On resilience of distributed routing in networks under cascade dynamics

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Abstract— We consider network flow over graphs between a single origin-destination pair, where the network state consists of flows *and* activation status of the links. The evolution of the activation status of a link is given by an irreversible transition that depends on the saturation status of that link and the activation status of the downstream links. The flow dynamics is determined by activation status of the links and node-wise routing policies under the flow balance constraints at the nodes. We formulate a deterministic discrete time dynamics for the network state, where the time epochs correspond to a change in the activation status of the links, and study network resilience towards disturbances that reduce link-wise flow capacities, under distributed routing policies. The margin of resilience is defined as the minimum, among all possible disturbances, of the link-wise sum of reductions in flow capacities, under which the links outgoing from the origin node become inactive in finite time. We propose a backward propagation algorithm to compute an upper bound on the margin of resilience for tree-like network topologies with breadth at most 2, and show that this bound is tight for trees with the additional property of having depth at most 2.

I. INTRODUCTION

Resilience is becoming a key consideration in the design and operation of critical infrastructure systems such as transportation, power, water and data networks. Due to their increasing scale and interconnectedness, these systems pose several new challenges. For example, small local disruptions can cascade through the network to cause massive failures, or local actions to mitigate disruptions can increase vulnerability of the other parts of the network. In this paper, we present an alternative framework to the traditional probabilistic frameworks for studying cascading phenomena in complex networks, and characterize network resilience under such a framework.

Formally, we consider network flow over graphs between a single origin-destination pair, where the network state consists of flows *and* activation status of the links. The evolution of the activation status of a link is given by an irreversible transition that depends on the saturation status of that link and the activation status of the downstream links. The flow dynamics is determined by activation status of the links and node-wise routing policies under flow balance constraints at the nodes. We formulate a deterministic discrete time dynamics for the network state, where the time

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epochs correspond to a change in the activation status of the links, and study network resilience towards disturbances that reduce link-wise flow capacities, under distributed routing policies. The margin of resilience is defined as the minimum, among all possible disturbances, of the link-wise sum of reductions in flow capacities, under which the links outgoing from the origin become inactive in finite time. The objective of this paper is to provide a tight characterization of margin of resilience under such a setting in terms of network parameters. The setting of this paper is to be contrasted with the *dynamical* flow network in our previous work [1], [2] where the state of a link is described by density, and whose dynamics is driven by the difference between the inflow and outflow on that link in such a way that the flow always remains within its capacity.

Models for cascades in general complex networks are given in [3], [4], [5], while domain-specific models are provided in [6] (power networks), [7] (financial networks), and [8] (supply networks). There has also been work on understanding the role of human decisions on such cascading phenomena, especially in the context of financial networks, e.g., see [9]. However, most of these models rely on a stochastic model for initiation of failure and its propagation. In contrast, in this paper we propose a deterministic dynamical framework for cascading failures that are particularly relevant for transportation networks. Other relevant references include [10] which considers optimal control of cascading failures in power grid; [11] which considers cascading node failures in the context of wireless networks; [12] for loading-dependent probabilistic models for cascading failures in power networks. The adversarial disturbance setting for network flows of this paper is also reminiscent of *network interdiction problems*, e.g., see [13], with the difference being that in our setting, we allow for control action in the form of routing in response to disturbance, and that this control action is distributed.

The contributions of the paper are as follows. First, we propose a novel model for cascade dynamics in flow networks under distributed routing policies, and formalize the notion of margin of resilience of networks within such a framework. Second, we propose an algorithm to compute an upper bound on the margin of resilience for trees of breadth at most 2, and illustrate by simple example, that the bounds obtained by this algorithm are sharper than those obtained in our previous work [14]. Our results are derived for networks with tree-like topologies, i.e., where every intermediate node lines on exactly one path from the origin to the destination. While many real networks do not have a tree-like topology, this assumption allows for an analytically tractable solution

which shows fundamental insight into the problem. Finally, we show that the upper bound computed by our algorithm is tight at least for trees with the additional property of having depth at most 2.

Before proceeding, we define some preliminary notation to be used throughout the paper. Let $\mathbb{R}$ be the set of real numbers, $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$ be the set of nonnegative real numbers. When $\mathcal{A}$ and $\mathcal{B}$ are finite sets, $|\mathcal{A}|$ will denote the cardinality of $\mathcal{A}$, $\mathbb{R}^{\mathcal{A}}$ (respectively, $\mathbb{R}_+^{\mathcal{A}}$) will stay for the space of real-valued (nonnegative-real-valued) vectors whose components are indexed by elements of $\mathcal{A}$. $\mathbf{1}$ and $\mathbf{0}$ stand for the all-one and all-zero vectors respectively, whose size will be clear from the context. A directed multigraph is the pair $(\mathcal{V}, \mathcal{E})$ of a finite set $\mathcal{V}$ of nodes, and of a multiset $\mathcal{E}$ of links consisting of ordered pairs of nodes (i.e., we allow for parallel links between a pair of nodes). If $e = (v, w) \in \mathcal{E}$ is a link, where $v, w \in \mathcal{V}$, we shall write $\sigma_e = v$ and $\tau_e = w$ for its tail and head node, respectively. The sets of outgoing and incoming links of a node $v \in \mathcal{V}$ will be denoted by $\mathcal{E}_v^+ := \{e \in \mathcal{E} : \sigma_e = v\}$ and $\mathcal{E}_v^- := \{e \in \mathcal{E} : \tau_e = v\}$, respectively. Moreover, we shall use the shorthand notation $\mathcal{R}_v := \mathbb{R}_+^{\mathcal{E}_v^+}$ for the set of nonnegative-real-valued vectors whose entries are indexed by elements of $\mathcal{E}_v^+$; for a given $\mu \geq 0$, $\mathcal{S}_v(\mu) := \{x \in \mathcal{R}_v : \sum_{e \in \mathcal{E}_v^+} x_e = \mu\}$; and $\mathcal{R} := \mathbb{R}_+^{\mathcal{E}}$ for the set of nonnegative-real-valued vectors whose entries are indexed by the links in $\mathcal{E}$. For $x \in \mathbb{R}$, we shall use the notation $[x]^+$ to mean $\max\{0, x\}$.

