TURIN POLYTECHNIC

First School of Engineering
M.Sc. Mathematical Engineering

Local Coordination-Global Congestion

Network Dynamics

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Introduction

In 1994, being inspired by the *El Farol* bar in Santa Fe, New Mexico, the economist Brian W. Arthur introduced the following problem (Arthur [3]):

"N people decide independently each week whether to go to a bar that offers entertainment on a certain night. For correctness, let us set N at 100. Space is limited, and the evening is enjoyable if things are not too crowded - specifically, if fewer than 60 percent of the possible 100 are present. There is no sure way to tell the numbers coming in advance; therefore a person or an agent goes (deems it worth going) if he expects fewer than 60 to show up or stays home if he expects more than 60 to go."

By introducing network structure above problem can be refined as follows. Every agent is represented by a node of the network and edges connecting nodes represent friend relationships:
I'm interested in going to the bar in case my friends will be there; however a too crowded place is discouraging me to get there.
Above new situation can be seen as an instance of a more general problem. To introduce this new general problem we need to define "externality" as any situation in which the welfare of an individual if affected by the actions of other individuals without a mutually agreed-upon compensation (Easley et al. [11]). For example, benefits for a single user of a social networking website are directly related the total number of users. His welfare is increased by each new user even though no explicit compensation accounts for this. Externalities can be of the positive type as above or negative, i.e. road users in an urban traffic situation. Problem definition is now as follows.
Agents have positive externalities from directly connected neighbours while at the same time suffer negative externality from the totality of the agents within the network.
In the sequel we will refer to this situation as a local coordination and global congestion case. Among possible applications the case presented in Mayer and Sinai paper (see [5]) about an air traffic problem can be considered: in hub airport the dominating hub airline might increase his marginal profits simply adding new market connections, each of them exponentially increasing connecting routes of the hub airline. However those benefits may be equated by marginal costs due to consequent major delays. A certain degree of delay is then necessarily

expected to represent the equilibrium outcome of an hub airline.
Within the paper this case has been defined as *tragedy of commons* and can be extended to each situation where individuals suffer the inconvenience of sharing public services with many other people in order to share common time with colleagues, friends and relatives. Other application cases are given herein.
Companies often benefit in case cost-reducing innovations are adopted by suppliers or other business partners. Nevertheless too many adopters may give rise to a negative externality.
Similarly, when choosing a location firms may prefer clusters with other firms in order to benefit from sharing local indivisible facilities. A congestion problem can arise in case of excess of cluster increase with the consequent reduction of the initial advantage of sharing the same location. Also social sciences offer similar application cases, i.e. the adoption of particular behaviors or beliefs, by young people aiming to be member of unique and exclusive groups.
Game theory can now be introduced to model the situation of local coordination and global congestion described above. Agents are seen as players involved in two games: the first with neighbouring connected players, the second with the total population. With reference to our initial *El Farol* problem following assumptions will be adopted. Players share the same two strategies. Furthermore they have no individual behavior. When evaluating a context single player does not care about who has a certain strategy. He only considers two percentages: the number of players having this strategy over the total number of neighbouring players and over the total number of players. Given those percentages there are no differences in single player's behavior when taking decisions. Above assumption about homogeneity of players never precludes that they may have a certain inertia in changing strategy. For example in *El Farol* case a player already in the bar can be influenced by the same fact of staying in the bar when taking his decision.
Results presented herein will deal with large size N of population. Traditional game-theory analyses deal with games in a static perspective: all configurations of strategies are evaluated the general assumption being that players follow some Nash equilibrium of the game at hand. Here because of the large number of players involved we approach the games in a dynamical perspective: at each time step a randomly selected player can adjust his strategy according to the current situation. We will consider how strategy configurations will result in long time perspective. As dynamic perspective is now introduced further analysis could be performed. Still remain to understand how players are currently refining their strategy. To sum up selected player will ground his decision on following data:

- the percentage of neighbouring players having a given strategy over the total of neighbouring players;
- the percentage of neighbouring players having a given strategy over the total of players;
- the current strategy of the player.

Players will behave in a probabilistic way on the base of above three inputs. This 'myopic' character of the players who evaluate only present situation of strategies when taking their decision, let us study our dynamic in a Markov Chains perspective. In the sequel strategy updates will be characterized in such a way that our Markov Chains will be irreducible and aperiodic. This will allow us to study their limiting time behavior through their unique invariant probability measure forgetting about initial conditions.
The thesis will procede as follows: in the first chapter we introduce the mathematical model of our dynamic and of the games on which this dynamic is based. We will then present particular tools for studying its behavior.
The second chapter will be focused on a particular type of coordination on a complete graph structure. The way players consider the congestion effect can be very general: interesting results emerge from their 'inertia' in changing strategies. This inertia will not be present in third chapter. Here, the peculiarity of our dynamics is that every player will be equally predisposed to the two strategies he can assume. In particular, given a certain strategy configuration, we define its symmetric form as the situation where each player has the opposite strategy. In third chapter we will assume that the probability that the selected player adopts a strategy under a certain strategy configuration will be equal to the probability that he adopts the other in the symmetric case. We call this situation 'symmetry of strategies'. We will provide results regarding network structures made of one and two complete connected components.
In the last chapter we will conclude with simulations based on the outcomes of chapters two and three.

