Ph.D. course on Network Dynamics

Homework 7

Due on Tuesday, November 19, 2013

Exercise 1 (Supercritical branching process)**.** *Consider a branching process Z_t with offspring distribution $p_k := \mathbb{P}(\xi_t^i = k)$, $k \geq 0$. Let $\mu := \sum_k k p_k$ be the expected number of offsprings and $\Phi(x) := \sum_k p_k x^k$ its generating function. Assume that $\mu > 1$, and $p_0 > 0$, so that that the extinction probability ρ_{ext} is the unique solution in $(0, 1)$ of $x = \Phi(x)$. Prove that*

(a) *the process conditioned on extinction, $\tilde{Z}_t$, is a branching process with offspring distribution having generating function*

$$\tilde{\Phi}(y) = \frac{\Phi(\rho_{ext} x)}{\rho_{ext}} ;$$

(hint: if $\tilde{\xi}_1^1$ is the number of first generation offsprings with a finite line of descent, then $\mathbb{P}(\tilde{\xi}_1^1 = k, ext) = p_k \rho_{ext}^k$, for $k \geq 0$)

(b) *show that, conditioned on survival, if one looks only at individuals that have an infinite line of descent, then one obtains a new branching process $\tilde{Z}_t$ with offspring distribution having generating function*

$$\tilde{\Phi}(y) = \frac{\Phi((1-\rho_{ext})y + \rho_{ext}) - \rho_{ext}}{1 - \rho_{ext}} .$$

(hint: if $\tilde{\xi}_1^1$ is the number of first generation offsprings with an infinite line of descent, then $\mathbb{P}(\tilde{\xi}_1^1 = k) = \sum_{j \geq k} p_j \binom{j}{k} (1-\rho_{ext})^k \rho_{ext}^{j-k}$, for $k \geq 1$)

Exercise 2 (Short cycles in Erdös-Rényi random graph)**.** *Consider the Erdös-Rényi ramdon graph $\mathcal{G}(n, p)$. Fix a node $v \in \{1, \ldots, n\}$. For $k \geq 3$, let $N_k(v)$ be the number of cycles of length k passing through node v in $\mathcal{G}(n, p)$.*

(a) *Prove that*

$$\mathbb{E}[N_k(v)] = \frac{1}{2}(n-1)(n-2)\dots(n-k+1)p^k\,;$$

(Hint: show that the possible cycles containing v *are* $(n-1)(n-2)\dots(n-k+1)/2$*, since one has to choose* $k-1$ *out of* $n-1$ *other nodes (beyond* v*) ...)*

(b) *Using Markov's inequality, prove that*

$$\mathbb{P}(\exists \textit{cycle of length } \leq k \textit{ containing } v) \leq \begin{cases} \dfrac{1}{n}\dfrac{\lambda^3}{2}\dfrac{\lambda^{k-2}-1}{(\lambda-1)} & \textit{if } \lambda \neq 1 \\ \dfrac{1}{n}\dfrac{k-2}{2} & \textit{if } \lambda = 1 \end{cases}$$

Conclude that:

(c) *if* $\lambda < 1$*, then*

$$\mathbb{P}(\exists \textit{cycle containing } v) \leq \frac{\lambda^3 n^{-1}}{2(1-\lambda)} \overset{n\to+\infty}{\longrightarrow} 0\,;$$

(d) *if* $\lambda > 1$*, then*

$$\mathbb{P}(\exists \textit{cycle of length } \leq a\log n \textit{ containing } v) \leq \frac{\lambda n^{a\log\lambda - 1}}{2(\lambda-1)} \overset{n\to+\infty}{\longrightarrow} 0\,,$$

for all $a < 1/\log\lambda$*.*

Observe that the above does not mean that there are no cycles of short length in the Erdos-Renyi random graph. In fact, for every given $k \geq 3$*, let* M_k *be the total number of cycles of length* k *in* $\mathcal{G}(n,\lambda/n)$*, and*

(e) *Prove that*

$$\lim_{n\to+\infty} \mathbb{E}[M_k] = \frac{1}{2k}\lambda^k\,.$$

In fact, one can prove that M_k *converges in distribution to a Poisson random variable, i.e.,*

$$\lim_{n\to\infty} \mathbb{P}(M_k = j) = e^{-\gamma}\frac{\gamma^j}{j!}\,, \qquad \forall j \geq 0\,,$$

where $\gamma := \frac{1}{2k}\lambda^k$*. ([1, Theorem 4.1])*

The following exercises will refer to the configuration model and its two-phase branching process approximation. Let us recall a bit of notation. We start with a (node-perspective) degree distribution $\{p_k : k \geq 0\}$, and let $\mu := \sum_k k p_k$ be its first moment, which will always be assumed finite. We also let

$$q_k := \frac{1}{\mu}(k+1)p_{k+1}\,, \quad k \geq 0\,, \qquad \nu := \sum_k q_k k = \frac{1}{\mu}\sum_k p_k k(k-1)\,,$$

be the link-perspective degree distribution and its first moment.