II. CASCADE DYNAMICS IN FLOW NETWORKS

The central object of study in this paper is a flow network which is formally defined as follows.

Definition 1 (Flow network): A *flow network* $\mathcal{N} = (\mathcal{T}, C)$ is the pair of a *topology*, described by a finite directed multigraph $\mathcal{T} = (\mathcal{V}, \mathcal{E})$, where $\mathcal{V} = \{0, 1, \ldots, n\}$ is the node set and $\mathcal{E}$ is the link multiset, and a vector $C \in \mathbb{R}_+^{\mathcal{E}}$ describing the maximum flow capacities on the links.

We shall use the notation $\mathcal{F}_v := \Pi_{e \in \mathcal{E}_v^+}[0, C_e]$ for the set of admissible flow vectors on outgoing links from node v, and $\mathcal{F} := \Pi_{e \in \mathcal{E}}[0, C_e]$ for the set of admissible flow vectors for the network. Throughout this paper, we shall restrict ourselves to network topologies satisfying the following:

Assumption 1: The topology $\mathcal{T}$ has a unique origin (i.e., a node $v \in \mathcal{V}$ such that $\mathcal{E}_v^-$ is empty), and a unique destination (i.e., a node $v \in \mathcal{V}$ such that $\mathcal{E}_v^+$ is empty). Throughout, the origin node will be assigned the label 0, and the destination node the label n. Moreover, there exists a path in $\mathcal{T}$ to the destination node n from every other node $0 \leq v < n$.

Let us define the set of *admissible flows* with inflow λ at the origin as $\mathcal{F}(\lambda) :=$
$$\left\{ f^* \in \mathcal{F} : \sum_{e \in \mathcal{E}_0^+} f_e^* = \lambda, \ \sum_{e \in \mathcal{E}_v^+} f_e^* = \sum_{e \in \mathcal{E}_v^-} f_e^*, \ 0 < v < n \right\}.$$
Then, it follows from the max-flow min-cut theorem (see, e.g., [15]), that $\mathcal{F}^*(\lambda) \neq \emptyset$ whenever λ is less than the min-cut capacity of $\mathcal{N}$. That is, the min-cut capacity equals the maximum flow that can pass from the origin to the

destination node while satisfying capacity constraints on the links, and conservation of flow at the intermediate nodes.

We shall often restrict ourselves to a specific subclass of topologies as characterized by the following assumption.

Assumption 2: For every $v \in \{1, \ldots, n-1\}$, there exists one and only one directed path from the origin node 0 to node v in $\mathcal{T}$.

We shall often refer to topologies $\mathcal{T}$ that satisfy Assumption 2 to be *tree topologies*. The *depth* of a tree-like topology is defined as the length of the longest path from the origin to the destination node. The *breadth* of a tree is defined to be equal to the cardinality of $\|\mathcal{E}_0^+\|$. We now describe a dynamical framework for studying evolution of flow on $\mathcal{N}$. For every link $e \in \mathcal{E}$, we describe its state by $(\xi_e, f_e) \in \{0, 1\} \times \mathcal{F}_e$. The binary variable ξ_e is used to denote the status of link e, i.e., link e is active if $\xi_e = 1$ and inactive otherwise. The variable f_e denotes the flow on link e. We use the shorthand notations $f = \{f_e : e \in \mathcal{E}\}$ and $f^v = \{f_e : e \in \mathcal{E}_v^+\}$. ξ and ξ^v are defined in a similar fashion. We assume that $\xi(0) = \mathbf{1}$, and $f(0) = f^*$ for some $f^* \in \mathcal{F}^*$. The network suffers a disturbance $\delta \in \mathbb{R}_+^{\mathcal{E}}$ at time $t = 0$. The effect of the disturbance is to reduce the link-wise maximum flow capacities. Formally, under the application of disturbance δ, the maximum flow capacity on link e decreases to $C_e - \delta_e$. The set of *feasible* disturbances is equal to the set of feasible flows $\mathcal{F}$. The response of the network to disturbance is through routing policies, one for every node, that determine the splitting of the inflow at a node among the links outgoing from that node. In this paper, we focus on *distributed* routing policies that rely only on local information around a node. The formal description is as follows.

