Ph.D. course on Network Dynamics

Homework 8

Due on Tuesday, November 26, 2013

Exercise 1 (Resilience of the configuration model with power-law degree distribution)**.** *Consider the configuration model with power-law degree distribution* $p_k = K_\beta k^{-\beta}$*, where* $\beta > 2$*, and* $K_\beta := (\sum_{k \geq 1} k^{-\beta})^{-1}$ *and finite size* n*. Now consider a random node removal process, where every node is removed with probability* $1 - \theta$ *and kept with probability* θ*, where* $\theta \in (0, 1)$ *is some constant. Define the following measure of 'fragility' of the network: let* θ^* *be the infimum value of* $\theta \in (0, 1)$ *such that a random node (selected with uniform probability) belongs to a connected component of size of order* n^α*, for some* $\alpha > 0$*, with high probability as* n *grows large.*

(a) *Show that* $\theta^*(\beta) = 0$ *for* $\beta \in (2, 3]$*, while, for* $\beta > 3$*,* $\theta^*(\beta) > 0$ *is an increasing function of* β*.*

(Hint: find the right branching process approximation, and use the fact that the graph looks like a tree in the neighborhood of a random node.)

Now, consider the same configuration model with a different model of node removal: assume that, for some $k_0 > 1$*, all the nodes of degree* $k > k_0$ *are removed, along with their incident links.*

(b) *Show that this is equivalent to removing links with probability* $q = \mu^{-1} \sum_{k > k_0} k p_k$*, where* $\mu := \sum_{k > 0} k p_k$*.*

Define $k_0^*(\beta)$ *as the smallest* k_0 *such that such that a random node (selected with uniform probability) belongs to a connected component of size of order* n^α*, for some* $\alpha > 0$*, with high probability as* n *grows large.*

(c) *Show that* $k_0^*(\beta) = \min\left\{k_0 : \sum_{1 \leq k \leq k_0} k(k-1) p_k > \mu\right\}$*.*

(Hint: find the right branching process approximation, and use the fact that the graph looks like a tree in the neighborhood of a random node.)

One interpretation of result (a) is that networks with a power law distribution with $\beta \leq 3$ (such as the Internet) are 'robust to random failures', while (c) is interpreted as saying that such networks are 'fragile to malicious attacks' (where the malicious attacker is supposed to target the most 'central' nodes of the network). See [1, 2]. Such results, and especially their interpretation, have generated also lots of critiques, see, e.g., [3, 4].

Exercise 2 (Completing the proof for the preferential attachment model)**.** *Consider the preferential attachment model described in class, where we start with a graph $\mathcal{G}_1 = (\mathcal{V}_1, \mathcal{E}_1)$ consisting of two nodes connected by a link, and at each time $t \geq 2$ we generate a graph $\mathcal{G}_t = (\mathcal{V}_t, \mathcal{E}_t)$ by adding a new node to $\mathcal{V}_{t_1}$ and add a link between such new node and another one chosen at random from $\mathcal{V}_{t-1}$with probability proportional to its degree in $\mathcal{G}_{t-1}$. Let $N_k(t)$ be the number of nodes of degree k in $\mathcal{G}_t$, and $n_k(t) := \mathbb{E}[N_k(t)]$ be its expected value. In class, we have shown that $n_k(t)/t \overset{t\to\infty}{\longrightarrow} 4/(k(k+1)(k+2))$ for all $k \geq 1$. In this exercise, we want to show that $N_k(t)$ and $n_k(t)$ are close to each other. We will make use of the following*

Hoeffding-Azuma Inequality *Let $M_0, M_1, \ldots$ be a martingale such that $\mathbb{P}(|M_i - M_{i-1}| \geq c_i) = 1$ for all $i \geq 1$. Then, for all $\varepsilon > 0$, and $t = 1, 2, \ldots$,*

$$\mathbb{P}\left(M_t - M_0 \geq \varepsilon\right) \leq \exp\left(-\varepsilon^2 / \left(2 \sum_{i=1}^{t} c_i^2\right)\right) .$$

Fix $k \geq 1$ and $t \geq 1$. For all $s = 1, \ldots, t$, define $M_s := \mathbb{E}[N_k(t)|\mathcal{G}_s]$.

(a) *Show that M_s, $s = 1, \ldots, t$, is a martingale;*

(b) *Show that $\mathbb{P}(|M_{s+1} - M_s| \leq 2) = 1$;*

(c) *Use rthe Hoeffding Azuma inequality to prove that, for all $t \geq 1$,*

$$\mathbb{P}\left(\left|\frac{N_k(t)}{t} - \frac{n_k(t)}{t}\right| \geq \sqrt{\frac{\log t}{t}}\right) \leq 2t^{-1/8} .$$

Hint: take $\varepsilon = \sqrt{t \log t}$.

Exercise 3. *The following model does not involve a graph, but can be studied using the same mean-field method as for the Albert-Barabasi preferential attachment model.*

A famous surrealist author is known to compose text as follows. She starts with a random word. Suppose that t words (not necessarily different) have already been written. The next word is chosen as follows:

- with probability α, it is a new word; - with probability $1-\alpha$, she chooses some j uniformly at random from the set of past instants $\{1, \ldots, t-1\}$ and copies the j-th word that she has alerady written. Let $n_i(t)$ be the expected number of distinct words that appear exactly i times, after the first t words have been written.

1. *Write down a recursion (in t) for the variables $n_i(t)$.*

2. *Assume (or, better, prove) that $n_i(t)/t$ converges to some $\beta_i \geq 0$ for all $i \geq 1$. Find equations that relate the β_i.*

3. *Show that β_i/β_{i+1} converges to 1 as i grows large, and that the β_i's correspond to a power law.*

References

[1] R. Albert, H. Jeong, and A.L. Barabasi, *Error and attack tolerance of complex networks*, Nature **406** (2000), 378–382.

[2] D.S. Callaway, M.E.J. Newman, S.H. Strogatz, and D.J. Watts, *Network robustness and fragility: Percolation on random graphs*, Physical Review Letters **85** (2000), 5468–5471.

[3] J.C. Doyle, D.L. Alderson, L. Li, S. Low, M. Roughan, S. Shalunov, R. Tanaka, and W. Willinger, *The "robust yet fragile" nature of the internet*, Proceedings of the National Academy of Science **102** (2005), no. 41, 14497–14502.

[4] W. Willinger, D. Alderson, and J.C. Doyle, *Mathematics and the internet: A source of enormous confusion and great potential*, Notices of the AMS **56** (2009), no. 5, 586–599.