{B}_\varepsilon(x^*)$ . ■

*Lemma 7:* Let  $\mathcal{G} = (\mathcal{V}, \mathcal{E})$  be a graph and  $J \in \mathbb{R}^{\mathcal{V} \times \mathcal{V}}$  be a weighted sublaplacian of  $\mathcal{G}$ . Then all the eigenvalues of  $J$  have negative real part except possibly eigenvalues in 0. Moreover, if  $\mathcal{S}$  is the set of  $v$  for which  $\sum_u J_{uv} < 0$ , then  $J$  is stable if for every  $u$  there exists a directed path in  $\mathcal{G}$  from  $u$  to a node  $v \in \mathcal{S}$ .

*Proof:* The first claim is straightforward by Gershgorin circle theorem. Concerning the second, let  $\alpha \in (0, 1)$  such that  $\bar{J} = \alpha J$  has all diagonal elements in absolute value strictly smaller than 1, and construct the matrix  $P = \begin{bmatrix} \bar{J}' & -\bar{J}' \mathbf{1} \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{|\mathcal{V}|+1 \times |\mathcal{V}|+1}$ . Clearly  $P$  is a stochastic matrix and if  $\mathcal{G}_P$  is the graph associated with  $P$  then it is an augmented version of  $\mathcal{G}$  in which all the edges that are edges of  $\mathcal{G}$  have reversed direction, and there is an additional node that is directly connected to all the nodes in  $\mathcal{S}$ . The graph theoretical condition in the statement ensures then that  $\mathcal{G}_P$  admits a spanning tree rooted at the additional node. Then by [16]  $P$  has a single eigenvalue in 1 and all the other eigenvalues strictly inside the unit circle. By construction of  $P$ , therefore, all eigenvalues of  $J$  have negative real part. ■

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