Ph.D. course on Network Dynamics

Homework 5

To be discussed on Tuesday, November 5, 2013

Recap In class, we have considered network dynamics of the following form. Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, a finite state space $\mathcal{X}$, and local transition kernels $\Psi_v(x'|x_v, x_{\mathcal{N}_v})$ for $v \in \mathcal{V}$, we consider a continuous-time Markov chain $X(t) = \{X_v(t) : v \in \mathcal{V}\}$ on $\mathcal{X}^{\mathcal{V}}$, whereby every $v \in \mathcal{V}$ gets activated at the ticking of an independent rate-1 Poisson clock, and, if activated at some time $t \geq 0$, it updates its state to $X_v(t^+) = j \in \mathcal{X}$ with conditional probability $\Psi_v(j|X_v(t), X_{\mathcal{N}_v}(t))$.

Let $\mathcal{P}(\mathcal{X})$ be the simplex of probability vectors over $\mathcal{X}$. For a length-k vector x with components in $\mathcal{X}$, define the type of x as

$$\theta(x) \in \mathcal{P}(\mathcal{X}), \qquad \theta_i(x) = \frac{1}{k} \left|\{l = 1, \ldots, k : x_l = i\}\right| .$$

We have argued that, if the population is totally mixed (i.e., $\mathcal{G}$ is complete with n nodes), and if the transition kernels are homogeneous, i.e., if

$$\Psi_v(j|i, x_{\mathcal{N}_v}) = P_{ij}(\theta(x_{\{v\}\cup\mathcal{N}_v})), \qquad \forall v \in \mathcal{V}$$

for some Lipschitz-continuous functions $P_{ij} : \mathcal{P}(\mathcal{X}) \to \mathbb{R}$ (adding or not v to $\mathcal{N}_v$ is rather arbitrary, and irrelevant as n grows large), then one can give up on keeping track of the states of the single nodes and rather take a 'Eulerian' viewpoint, looking at the evolution of the empirical density of the agents' opinions, defined as

$$\rho^n(t) := \theta(X(t)).$$

In fact, under such assumptions of total mixing of the population and homogeneity of agents' behavior, $\rho^n(t)$ is a Markov chain itself on the space of types $\mathcal{P}_n(\mathcal{X}) := \{\rho \in \mathbb{R}^{\mathcal{X}} : n\rho_i = n_i \in \mathbb{Z}_+, \sum_i n_i = n\}$.

Moreover, Kurtz's theorem guarantees that there exist positive constants K, K' such that, for all $T > 0$, $\varepsilon > 0$, if $\lim_n \rho^n(0) = \rho(0) \in \mathcal{P}(\mathcal{X})$, then

$$\mathbb{P}\left(\sup_{t\in[0,T]} ||\rho^n(t) - \rho(t)|| \geq \varepsilon\right) \leq K' \exp(-Kn\varepsilon^2/T)\,,$$

where $\rho(t)$ is the solution of the Cauchy problem associated to the mean-field ODE

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho = P'(\rho)\rho - \rho\,,$$

with initial condition $\rho(0)$.

Exercise 1 (Cardinality of the space of types)**.** *Prove that*

$$|\mathcal{P}_n(\mathcal{X})| = \binom{n + |\mathcal{X}| - 1}{|\mathcal{X}| - 1} \leq (n + |\mathcal{X}| - 1)^{|\mathcal{X}|-1}\,.$$

*(*Hint: one has to count the different ways of assigning n identical balls to $|\mathcal{X}|$ distinguished urns... *)*

Exercise 2 (mean-field limit of the noisy majority-rule dynamics)**.** *The noisy majority-rule dynamics is characterized by state space $\mathcal{X} = \{0,1\}$ and transition kernel $\Psi_v(1|i, x_{\mathcal{N}_v}) = \Phi_\beta(d_v^{-1}\sum_{w\in\mathcal{N}_v} x_w)$, $i = 0,1$, where*

$$\Phi_\beta(\rho) := \frac{\exp(\beta\rho)}{\exp(\beta\rho) + \exp(\beta(1-\rho))}\,, \qquad \beta \geq 0\,.$$

(a) *Prove that,*

$$\lim_{\beta\to\infty} \Phi_\beta(\rho) = \begin{cases} 0 & \textit{if} \quad \rho \in [0, 1/2) \\ 1/2 & \textit{if} \quad \rho = 1/2 \\ 1 & \textit{if} \quad \rho \in (1/2, 1]\,. \end{cases}$$

(this justifies the name 'noisy majority rule', with $1/\beta$ to be interpreted as a measure of the noise level)