Definition 2: (Distributed routing policy) A *distributed routing policy* for a network $\mathcal{N}$ is a family of functions $\mathcal{G} := \{G^v : \{0, 1\}^{\mathcal{E}_v^+} \times \mathcal{F}_v \times \mathbb{R}_+ \to \mathcal{R}_v; (\xi^v, \delta^v, \lambda_v) \mapsto G^v(\xi^v, \delta^v, \lambda_v) \in \mathcal{S}_v(\lambda_v) \cup \mathbf{0}\}_{0 \leq v < n}$ describing the splitting of the incoming flow from each non-destination node v among its outgoing link set $\mathcal{E}_v^+$, as a function of the observed status and the magnitude of disturbances on the outgoing links, and the incoming flow. We also implicitly assume that every G^v, $v \in \{0, \ldots, n-1\}$, has a priori information about the vector of link flow capacities C. Moreover, we will adopt the conventions that, for all $v \in \{0, \ldots, n-1\}$, $\delta^v \in \mathcal{F}_v$, and $\lambda_v \geq 0$: for all $e \in \mathcal{E}_v^+$, if $\xi_e = 0$ then $G_e^v(\xi^v, \delta^v, \lambda_v) = 0$, and $G^v(\xi^v, \delta^v, \lambda_v) = \mathbf{0}$ if and only if $\xi^v = \mathbf{0}$.

Remark 1: The two conventions established at the end of Definition 2 imply that the routing policy does not send any flow to an outgoing link once it becomes inactive, and respectively, in the case when all the outgoing links become inactive, the routing policy does not transmit any of the incoming flow.

We are now ready to define the dynamics of flow on the network. For a flow network as per Definition 1, a distributed routing policy as per Definition 2, a constant outflow at the origin $\lambda > 0$, a feasible disturbance $\delta \in \mathcal{F}$ and an admissible flow vector $f^* \in \mathcal{F}(\lambda)$, we consider the following discrete-time dynamics in this paper.

For all $e \in \mathcal{E}$, $t \geq 0$,

$$
\begin{aligned}
f_e(t+1) &= G_e^{\sigma_e}\big(\xi^{\sigma_e}(t), \delta^{\sigma_e}, \lambda_{\sigma_e}(t)\big), \\
\xi_e(t+1) &= \chi_e(t+1) \cdot \psi_{\tau_e}(t) \cdot \xi_e(t),
\end{aligned} \tag{1}
$$

where $\chi_e(t) := \mathbf{1}_{f_e(t) \leq C_e - \delta_e}$ for $t \geq 1$ is a binary variable that defines the saturation status of link e,

$$
\psi_v(t) := 1 - \Pi_{e \in \mathcal{E}_v^+}(1 - \xi_e(t)), \ t \geq 0
$$

is a binary variable that describes the active status of node v, and

$$
\lambda_v(t) := \begin{cases} \lambda & \text{if} \quad v = 0 \\ \sum_{e \in \mathcal{E}_v^-} f_e(t) & \text{if} \quad v > 0\,, \end{cases} \tag{2}
$$

is the incoming flow at node $v \in \mathcal{V}$, and the initial conditions are $\xi(0) = \mathbf{1}$ and $f(0) = f^*$. Note that, from the definition of $\chi_e(t)$, the flow capacity is modeled to be $C_e - \delta_e$. With a slight abuse of terminology with respect to standard literature, e.g., [15], we shall refer to the system defined by (1) and (2) as a flow network under a routing policy $\mathcal{G}$, an initial flow $f^* \in \mathcal{F}(\lambda)$, and a constant total outflow at the origin node $\lambda > 0$. We are interested in the dynamics of the link-wise flows under feasible disturbances $\delta \in \mathcal{F}(\lambda)$.

Remark 2 (Cascading failure): The dynamics in (1) can be given the following interpretation: at each time instant $t \geq 0$, each non destination node $0 \leq v < n$ splits its inflow among its outgoing links according to the routing function. Such routing function takes into account only local information, and in particular the current inflow to node v, as well as the current activation status of the outgoing inks of v, and their current maximum flow capacity. Such a model potentially originates cascaded failures in the following way: if, at time t a link e is fed with flow exceeding its maximum flow capacity C_e, it saturates, i.e., $\chi_e(t) = 0$ and consequently becomes inactive, i.e., $\xi_e(t) = 0$ and remains inactive thereafter due to the irreversible transition of ξ_e from 1 to 0. When a node v has all its outgoing links inactive, it also becomes inactive, i.e., its status becomes $\psi_v(t) = 0$, and this causes all of its incoming links to become inactive in turn. Such failures are irreversible, and can propagate both downstream, since the inactivation of a link forces its tail node to route more flow towards other links, potentially overloading either them, or the portion of the network downstream to them, and upstream, since the failure of a node implies the inactivation of all of its incoming links with effects on the portion of the network upstream to such links. This discrete-time dynamics is related to the continuous-time switched system proposed in [14], and in particular to the sequence of states at the switching time of it. However, we will not attempt to make this connection rigorous here.

The effect of cascading failures is that the network flow vector $f(t)$ may not remain feasible for all $t \geq 0$. We formalize the corresponding dichotomy in the network behavior by the notion of transfer efficiency as follows.

