Ph.D. course on Network Dynamics

Homework 6

Due on Tuesday, November 12, 2013

Exercise 1 (SIR epidemics on infinite regular tree)**.** *Consider an infinite k-regular tree, where $k \geq 2$. Consider the SIR epidemics with constant recovery time $\tau = 1$, and infection rate γ on every link. Assume that, at time 0, the root is infected and every other node is susceptible.*

(a) Find the largest $\gamma > 0$ for which, with probability 1, the system will end up having a finite number of recovered nodes.

(b) Answer point (a) in the case when the recovery time is random with exponential distribution of rate 1.

(c) Answer point (a) in the case when the recovery time is random with arbitrary probability distribution and expected value 1.

Exercise 2 (Mean-field limit for noisy best response in congestion games)**.** *Consider the following stylized model of traffic congestion. There is a finite set of routes $\mathcal{R} := \{1, 2, \ldots, k\}$ connecting one city (the origin) to another (the destination). There is a population of commuters: each commuter uses one route. The delay incurred by each commuter who is using route $i \in \mathcal{R}$ is given by $d_i(\rho_i^n)$, where ρ_i is the fraction of commuters using road i, and*

$$d_i : [0, 1] \to (0, +\infty)$$

is a differentiable increasing function. A Wardrop equilibrium *is a probability vector $\rho^* \in \mathcal{P}(\mathcal{R})$ such that, for every $i \in \mathcal{R}$, one has that*

$$\rho_i^* > 0 \qquad \Longrightarrow \qquad d_i(\rho_i^*) \leq d_j(\rho_j^*)\,, \quad \forall j \in \mathcal{R}\,.$$

(i.e., the delay on every route used by a nonzero fraction of commuters is not higher than the delay on any other route: the interpretation is that, in such a situation, no commuter has an incentive to modify his/her route choice.) *Define*

$$\Psi(\rho) := \sum_{i\in\mathcal{R}} \int_0^{\rho_i} d_i(s)\mathrm{d}s\,, \qquad \rho \in \mathcal{P}(\mathcal{R})\,. \tag{1}$$

(a) *Show that $\Psi(\rho)$ is strictly convex on $\mathcal{P}(\mathcal{R})$.*

(b) *Show that ρ^* is a minimum of $\Psi(\rho)$ on $\mathcal{P}(\mathcal{R})$ if and only if ρ^* is a Wardrop equilibrium.* (hint: you are looking at the minimization of a convex function with linear constraints)

(c) *Use (a) and (b) to conclude that there exists a unique Wardrop equilibrium $\rho^* \in \mathcal{P}(\mathcal{R})$.*

Now, assume that the population has finite size n, and consider the following dynamics. Commuters get activated at the thinking of independent rate-1 Poisson clocks. If commuter j is activated at time t, he/she gets the possibility of choosing a new route and switches to route i with probability $g_i(\rho^n(t))$, where $\rho^n(t) \in \mathcal{P}(\mathcal{X})$ is the vector of the fractions of commuters currently using the different routes and

$$g_i(\rho) := \frac{\exp(-\beta d_i(\rho_i^n))}{\sum_{r\in\mathcal{R}} \exp(-\beta d_r(\rho_r^n))}\,, \qquad r \in \mathcal{R}\,,$$

where $\beta > 0$ is a parameter whose inverse $1/\beta$ is a measure of noise. Let $\rho(t)$ be the solution of the Cauchy problem associated to the ODE

$$\dot{\rho} = g(\rho) - \rho\,, \tag{2}$$

with a given initial condition $\rho(0) \in \mathcal{P}(\mathcal{R})$.

(d) *Prove that, if $\rho^n(0) \to \rho(0)$ as n grows large, then $\rho^n(t)$ converges to $\rho(t)$ in probability, on every finite time interval $[0, T]$.*

Now, we focus on the behavior of the solution of (2) itself. Define

$$V_\beta(\rho) := \Psi(\rho) - \frac{1}{\beta}\,\mathrm{H}(\rho)\,,$$

where $\Psi(\rho)$ is as in (1) and $\mathrm{H}(\rho) := -\sum_i \rho_i \log \rho_i$ is the entropy of ρ.[1]

[1]Here, we adopt the standard convention that $0 \log 0 = 0$.

(e) *Show that the entropy function is strictly concave, so that* $V_\beta(\rho)$ *is strictly convex.*

Let ρ^β *be the unique minimum of* $V_\beta(\rho)$ *on* $\mathcal{P}(\mathcal{R})$*.*

(f) *Show that* $\rho^\beta \to \rho^*$ *as* $\beta \to \infty$*.*

(g) *Prove that* $\rho_i^\beta > 0$ *for all* $i \in \mathcal{R}$*.*

(h) *Use (g) to prove that* $g(\rho^\beta) = \rho^\beta$*, i.e.,* ρ^β *is an equilibrium for* (2)*.*

(i) *Prove that, for every initial condition* $\rho(0) \in \mathcal{P}(\mathcal{R})$*, the solution of the Cauchy problem associated to (2) satisfies*

$$\frac{\mathrm{d}}{\mathrm{d}t} V_\beta(\rho(t)) \leq 0\,,$$

with equality of and only if $\rho(t) = \rho^\beta$*.* (Hint: you may find it useful that

$$(\nabla \mathrm{H}(\rho') - \nabla \mathrm{H}(\rho))(\rho' - \rho) < 0\,, \qquad \forall \rho \neq \rho' \in \mathcal{P}(\mathcal{R})\,,$$

because of the strict concavity of $\mathrm{H}(\rho)$) *Conclude that* ρ^β *is a globally attractive equilibrium for the dynamical system (2) on* $\mathcal{P}(\mathcal{R})$*.*

Now, you can go back to the stochastic finite-population process $\rho^n(t)$*. Note that* $\rho^n(t)$ *has a unique invariant probability measure* μ_n *over* $\mathcal{P}_n(\mathcal{R}) \subseteq \mathcal{P}(\mathcal{R})$*.*

(l) *Prove that* μ_n *concentrates on* ρ^β *as* $\beta \to \infty$*, i.e.,*

$$\lim_{n\to\infty} \mu_n(O) = 1\,,$$

for every open subset $O \subseteq \mathcal{P}(\mathcal{R})$ *such that* $\rho^\beta \in O$*.*

Exercise 3. *Consider the following heterogeneous voter model. There are two populations consisting of finite size* n_1 *and* n_2*, respectively. Individuals in both population get activated at the thinking of independent rate-*1 *Poisson clocks. If an individual from population* $i \in \{1, 2\}$ *gets activated, with probability* $q_i \in (0, 1)$ *he/she selects another individual uniformly at random from the union of the two populations and takes his/her opinion, while with probability* $1 - q_i$ *he/she keeps his/her opinion.*

(a) *Propose a meaningful mean-field approach to this model, find the system of differential equations describing the limit when* $n = n_1 + n_2 \to +\infty$ *under the assumption that* $n_1/n \to \nu \in (0, 1)$.

(b) *Study the asymptotic behavior of the system of differential equations found in item (a) as a function of the four parameters* ν, q_1, *and* q_2.