Contents

Chapter 1

Games and Markov Chains

1.1 Games on Network

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an undirected graph with a finite set of vertices $\mathcal{V}$ of cardinality N. Self loops won't be considered in our work.
Nodes will represent players: for every player v we define the set of its neighbors as

$$N_v = \{w \in \mathcal{V} \ s.t. \ \{v, w\} \in \mathcal{E}\}$$

which has cardinality d_v.
Furthermore, every player v has a set of strategies $\mathcal{X}$ with generic element s_v. Given a strategy configuration $\underline{s} = (s_1, \ldots, s_N) \in \mathcal{X}^{\mathcal{V}}$ we define a function

$$u_{v,\mathcal{G}} : \mathcal{X}^{\mathcal{V}} \to \mathbb{R}$$

such that $u_{v,\mathcal{G}}(s_1, \ldots, s_N)$ measures the satisfaction of player v if the strategy configuration $\underline{s}$ gets realized. $u_{v,\mathcal{G}}$ is called player v's payoff function.

Definition 1.1.1. *Given an undirected graph $\mathcal{G}$ with N nodes, a game on $\mathcal{G}$ is a set*

$$B = \{\mathcal{X}, u_{1,\mathcal{G}}, \ldots, u_{N,\mathcal{G}}\}. \tag{1.1}$$

In our work we deal with games with a binary set of strategies, namely

$$\mathcal{X} = \{0, 1\}.$$

Furthermore, we assume that the way the graph affects our game is that every player is only influenced by its neighborhood. This assumption and the homogeneity of players leads us to the following simplifications: given a positive integer P, let

$$F_P = \left\{ \frac{k}{P} : k = 0, \ldots, P \right\} \tag{1.2}$$

be a discrete set contained in the interval $[0,1]$. Given a node $v \in \mathcal{V}$ and a strategy configuration $\underline{s} = (s_1, \dots, s_N)$, we define v's local fraction of players adopting strategy one as

$$f^v_{loc} = \frac{1}{d_v} \sum_{w \in N_v} s_w \in F_{d_v}. \tag{1.3}$$

Then

$$u_{v,\mathcal{G}}(s_1, \dots, s_N) = p_{s_v}(f^v_{loc}).$$

Notice that, with this simplification, every player is characterized by its current strategy and its position in the network $\mathcal{G}$. All the information required for the definition of game B is given by $\mathcal{G}$ and by the two payoff functions p_0 and p_1. We will be then interested in defining these payoff functions for our local coordination and global congestion games.
Given a positive integer P, let K_P be the complete graph with M vertices. In case $\mathcal{G} = K_2$ the payoffs can be provided through a matrix $A \in \mathbb{R}^{\mathcal{X} \times \mathcal{X}}$ such that, given $s, s' \in \mathcal{X}$,

$$A_{ss'} = p_s(s').$$

These binary games are the base for more complex situations. Indeed if we consider an arbitrary set of nodes $\mathcal{V}$ and two players $v, w \in \mathcal{V}$, we say that they coordinate if they play a binary game with

$$A^{coor} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and anticoordinate if they play a binary game with

$$A^{cong} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The concept of local coordination that we adopt in our work directly follows from the binary coordination game. In this situation, a player v plays a unified coordination game with respect to all his neighborhood: the payoffs of such unified game will be averages of the ones related to the binary coordination games corresponding to each edge involving v.
In particular, given a strategy $s \in \mathcal{X}$, the local coordination payoff will be

$$p^l_s(f^v_{loc}) = A^{coor}_{s0}(1 - f^v_{loc}) + A^{coor}_{s1} f^v_{loc} \quad \forall s \in \mathcal{X}. \tag{1.4}$$

Similarly, a global congestion game is a unified anticoordination game that a vertex v plays with all the other vertices. This 'globality' can be obtained by considering the neighborhood of a complete graph K_N. Given a strategy configuration $\underline{s} = (s_1, \dots, s_N)$ we define, for every $v \in \mathcal{V}$,

$$f^v = \frac{1}{N-1} \sum_{w \in V \setminus \{v\}} s_w \in F_{N-1}. \tag{1.5}$$

The payoffs of the global congestion game will be

$$p_s^l(f^v) = A_{s0}^{cong}(1 - f^v) + A_{s1}^{cong} f^v \quad \forall s \in \mathcal{X}. \tag{1.6}$$

Notice that, once a topology is specified, our games will be well defined. From now on we will assume every player to be involved in this two games of local coordination and global congestion. The way this involvement occurs and the players behave will be discussed case by case.