Definition 3 (Transfer efficiency of the network): Let $\mathcal{N}$ be a dynamical flow network with a distributed routing policy $\mathcal{G}$, an initial flow $f^* \in \mathcal{F}(\lambda)$, and $\lambda > 0$ a constant total outflow at the origin node. The network is called transferring under disturbance δ if there exists $T \geq 0$ such that $\lambda_n(t) = \lambda$ for all $t \geq T$.

The following simple proposition, which we state without proof, shows that the network being transferring is equivalent to the origin node being active all the time.

Proposition 1: Consider a dynamical flow network with a distributed routing policy $\mathcal{G}$, and $\lambda > 0$ a constant total outflow at the origin node. The network is transferring under disturbance δ if and only if $\psi_0(t) = 1$ for all $t \geq 0$.

We define the *magnitude of a disturbance* $\delta \in \mathcal{F}$ to be its 1-norm, i.e., $\|\delta\|_1 = \sum_{e \in \mathcal{E}} \delta_e$. With a slight abuse of notation, we shall use δ to refer to the vector of link-wise disturbance as well as its 1-norm, with the meaning being clear from the context.

Definition 4 (Margin of resilience of the network): Let $\mathcal{N}$ be a flow network satisfying Assumption 1, with a distributed routing policy $\mathcal{G}$, an initial flow $f^* \in \mathcal{F}(\lambda)$, and $\lambda > 0$ a constant total outflow at the origin node. The margin of resilience of the network is defined as the infimum of the magnitude of disturbances under which the network is not transferring.

Our objective in this paper is to compute the margin of resilience of flow networks satisfying Assumption 1 and operating under distributed routing policies.

III. MAIN RESULTS

Algorithm 1: Backward Propagation Algorithm

1: $R_n(\mu) := +\infty$ for all $\mu \geq 0$ {destination node}
2: **for** $v = n-1, n-2, \ldots, 0$ **do** {construct a series of intermediate functions for every node starting with $n-1$, and going backward up to the origin}
3:     put $S_\emptyset(\gamma, \mu) \equiv 0$ and iteratively compute for $\mathcal{J} \subseteq \mathcal{E}_v^+$ of increasing size:

$$
S_\mathcal{J}(\gamma, \mu) = 0 \quad \text{if} \sum_{j \in \mathcal{J}}(C_j - \gamma_j) \leq \mu, \tag{3}
$$

and, if $\sum_{j \in \mathcal{J}}(C_j - \gamma_j) > \mu$,

$$
S_\mathcal{J}(\gamma, \mu) := \max_x \min_{e \in \mathcal{J}}\big\{R_{\tau_e}(x_e) + S_{\mathcal{J}\setminus\{e\}}(\gamma, \mu)\big\} \tag{4}
$$

where the $\max$ is over $x \in \prod_{j \in \mathcal{J}}[0, C_j - \gamma_j]$ such that $\sum_{j \in \mathcal{J}} x_j = \mu$.

4:

$$
R_v(\mu) := \inf_{\gamma \in \mathcal{F}_v}\left\{\sum_{e \in \mathcal{E}_v^+} \gamma_e + S_{\mathcal{E}_v^+}(\gamma, \mu)\right\}. \tag{5}
$$

5: **end for**

For a network $\mathcal{N} = (\mathcal{T}, C)$ satisfying Assumption 1, we associate $R_v : \mathbb{R}_+ \to \mathbb{R}_+$ to every node $v \in \{0, \ldots, n\}$ that is representative (following subsequent technical results in the paper) of the maximum disturbance that the tree rooted at node v can suffer when the inflow at node v is μ, before

it looses its transferring property. The construction of such functions is formally described by the following algorithm, called the *Backward Propagation Algorithm*.

In the remainder of this paper, the quantity $R_0(\lambda)$ will be related to the margin of resilience for dynamical flow networks with tree like topologies. We first state $R_0(\lambda)$ as an upper bound on the margin of resilience.

Theorem 1: Consider a flow network $\mathcal{N} = (\mathcal{T}, C)$ with tree topology satisfying assumptions 1 and 2, and having breadth at most 2, a distributed routing policy $\mathcal{G}$, an initial flow $f^* \in \mathcal{F}(\lambda)$, and $\lambda > 0$ a constant total outflow at the origin node. Then its margin of resilience does not exceed $R_0(\lambda)$.

We next show that $R_0(\lambda)$ is in fact *equal* to the margin of resilience for tree topologies of depth 2 under a specific distributed routing policy which mimics the local optimization step (4) involved in the Backward Propagation Algorithm. Formally, for a given $\xi^v \in \{0,1\}^{\mathcal{E}_v^+}$ and $\mu \geq 0$, $v \in \{1, \ldots, n-1\}$, if we let $\mathcal{H}_v := \{e \in \mathcal{E}_v^+ \mid \xi_e = 1\}$, and $\mathcal{X}_v := \{x \in \mathcal{S}_v(\mu) \mid x_e = 0 \ \forall e \in \mathcal{E}_v^+ \setminus \mathcal{H}_v\}$, then we shall consider a distributed routing policy that satisfies the following for all $\xi^v \in \{0,1\}^{\mathcal{E}_v^+}$, $\delta^v \in \mathcal{F}_{\mathcal{E}_v^+}$, $v \in \{1, \ldots, n-1\}$: if $\sum_{j \in \mathcal{H}_v} (C_j - \delta_j) > \lambda_v$, then

$$G_{\mathcal{H}_v}^v(\xi^v, \delta^v, \lambda_v) \in$$
$$\operatorname*{argmax}_{x \in \prod_{j \in \mathcal{H}_v}[0, C_j - \delta_j]} \min_{e \in \mathcal{H}_v} \left\{ R_{\tau_e}(x_e) + S_{\mathcal{H}_v \setminus \{e\}}(\delta, \lambda_v) \right\},$$
(6)

where $G_{\mathcal{H}_v}^v$ is the restriction of G^v to $\mathcal{H}_v$.

