Ph.D. course on Network Dynamics

Homework 3

To be discussed on Tuesday, October 22, 2013

Exercise 0 Do Exercise 6 of Homework 2.

Exercise 1 (Mixing time on the hyper-cube)**.** *For $d \geq 1$, let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be the d-dimensional hypercube (with node set $\mathcal{V} = \{0,1\}^d$), and let P be the stochastic matrix associated to the lazy random walk on $\mathcal{G}$. Consider the Markov chain $(Z^1(t), Z_2(t))$ on $\mathcal{V}\times\mathcal{V}$ obtained by sampling at each time $t \geq 0$ a uniformly distributed component $I(t) \in \{1, \dots, d\}$ and a an independent uniform binary variable $B(t)$, and putting $Z^1_{I(t)}(t+1) = Z^2_{(I(t))}(t+1) = B(t)$ and $Z^j_i(t+1) = Z^j_i(t)$ for all $i \in \{1, \dots, d\} \setminus \{I(t)\}$ and $j \in \{1, 2\}$. Show that this is a Markov coupling for P, and use it to obtain an upper bound on the mixing time of P. (*Hint: use Exercise 4 of Homework 2.*)*

Exercise 2 (Voter model and 'conservation of the total magnetization')**.** *Let $\mathcal{G} := (\mathcal{V}, \mathcal{E})$ be a strongly connected directed graph. Consider the following Markov chain $X(t)$ on $\{0,1\}^{\mathcal{V}}$: at time $t \geq 0$ a directed link $(i,j) \in \mathcal{E}$ is sampled with uniform probability from $\mathcal{E}$, and the state is updated as $X_i(t+1) = X_j(t)$, and $X_k(t+1) = X_k(t)$ for all $k \in \mathcal{V} \setminus \{i\}$ (i.e., node i copies node j's state, and the other states do not change). This is (a special of) the voter model. For every choice of the initial state $X(0) \in \{0,1\}^{\mathcal{V}}$,*

(a) show that, with probability one, $X(t)$ converges to one of the two absorbing states, the all-zero vector $\mathbf{0}$ or the all-1 vector $\mathbf{1}$.

(b) prove that, if $\mathcal{G}$ is undirected (in the sense that $(i,j) \in \mathcal{E}$ if and only if $(j,i) \in \mathcal{E}$), then

$$\mathbb{P}\left(X(t) \stackrel{t\to\infty}{\longrightarrow} \mathbf{1} | X(0)\right) = \frac{1}{n}\sum_{v\in\mathcal{V}} X_v(0)\,.$$

*(*Hint: prove that $n^{-1}\sum_v \mathbb{E}[X_v(t)]$ is constant in t: in the statistical physics jargon, such a property is sometimes referred to as 'conservation of the total magnetization'.*)*

(c) generalize (b) to directed graphs by proving that*

$$\mathbb{P}\left(X(t) \stackrel{t\to\infty}{\longrightarrow} \mathbf{1} | X(0)\right) = \sum_{v\in\mathcal{V}} \pi_v X_v(0)\,,$$

where π is the stationary distribution of a suitably defined irreducible stochastic matrix P. (you should explicitly construct such P!)

Exercise 3 (Universal bounds on the mixing time for random walks)**.** *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a connected undirected graph with $n := |\mathcal{V}|$ nodes and $m := |\mathcal{E}|$ links. Let P be the stochastic matrix associated to the lazy random walk on $\mathcal{G}$.*

(a) Use Cheeger's inequality to prove that the mixing time of P satisfies $\tau_{\text{mix}} \le Cm^2 \log m$, for $m \ge 2$, where $C > 0$ is an absolute constant;

*(b**) Prove that the spectral gap of P satisfies $1 - \lambda_2 \ge C'/(nm)$, where $C' > 0$ is an absolute constant,.* (hint: first observe that, for all x such that $\pi' x = 0$ and $\sum_v \pi_v x_v^2 = 1$, one has that $x^* - x_* \ge 1/\sqrt{2m}$, where $x^* := \max_v x_v$ and $x_* := \min_v x_v$. Then, find a simple path $\mathcal{P} = (i = v_0, v_1, \ldots, v_k = j\}$ in $\mathcal{G}$ joining $i := \operatorname{argmax}_v x_v$ to $j := \operatorname{argmin}_v x_v$ and use Cauchy-Schwartz's inequality to prove that $k\sum_{1\le l\le k}(x_{v_l} - x_{v_{l-1}})^2 \ge 1/(2m)$. Use the variational characterization of $1 - \lambda_2$.)

(c) Use (b) to show that $\tau_{\text{mix}} \le C'' n^3 \log n$ for $n \ge 2$ where $C'' > 0$ is some absolute constant.