(b) *Prove that the mean-field limit ODE is given by*

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_1 = \Phi_\beta(\rho_1) - \rho_1\,; \tag{1}$$

(c) *For arbitrary $\beta \geq 0$, find the equilibria of (1), discuss their stability and characterize their region of attraction.*

Exercise 3 (k-majority rule dynamics)**.** *Let k be a positive odd integer and $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ a graph such that $d_v \geq k$ for all $v \in \mathcal{V}$. Consider the k-majority rule dynamics whereby each agent, when activated, selects k distinct neighbors uniformly at random and moves towards the opinion held by the majority of them. Find the mean-field ODE and study the asymptotic behavior (as t grows large) of the associated initial value problem.*

Exercise 4 (mean-field limit of the SIS epidemics)**.** *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a connected graph. Consider the SIS epidemics with rate-($\gamma/(n-1)$) transmission time on every link with one infected and one susceptible end-node, and rate-($1-\gamma$) exponential recovery time, where $\gamma \in (0,1)$. Let $\mathcal{X} = \{S, I\}$.*

(a) *Write down the transition kernel $\Psi_v(j|i, x_{\mathcal{N}_v})$ for $v \in \mathcal{V}$ (hint: assume that a node gets activated whenever it has a potential recovery or gets a potential decease transmission from one of the other $n-1$ nodes)*

(b) *What are all the absorbing states in $\mathcal{X}^{\mathcal{V}}$ of the Markov chain $X(t)$?*

(c) *Prove that the mean-field limit ODE is*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\rho_S = (1-2\gamma)\rho_I + \gamma\rho_I^2 \\ \frac{\mathrm{d}}{\mathrm{d}t}\rho_I = -(1-2\gamma)\rho_I - \gamma\rho_I^2 \end{cases} \tag{2}$$

(d) *What are the equilibria of (2)? Discuss their stability as γ varies in $(0,1)$.*

Exercise 5 (mean-field limit of the SIR epidemics)**.** *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a connected graph and $\gamma \in (0,1)$. Consider the SIR epidemics with rate-$\frac{\gamma}{n-1}$ transmission time on every link with one infected and one susceptible end-node, and rate-($1-\gamma$) exponential recovery time. Let $\mathcal{X} := \{S, I, R\}$.*

(a) *Write down the transition kernel $\Psi_v(j|i, x_{\mathcal{N}_v})$ for $v \in \mathcal{V}$ (hint: assume that a node gets activated whenever it has a potential recovery or gets a potential decease transmission from one of the other $n-1$ nodes)*

(b) *What are all the absorbing states in $\mathcal{X}^{\mathcal{V}}$ of the Markov chain $X(t)$? And what are they if $\mathcal{G}$ is the complete graph?*

(c) *Prove that the mean-field limit ODE is*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}\rho_S &= -\gamma\rho_S\rho_I \\ \frac{\mathrm{d}}{\mathrm{d}t}\rho_I &= \gamma\rho_S\rho_I - (1-\gamma)\rho_I \\ \frac{\mathrm{d}}{\mathrm{d}t}\rho_R &= (1-\gamma)\rho_I \end{cases} \tag{3}$$

(d) *Plot (or sketch) the phase portrait of (3);*

(e) *What are the equilibria of (3)?*

(f) *Prove that every trajectory of (3) is convergent to some limit* $\rho^* \in \mathcal{P}(\mathcal{X})$*, which depends on the initial condition* $\rho(0)$*; (hint:* $\rho_S(t)$ *is non-increasing,* $\rho_R(t)$ *non-decreasing, and the trajestoris belong to the compact* $\mathcal{P}(\mathcal{X})$*)*

(g) *Let* $\mathcal{R} := \{\rho \in \mathcal{P}(\mathcal{X}) : \rho_I = 0\}$, $\mathcal{R}_\gamma^S := \{\rho \in \mathcal{R} : \rho_S \le 1/\gamma - 1\}$*, and* $\mathcal{R}_\gamma^U := \mathcal{R}\backslash\mathcal{R}_\gamma^S$*. Prove that, for every initial condition* $\rho(0) \in \mathcal{P}(\mathcal{X})\backslash\mathcal{R}_\gamma^U$*, the solution of (3) satisfies*

$$\lim_{t\to+\infty} \rho(t) \in \mathcal{R}_\gamma^S\,;$$

(h) *Conclude that, for* $\gamma > 1/2$,

$$\rho_I(0) > 0 \quad \Longrightarrow \quad \lim_{t\to\infty} \rho_R(t) \ge 2 - 1/\gamma > 0\,.$$