Theorem 2: Consider a flow network $\mathcal{N} = (\mathcal{T}, C)$ satisfying assumptions 1 and 2, and having depth of at most 2, a distributed routing policy $\mathcal{G}$ satisfying (6), an initial flow $f^* \in \mathcal{F}(\lambda)$, and $\lambda > 0$ a constant total outflow at the origin node. Then its margin of resilience is at least equal to $R_0(\lambda)$.

Theorems 1 and 2 imply that $R_0(\lambda)$ is a tight characterization of the margin of resilience for tree topologies of breadth and depth at most 2, and the routing policy satisfying (6) is maximally resilient within the class of distributed routing policies for such tree topologies. The proof of Theorems 1 and 2 are provided in Sections IV and V respectively. It is worth to compare the quantity $R_0(\lambda)$ computed by the backward propagation algorithm above with the upper bound $\Gamma(\mathcal{N}, f^*)$ derived in [14] for the resilience of a network $\mathcal{N}$ with initial equilibrium flow $f^* \in \mathcal{F}(\lambda)$. The comparison is best appreciated in the following simple example.
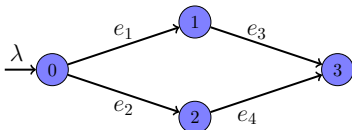

Fig. 1. A tree with breadth 2 and depth 2.

Example 1: Consider a network $\mathcal{N} = (\mathcal{T}, C)$ whose topology is the one depicted in Figure 1. Assume that $C_{e_i} = 1$ for all $i \in \{1, 2, 3, 4\}$. Then, for every $\lambda \geq 0$,

one has

$$R_0(\lambda) = \begin{cases} 2 - 3\lambda/2 & \text{if} \quad \lambda \in [0, 1] \\ [1 - \lambda/2]^+ & \text{if} \quad \lambda \geq 1, \end{cases}$$

while $\Gamma(\mathcal{N}, f^*) = 2 - \lambda$ for every initial equilibrium flow $f^* \in \mathcal{F}(\lambda)$. This shows that the new bound $R_0(\lambda)$ is stronger than $\Gamma(\mathcal{N}, f^*)$ derived in [14].

The following example illustrates that it is not possible to extend Theorem 2 to trees of depth more than 2.
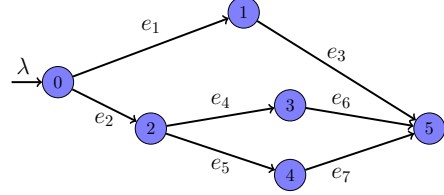

Fig. 2. A tree with breadth 2 and depth 3.

Example 2: Consider the graph topology illustrated in Figure 2. Let the maximum flow capacities be $C_{e_1} = 0.15$, $C_{e_2} = 10$, $C_{e_3} = 0.2$, $C_{e_4} = 1$, $C_{e_5} = 0.5$, $C_{e_6} = C_{e_7} = 2$. Consider a constant inflow $\lambda = 0.55$ and a disturbance supported only on links e_3 and e_6, of magnitude $\delta_{e_3} = 0.15$ and $\delta_{e_6} = 1.8$. We will show that, in spite of the fact that $R_0(\lambda) > \delta = 1.95$, the network will not be transferring under the routing policy satisfying (6).

First observe that $R_3(\mu) = R_4(\mu) = [2 - \mu]^+$ for all $\mu \geq 0$. For brevity in notation, let $x_4^*(\mu) := G_{\mathcal{E}_2^+, e_4}^2(\mathbf{1}, \delta^2, \mu)$ and $x_5^*(\lambda) = \mu - x_4^*(\mu)$ be the flow on links 4 and 5 respectively, under a routing policy satisfying (6) for all $0 \leq \mu \leq 1.5$. One can verify that, for $0 \leq \mu < 0.5$,

$$x_4^*(\mu)$$
$$= \operatorname*{argmax}_{z \in [0, \mu]} \min \left\{ (2-z) + (2-\mu), (2+z-\mu) + (2-\mu) \right\}$$
$$= \operatorname*{argmax}_{z \in [0, \mu]} \min \{4 - \mu - z, 4 - 2\mu + z\} = \mu/2,$$

for $\mu \in (0.5, 1]$,

$$x_4^*(\mu) = \operatorname*{argmax}_{z \in [\mu - 0.5, \mu]} \min \{2 - z, (2 - \mu + z) + (2 - \mu)\}$$
$$= \operatorname*{argmax}_{z \in [\mu - 0.5, \mu]} \min \{2 - z, 4 - 2\mu + z\} = \mu - 0.5,$$

and for $\mu \in (1, 1.5]$,

$$x_4^*(\mu) = \operatorname*{argmax}_{z \in [\mu - 0.5, 1]} \min \{2 - z, 2 - \mu + z\} = \mu - 0.5.$$

