Ph.D. course on Network Dynamics

Homework 4

To be discussed on Tuesday, October 29, 2013

Exercise 1 (Stationary fluctuations in the noisy voter model)**.** *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a strongly connected directed graph, and $\alpha \in (0,1)$, $\beta \in (0,1]$. Consider the following noisy voter model process $X(t) \in \{0,1\}^{\mathcal{V}}$: at each time t, a link $(i,j) \in \mathcal{E}$ is chosen at random with uniform probability, then with probability $(1-\beta)$ node i copies node j's state, and with probability β node i updates its state to an independent Bernoulli(α) random variable. (The case $\alpha = 1/2$ was discussed in class.)*

(a) Show that $X(t)$ is an irreducible Markov chain on $\{0,1\}^{\mathcal{V}}$;

(b) Prove that

$$\mathbb{P}(X_i(t) = 1 | X(0)) \stackrel{t\to\infty}{\longrightarrow} \alpha\,, \qquad \forall i \in \mathcal{V},$$

*for every initial condition $X(0) \in \{0,1\}^{\mathcal{V}}$; (*hint: use point (a) and duality for $\mathbb{P}(X_i(t) = 1|X(0)) = \mathbb{E}[X_i(t)|X(0)]$*)*

Now, let $\overline{X}(t) := n^{-1}\mathbb{1}'X(t)$ be the mean state (or 'barycenter') at time t, and let

$$\sigma^2 := \lim_{t\to\infty} \mathbb{E}\left[(\overline{X}(t) - \mathbb{E}[\overline{X}(t)])^2\right]$$

be its asymptotic variance. Further, let

$$\Delta^2 := \lim_{t\to\infty} \frac{1}{n^2} \sum_{i,j\in\mathcal{V}} \mathbb{E}\left[(X_i(t) - X_j(t))^2\right]$$

be the asymptotic mean square disagreement. The physical interpretation is that σ^2 is a measure of the amplitude of synchronous oscillations of the stationary states, while Δ^2 measures the amplitude of asynchronous oscillations.

(c) *Prove that*

$$\frac{\Delta^2}{2} + \sigma^2 = \alpha(1-\alpha)$$

*(*Hint: show that $\sigma^2 = \frac{1}{n^2}\sum_{i,j}\lim_t \mathrm{Cov}[X_i(t), X_j(t)]$, and that

$$\frac{\Delta^2}{2} = \lim_t \frac{1}{n}\sum_i \mathrm{Var}[X_i(t)] + \frac{1}{2n^2}\sum_{i,j}(\mathbb{E}[X_i(t) - X_j(t)])^2 - \frac{1}{n^2}\sum_{i,j}\mathrm{Cov}[X_i(t), X_j(t)].$$

You may find a shorter way.)

(d***) *Prove that, if $\mathcal{G}$ is undirected and $n\beta \to \infty$, then $\sigma^2 \to 0$ (*hint: find the stochastic matrix Q such that $y(t+1) = Qy(t)$, and prove that if $(V_1(0), V_2(0))$ is uniformly distributed over $\mathcal{V}^2$ and $(V_1(t), V_2(t))$ move with transition probability matrix Q, then $T_{couple} >> T_{noise}$ with high probability as $n \to \infty$, where T_{couple} is the first time $V_1(t) = V_2(t)$, while T_{noise} is the first time any between $V_1(t)$ and $V_2(t)$ is updated to an independent Bernoulli(α) variable*)*

Exercise 2 (Voter model with stubborn nodes [1])**.** *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a connected undirected graph. Let $s_0 \neq s_1 \in \mathcal{V}$ be such that $\{s_0, s_1\} \notin \mathcal{E}$ and consider the voter model process $X(t)$ on $\{0,1\}^{\mathcal{V}}$ with stubborn nodes s_0 and s_1: at each time t, a link $\{i,j\} \in \mathcal{E}$ is randomly selected with uniform probability from $\mathcal{E}$, and node i copies node j or vice versa with probability $1/2$, with the exception of the stubborn nodes which are copied with probability $1/2$ when a link incident on them is activated but never change their state equal to 1 for s_1 and to 0 for s_0.*

(a) *Prove that, for every initial condition $X(0) \in \{0,1\}^{\mathcal{V}}$ such that $X_{s_0}(0) = 0$ and $X_{s_1}(0) = 1$, the vector $x := \lim_{t\to\infty}\mathbb{E}[X(t)|X(0)]$ satisfies*

$$(I-P)x = 0 \text{ on } \mathcal{V}\setminus\{s_0, s_1\}\,, \qquad x_{s_0} = 0\,, \qquad x_{s_1} = 1\,,$$

where P is the stochastic matrix associated to the (lazy) random walk on $\mathcal{G}$;

(b) *Use (a) to show that, for all $i \in \mathcal{V}$,*

$$x_i = \mathbb{P}(V(T_{\mathcal{S}}) = s_1 | V(0) = i)\,,$$

where $V(t)$ is a (lazy) random walk on $\mathcal{G}$, and $T_{\mathcal{S}}$ is the corresponding hitting time on $\mathcal{S} := \{s_0, s_1\}$;

(c)* *Show that, for all* $i \in \mathcal{V}$, *and* $t \geq 0$,

$$|x_i - \pi' x| \leq \mathbb{P}(T_{\mathcal{S}} < t | V(0) = i) + \exp(-\lfloor t/\tau_{\text{mix}} \rfloor)\,,$$

where π *and* τ_{mix} *are the invariant distribution and, respectively, the mixing time of* P.