Exercise 3 (Short cycles in the configuration model)**.** *For $k \geq 3$ and even $n \geq 2$, let $\mathcal{G}_n$ be the random (hyper-)graph generated by the configuration model with fixed degree $k \geq 2$. Fix an arbitrary node v.*

(a) *Prove that*

$$\mathbb{P}(v \textit{ has a self-loop}) \leq \frac{k(k-1)}{kn-1}$$

(hint: compute the probability that any of the k half-links stemming from k is connected to some other half-link stemming from v)

Now, for $l \geq 1$, let E_l be the event that $\mathcal{G}_n$ does not contain cycles of length less than or equal to $2l$ passing through node v.

(b) *Show that, conditioned on E_l, v has exactly $k(k-1)^{l-1}$ nodes at distance l, and exactly $1 + k\frac{(k-1)^l - 1}{k-2}$ nodes at distance less than or equal to l.*

(c) *Prove that*

$$\mathbb{P}(E_{l+1}|E_l) \geq 1 - k^2(k+1)\frac{(k-1)^{2l}}{n - (k-1)^{l+1}/(k-2)}$$

(hint: proceed by matching the $k(k-1)^l$ half-links connecting the nodes at distance l to nodes at distance $\geq l$; show that, independently from how the first $j-1$ half-links are matched, the probability that the j-th half-link is matched in such a way to unveil a cycle of length $2l+1$ or $2l+2$ is not more than $2k^2(k-1)^l/(nk - 2k(k-1)^l(k-1)/(k-2))$, where the numerator is an upper bound on the number of potential matches that create cycles of length $2l+1$ or $2l+2$, and the denominator is a lower bound on the total number of potential matches)

(d) *Use (c) to show that*

$$\lim_{n\to\infty} \mathbb{P}\left(E_{\alpha \log n}\right) = 0\,, \qquad \forall \alpha \in [0, 1/\log(k-1))\,,$$

i.e., for every $\alpha < 1/(\log(k-1))$ the $(\alpha \log n)$-distance neighborhood of node v is tree-like with high probability as n grows large.

Exercise 4 (Link-oriented versions of power laws and Poisson distributions)**.** *Let $p_0 = 0$ and, for $k \geq 1$, $p_k = C_\beta k^{-\beta}$ with $\beta > 1$, and $C_\beta := \left(\sum_{k\geq 1} k^{-\beta}\right)^{-1}$.*

(a) *prove that μ is finite if and only if $\beta > 2$, while ν is finite if and only if $\beta > 3$;*

Let $\lambda > 0$, and $p_k = e^{-\lambda}\lambda^k/k!$, for $k \geq 0$;

(b) *prove that $q_k = p_k$, for $k \geq 0$.*

Conversely, assume that $q_k = p_k$, for $k \geq 0$, and

(c) *prove that $p_k = e^{-\lambda}\lambda^k/k!$, for $k \geq 0$, where $\lambda = \sum_k kp_k = \mu$.*

I.e., the Poisson distribution is the unique distribution which is the same from node and link perspective.

Exercise 5 (SIR epidemics on the configuration model)**.** *Let $\{p_k\}_{k\geq 0}$ be a degree distribution with finite first moment $\mu := \sum_k p_k k$. Let $\mathcal{T}$ be a random infinite tree generated by the branching process with offspring distribution $\{p_k\}$ Consider the SIR epidemics with constant recovery time $\tau = 1$, and rate-γ Poisson infections on every link, on $\mathcal{T}$. Assume that, at time 0, the root node is infected and every other node is susceptible. Also, assume that randomness of the graph generation and of the infection process are independent.*

(a) Find the largest $\gamma > 0$ for which, with probability 1, the system will end up having a finite number of recovered nodes.

Now, let $\mathcal{G}_n$ be the random graph with $n \geq 1$ nodes, generated by the configuration model with degree distribution $\{p_k\}$. Consider the SIR epidemics as above on $\mathcal{G}_n$, assuming that, at time 0, one node v is infected and every other node is susceptible.

(b) *Find the largest* $\gamma > 0$ *for which the final number of recovered nodes* R_∞ *is bounded with high probability as* n *grows large, i.e.,*

$$\lim_{m\to\infty}\lim_{n\to\infty}\mathbb{P}(R_\infty \le m) = 0$$

Hint: use the two-phase branching process approximation to conjecture the result, then assume the generalization of Exercise 3 holds true for the configuration model with degree distribution $\{p_k\}$ holds true, in order to make your argument rigorous.

References

[1] B. Bollobás, *Random graphs*, Cambridge University Press, 2001.