One can also verify that the computations in the Backward Propagation Algorithm give

$$R_2(\mu) = \begin{cases} 2.5 - \mu & \text{if} \quad \mu \in [0, 1.5] \\ 0 & \text{if} \quad \mu > 1.5. \end{cases}$$
(7)

Observe that $x_4^*(\mu)$ and $x_5^*(\mu)$ are continuous and monotonically non-decreasing over $[0, 1.5)$ except at $\mu = 0.5$, where $x_4^*(\mu)$ (resp, $x_5^*(\mu)$) is discontinuous and has a negative (resp, positive) jump. $R_2(\mu)$ is discontinuous at $\mu = 1.5$, but monotonically decreasing for $\mu \geq 0$. Since $C_{e_2} = 10$,

(4) implies that $S_{e_2}(0, \lambda) = R_2(\lambda) = 1.95$. On the other hand, since $\lambda > C_{e_1}$, we get $R_1(\mu) = [0.2 - \mu]^+$ and (4) implies that $S_{e_1}(0, \lambda) = 0$. One can verify that, for $R_0(\lambda)$, the infimum in (5) is achieved at $\gamma = \mathbf{0}$. Therefore,

$$
\begin{aligned}
R_0(\lambda) \\
= \max \min \left\{ R_1(\lambda - x) + S_{e_2}(0, \lambda), R_2(x) + S_{e_1}(0, \lambda) \right\} \\
= 2.05,
\end{aligned}
$$

where the max is over $x \in [\lambda - 0.15, \lambda] = [0.4, 0.55]$, and is achieved at $x = 0.45$. Moreover, these calculations also show that $G^0_{e_2}(\mathbf{1}, \mathbf{0}, \lambda) = 0.45$, i.e., the flow on link e_2 at time 1, $f_{e_2}(1) = 0.45 < 10 = C_{e_2} - \delta_{e_2}$, and hence $f_{e_1}(1) = \lambda - f_{e_2}(1) = 0.1 < 0.15 = C_{e_1} - \delta_{e_1}$. Similarly, based on the above calculations, one sees that $f_{e_3}(1) = 0.1 > 0.05 = C_{e_3} - \delta_{e_3}$, $f_{e_4}(1) = G^2_{e_4}(\mathbf{1}, \delta^2, 0.45) = x_4^*(0.45) = 0.225 < 1 = C_{e_4} - \delta_{e_4}$, $f_{e_5}(1) = 0.225 < 0.5 = C_{e_5} - \delta_{e_5}$, $f_{e_6}(1) = 0.225 > 0.2 = C_{e_6} - \delta_{e_6}$, and $f_{e_7}(1) = 0.225 < 2 = C_{e_7} - \delta_{e_7}$. Hence, under the dynamics in (1), for all $s \geq 1$, $\xi_{e_i}(s) = \chi_{e_i}(s) = 1$ for all $i \in \{1, 2, 4, 5, 7\}$ and $\xi_{e_i}(s) = \chi_{e_i}(s) = 0$ for $i \in \{3, 6\}$. This implies that $\xi_{e_1}(s) = \xi_{e_4}(s) = 0$ for all $s \geq 2$. At $t = 2$, the routing policy at origin has to route all of λ towards e_2, so that $f_{e_2}(3) = 0.55$ Since $\xi_{e_4}(4) = 0$, at $t = 4$, the routing at node 2 has to route all the incoming flow of 0.55 to link e_5. This implies that $f_{e_5}(4) = 0.55 > 0.5 = C_{e_5}$, so that $\xi_{e_5}(s) = 0$ for all $s \geq 4$, and hence $\xi_{e_2}(s) = 0$ for all $s \geq 5$. This combined with the earlier established fact that $\xi_{e_1}(s) = 0$ for all $s \geq 2$ implies that $\psi_0(t) = 0$ for all $t \geq 5$, and hence the network is not transferring by Proposition 1.

It is interesting to note the reason for the non-transferring of the network in light of Theorem 2. The ultimate cause for the non-transferring property of the network is the non-transferring nature of the sub-network rooted at node 2. The maximum inflow to the sub-network, which is a tree of depth 2, is 0.55. Note that (7) implies that $R_2(0.55) = 1.95$ which is greater than the total disturbance on the sub-network, $\delta_{e_6} = 1.8$. Therefore the non-transferring of the sub-network gives an impression of an inconsistency with Theorem 2 applied to the sub-network rooted at node 2. It is important however to note that Theorem 2 is valid for a *constant* inflow, and in our example the inflow at the sub-network is 0.4 for $t \leq 2$ and 0.55 thereafter. This phenomenon of non-transferring property of the sub-network under a dynamic inflow versus transferring property under a constant inflow that upper bounds the dynamic inflow is attributed to the non-monotonicity of the routing policy at node 2. In summary, it is the non-monotonicity property of routing policy satisfying (6) that prevents generalization of Theorem 2 to trees with depth greater than 2.