*(d**)* *Conclude that, if* $(\pi_{s_0} + \pi_{s_1})\tau_{\text{mix}} \to 0$ *and the maximal degree* d_{max} *in* $\mathcal{G}$ *remains bounded as* $n \to \infty$, *then for all* $\varepsilon > 0$ *the fraction of nodes* i *such that* $|x_i - \pi' x| \geq \varepsilon$ *vanishes as* $n \to \infty$, *i.e., the stubborn nodes have a* homogeneous influence *on the rest of the nodes.*

Exercise 3 (Evolutionary dynamics on graphs [2])**.** *The following stochastic model for the evolution of a finite population of constant size* n *has been considered in the literature. Individuals occupy the nodes of a srongly connected directed graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. *At time* $t = 0$, *each node* i *is occupied by an individual of specie* $S_i(0) \in \mathcal{S}$. *Each species* $s \in \mathcal{S}$ *is characterized a 'fitness' parameter* $f_s \in (0, 1]$. *At each time* $t \geq 0$, *a link* (i, j) *is selected with uniform probability from* $\mathcal{E}$*: then, the individual currently occupying node* i *generates an offspring of its own specie* $S_i(t)$ *which, with probability* $f_{S_i(t)}$, *replaces the individual currently occupying node* j.

(a) *Show that, when* $f_s = 1$ *for all* $s \in \mathcal{S}$, *the process* $S(t) \in \mathcal{S}^{\mathcal{V}}$ *described above coincides with the voter model* $X(t)$ *on a directed graph* $\tilde{\mathcal{G}}$ *(specify* $\tilde{\mathcal{G}}$*).*

Now, assume that $\mathcal{G}$ *is undirected, and that there are only two species, say* $\mathcal{S} = \{0, 1\}$. *Let* $M(t) := \mathbb{1}' S(t)$ *be the number of individuals of specie* 1, *and* $r := f_1 / f_0$.

(b) *Prove that*

$$\mathbb{P}(M(t+1) - M(t) = 1 | S(t)) = r \mathbb{P}(M(t+1) - M(t) = -1 | S(t))\,,$$

for all $t \geq 0$.

Now, assume that $r > 1$, *i.e., specie* 1 *has a higher fitness than specie* 0. *For* $i \in \mathcal{V}$, *let*

$$\rho_i := \mathbb{P}\left(S(t) \stackrel{t \to \infty}{\longrightarrow} \mathbf{1} | S_i(0) = 1,\, S_j(0) = 0\, \forall j \neq i\right)\,, \qquad i \in \mathcal{V}\,,$$

be the fixation probability, *i.e., the probability that a single individual of the most fit specie* 1 *(a* mutant*) initially present in node* i *will eventually take over a population of individuals initially all of the less fit specie* 0.

(c*) *Prove that, for all* $i \in \mathcal{V}$,

$$\rho_i = \frac{1 - 1/r}{1 - 1/r^n}\,. \tag{1}$$

*(*hint: use point (b) to show that ρ_i coincides with the probability that a birth and death chain on $\{0, 1, \ldots, n\}$ with birth/death ratio r, started at 1 will hit node n before node 0.*)*

The remarkable property of formula (1) *is that* ρ_i *is independent both of the node* i *where the mutant is initially placed, and of the graph* $\mathcal{G}$.

(d*) *Generalize formula* (1) *to the case when* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ *is a directed weighted graph, with weights* $\{w_e > 0 : e \in \mathcal{E}\}$ *such that*

$$\sum_j w_{(i,j)} = \sum_j w_{(j,i)}\,, \qquad \forall i \in \mathcal{V}$$

(here we use the convention that $w_{(i,j)} = 0$ *for every pair* $(i, j) \notin \mathcal{E}$*).*

References

[1] D. Acemoglu, G. Como, F. Fagnani, and A. Ozdaglar. Opinion fluctuations and disagreement in social networks. *Mathematics of Operation Research*, 38(1):1–27, 2013.

[2] E. Lieberman, C. Hauert, and M.A. Nowak. Evolutionary dynamics on graphs. *Nature*, 433:312–316, January 2005.