IV. PROOF OF THEOREM 1

First consider a depth-1 tree of arbitrary breadth $|\mathcal{E}_0^+| \geq 1$, and in particular for breadth at most 2. Then, for all $e \in \mathcal{E}_0^+$, $R_{\tau_e}(x_e) = \infty$ for all $x_e \in [0, C_e]$, so that $R_0(\lambda) = [\sum_{e \in \mathcal{E}_0^+} C_e - \lambda]^+$. The claim then follows simply

by choosing[1] $\delta_e \in [0, C_e]$ for all $e \in \mathcal{E}$ such that $\sum_{e \in \mathcal{E}} \delta_e = [\sum_{e \in \mathcal{E}_0^+} C_e - \lambda]^+$.

Now assume the claim is true for all trees of depth less than or equal to some $n \geq 1$ and breadth at most 2. Consider a depth-$(n+1)$ tree of breadth $|\mathcal{E}_0^+| = 1$. Then $R_0(\lambda) = \min\{[C_e - \lambda]^+, R_{\tau_e}(\lambda)\}$ where e is the unique element of $\mathcal{E}_0^+$. If $R_0(\lambda) = [C_e - \lambda]^+$, then choose a disturbance with $\delta_e = [C_e - \lambda]^+$, and $\delta_j = 0$ for all $j \in \mathcal{E} \setminus \{e\}$; otherwise, choose a disturbance such that $\delta_e = 0$, and δ coincides on $\mathcal{E} \setminus \{e\}$ with an (optimal) disturbance of size $R_{\tau_e}(\lambda)$ which makes the depth-n subtree rooted in τ_e with inflow $\lambda_{\tau_e} = \lambda$ (whose existence is guaranteed by the induction hypothesis) not transferring. This proves that the claim is true for all trees of depth $n+1$ and $|\mathcal{E}_0^+| = 1$.

Now, consider a tree $\mathcal{T}$ of depth-$(n+1)$ and breadth $|\mathcal{E}_0^+| = 2$. If $R_0(\lambda) = [\sum_{e \in \mathcal{E}_0^+} C_e - \lambda]^+$, then just choosing a disturbance δ such that $\delta_e \in [0, C_e]$ for all $e \in \mathcal{E}_0^+$ with $\sum_{e \in \mathcal{E}_0^+} \delta_e = [\sum_{e \in \mathcal{E}_0^+} C_e - \lambda]^+$, and $\delta_e = 0$ for all $e \in \mathcal{E} \setminus \mathcal{E}_0^+$, suffices to make the network not transferring under any routing policy at node 0. Otherwise, consider a minimizing sequence $\{\gamma\} \subseteq \prod_{e \in \mathcal{E}_0^+} [0, C_e]$ for the infimization problem in the right-hand side of (5) for $v = 0$. For all γ in the sequence, let the disturbance δ^γ coincide with γ on $\mathcal{E}_0^+$, and let $y^* := G^0(\mathbf{1}, \gamma, \lambda)$ for an arbitrary distributed routing policy $\{G^v\}_{v \in \mathcal{V}}$. Let $j \in \mathcal{E}_0^+$ be such that $S_0(\gamma, \lambda) \geq R_{\tau_j}(y_j^*) + S_i(\gamma, \lambda)$, where i is the unique element in $\mathcal{E}_0^+ \setminus \{j\}$. Let $\mathcal{T}_j$ be the depth-n subtree rooted in τ_j, and $\mathcal{T}_i$ be the depth-n subtree rooted at τ_i. Note that this implies that $\mathcal{E} = \mathcal{E}_0^+ \cup \mathcal{T}_i \cup \mathcal{T}_j$. Complete the perturbation δ^γ by choosing its restriction on $\mathcal{T}_j$ to have magnitude $R_{\tau_j}(y_j^*)$ and to make this subtree non-transferring when its inflow is y_j^* (whose existence is guaranteed by the induction hypothesis on n); and, on $\mathcal{T}_i$, it has magnitude $S_i(\gamma, \lambda)$ and makes the union of these subtrees non-transferring when its inflow is λ (whose existence is guaranteed by the inductive hypothesis on k). Under this disturbance, the routing policy at node 0 will not change the way it splits its flow λ until the inactivation of some link $e \in \mathcal{E}_0^+$. If the first link in $\mathcal{E}_0^+$ to become inactive is i, the inflow on link j after that is $\lambda \geq y_j^*$. Since the restriction of δ^γ on $\mathcal{T}_j$ has been chosen to make $\mathcal{T}_j$ non-transferring when its inflow is y_j^*, it will also be non-transferring when the inflow on it is $\lambda \geq y_j^*$ by the monotonicity of $R_{\tau_j}(x)$. If i does not become inactive before j, then j necessarily becomes inactive because of the choice of δ^γ restricted to $\mathcal{T}_j$. Thereafter, the flow on link i is λ, and it fails because the choice of δ^γ restricted to $\mathcal{T}_i$ is such that $\mathcal{T}_i$ is non-transferring when the inflow to it is λ. This makes $\mathcal{T}$ non-transferring. The claim now follows by observing that $\limsup_\gamma \|\delta^\gamma\|_1 \leq R_0(\lambda)$.

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[1]Rigorously speaking, we need to choose δ_e to be arbitrarily larger than the specified value in order to cause link inactivation. This changes the value of margin of resilience by an arbitrarily small amount. Hence, for conciseness in presentation, we do not deal with this issue explicitly and rather implicitly assume that the disturbances are chosen to be arbitrarily larger than the specified values in this proof.

V. Proof of Theorem 2

In the following, we prove that for any $\delta < R_0(\lambda)$, a flow network over a tree of depth at most 2 is transferring under a routing policy satisfying (6). This combined with Theorem 1 establishes Theorem 2.

The proof for trees of depth 1 is trivial. For trees with depth 2, we proceed by induction on number of links in $|\mathcal{E}_0^+|$. Assume that the claim is true for all trees of depth 2 with $|\mathcal{E}_0^+| \leq k$, for some $k \geq 1$. Consider a tree of depth 2 with $|\mathcal{E}_0^+| = k+1$. For $1 \leq j \leq k+1$, let e_j be the outgoing links from 0, and $v_j := \tau_{e_j}$ be the head nodes of those links.

Consider a disturbance $\delta \in \mathcal{F}(\lambda)$ such that $\delta < R_0(\lambda)$. (5) then implies that

$$\delta < \sum_{1 \leq j \leq k+1} \gamma_j + S_{\mathcal{E}_0^+}(\gamma, \lambda), \qquad (8)$$

for all $\gamma \in \mathcal{F}_0$. Following this observation, for the rest of the proof, we shall use (8) for a fixed $\gamma_j := \delta_{e_j}$ and let $\delta^j := \sum_{e \in \mathcal{E}_{v_j}^+} \delta_e$ and $\overline{\delta}^j := \sum_{i \neq j} \sum_{e \in \mathcal{E}_{v_i}^+} \delta_e$. In particular, (8) implies that $\sum_{1 \leq j \leq k+1} \gamma_j < \sum_{1 \leq j \leq k+1} C_{e_j} - \lambda$. For $1 \leq j \leq k+1$, the tree rooted at v_j is of depth 1, and hence $R_{v_j}(\mu) = \left[\sum_{e \in \mathcal{E}_{v_j}^+} C_e - \mu \right]^+$, and under a routing policy satisfying (6) at v_j,

$$\delta^j < R_{v_j}(f_{e_j}(t)) \ \forall t \geq 0 \implies \xi^{v_j}(t) = \mathbf{1} \ \forall t \geq 0. \quad (9)$$

When the routing policy at 0, $G^0(\mathbf{1}, \delta^0, \lambda)$, also satisfies (6), then (4), (8), and (6) imply that

$$\delta^j + \overline{\delta}^j = \delta - \sum_i \gamma_i < R_{v_j}\left(G_{e_j}^0(\mathbf{1}, \delta^0, \lambda) \right) + S_{\mathcal{E}_0^+ \setminus \{e_j\}}(\gamma, \lambda), \quad (10)$$

for all $1 \leq j \leq k+1$, and the summation is over $1 \leq i \leq k+1$. First, consider the case when $\delta^j < R_{v_j}\left(G_{e_j}^0(\mathbf{1}, \delta^0, \lambda) \right)$ for all $1 \leq j \leq k+1$. Then, it follows from (9) that, for all $1 \leq j \leq k+1$, $\xi_{e_j}(t) = 1$ for all $t \geq 0$ and hence the network is transferring. On the other hand, assume that $\delta^i \geq R_{v_i}\left(G_{e_j}^0(\mathbf{1}, \delta^0, \lambda) \right)$ for some $i \in \{1, 2, \ldots, k+1\}$, while (10) holds true for $j \in \{1, 2, \ldots, k+1\} \setminus \{i\}$. In this case, (10) implies that

$$\overline{\delta}^i < S_{\mathcal{E}_0^+ \setminus \{e_i\}}(\gamma, \lambda). \qquad (11)$$

Then, (9) implies that no link $e_j \in \mathcal{E}_0^+ \setminus \{e_i\}$ becomes inactive before e_i possibly becomes inactive, while (11) and the induction hypothesis on $k = |\mathcal{E}_0^+ \setminus \{e_i\}|$ guarantee that, even when possibly e_i becomes inactive, the rest of the network will continue to remain transferring under constant inflow λ. The case when $\{j : \delta^j \geq R_{v_j}(G^0(\delta^0, \mathcal{E}_0^+, \lambda))\}$ has cardinality 2 or more can be handled in a similar way, thus showing that, under routing policies satisfying (6), the network is transferring under any disturbance $\delta < R_0(\lambda)$.

VI. Conclusions and Future Work

In this paper, we considered deterministic cascade dynamics for flow networks over tree topologies between a single origin-destination pair under distributed routing policies, where the network state consists of flows *and* activation status of the links. We studied resilience of such network dynamics to disturbances that reduce link-wise capacities. The margin of resilience is defined as the minimum, among all possible disturbances, of the link-wise sum of reductions in flow capacities, under which the links outgoing from the origin become inactive in finite time. We proposed an algorithm to compute an upper bound on the margin of resilience for tree topologies of breadth at most 2, and show that the resultant value is tight for trees with the additional property of having depth at most 2.

In future, we plan to consider alternate routing policies and algorithms for tighter characterization of margins of resilience on arbitrary tree topologies and then extending it to general network topologies, multi-commodity flows, stochasticity in the cascade dynamics and probabilistic disturbances. We also plan to extend the formulation here to continuous time framework in the context of dynamical flow networks as pursued in our previous and current work [1], [2], [16].

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