1.2 The Dynamic

Once defined the games we should define the dynamic that describes how they are played. It proceeds as follows: at every discrete time step $t \in \mathbb{N}$ a player v is randomly selected and adopts a new strategy in a probabilistic way depending on its current one and the payoffs of the local coordination and global congestion games he is involved in. In particular, let $f_{loc}^v(t)$ and $f^v(t)$ be the local and global fraction of strategy one players for node v at time t: $p_s^l(t)$ and $p_s^g(t)$ will be their associated local coordination and global congestion payoffs. Given $s \in \mathcal{X}$ and the function

$$\tilde{\Phi}_s : [-1, 1]^2 \to [0, 1], \tag{1.7}$$

our player v currently adopting strategy s will have strategy 1 at the time step $t + 1$ according to the probability

$$\tilde{\Phi}_s(p_1^l(t) - p_0^l(t), p_1^g(t) - p_0^g(t)). \tag{1.8}$$

To be coherent with the payoffs we will assume $\tilde{\Phi}_s$ non decreasing in its variables. Since from (1.4)

$$p_1^l(t) - p_0^l(t) = 2f_{loc}^v(t) - 1$$

and from (1.6)

$$p_1^g(t) - p_0^g(t) = 1 - 2f^v(t),$$

we can simplify the notation by defining a function Φ_s such that

$$\Phi_s(f_{loc}^v(t), f^v(t)) = \tilde{\Phi}_s(p_1^l(t) - p_0^l(t), p_1^g(t) - p_0^g(t)).$$

This will be non decreasing in its first variable and non increasing in its second one. For an operative simplification in the sequel we will assume some regularities on Φ_s. The definition is formalized as follows.

Definition 1.2.1. *(Local Coordination-Global Congestion interaction kernels) A Local Coordination-Global Congestion interaction kernel for a strategy $s \in \mathcal{X}$ is a function*

$$\Phi_s : [0, 1]^2 \to [0, 1] \qquad \Phi_s \in C^1([0, 1]^2)$$

that satisfies following conditions:

- $\dfrac{\partial \Phi_s}{\partial x_1}(x_1, x_2) \geq 0$ *in* $[0,1]^2$ *(local coordination);*
- $\dfrac{\partial \Phi_s}{\partial x_2}(x_1, x_2) \leq 0$ *in* $[0,1]^2$ *(global congestion).*

In the sequel we will simply use the term interaction kernels when referring to Local Coordination-Global Congestion interaction kernels.
Our dynamic can be modeled as a discrete time Markov Chain on the global configuration space $\mathcal{X}^{\mathcal{V}}$. The transition probability matrix P will be such that given $\underline{s}$ and $\underline{s}' \in \mathcal{X}^{\mathcal{V}}$, $P_{\underline{s},\underline{s}'} = 0$ if $\underline{s}$ and $\underline{s}'$ differ in more than one component. Otherwise, if $\underline{s}$ and $\underline{s}'$ differ in the only component v we define

- $P_{\underline{s},\underline{s}'} = \dfrac{1}{N}\Phi_0\left(\dfrac{1}{d_v}\displaystyle\sum_{w \in N_v} s_w, \dfrac{1}{N-1}\sum_{w \in V\setminus\{v\}} s_w\right)$ if $s_v = 0$ and $s_{v'} = 1$;
- $P_{\underline{s},\underline{s}'} = \dfrac{1}{N}\left(1 - \Phi_1\left(\dfrac{1}{d_v}\displaystyle\sum_{w \in N_v} s_w, \dfrac{1}{N-1}\sum_{w \in V\setminus\{v\}} s_w\right)\right)$ if $s_v = 1$ and $s_{v'} = 0$.

To complete the definition of the stochastic matrix P we set $P_{\underline{s},\underline{s}} = 1 - \displaystyle\sum_{\underline{s}' \neq \underline{s}} P_{\underline{s},\underline{s}'}$.
Let $\underline{S}(t)$ be the random variable representing the strategy configuration at time t. The sequence $(\underline{S}(t))_{t \geq 0}$ will be a Markov Chain with conditional probabilities

$$\mathbb{P}(\underline{S}(t+1) = \underline{s}' | \underline{S}(t) = \underline{s}) = P_{\underline{s},\underline{s}'}.$$

Notice that our dynamic will be well defined once provided a network $\mathcal{G}$, two interaction kernels Φ_0 and Φ_1 and an initial condition $\underline{S}(0)$.
To study the limiting time behavior of our players, we will consider the states of our Markov Chains for large t. In the sequel we will always deal with cases where

$$0 < \Phi_s(x_1, x_2) < 1 \qquad \forall (x_1, x_2) \in [0,1]^2. \tag{1.9}$$

This implies that aperiodic and irreducible Markov Chains will be matters of our studies. In particular, each of them will have a unique limiting time behavior independent from the initial condition. This limiting time behavior coincides with the unique invariant probability measure of the Markov Chain (see Levin et al.[8]).

1.3 Mean Field Markov Chains

Let d be a divisor of N. in the sequel we will assume that every graph underlying our dynamics will be a union of d complete connected components K_M, where $M = \dfrac{N}{d}$; each connected component represents a group of players.
Given the homogeneity of the players, we can convey all the information of a

general state $\underline{s} \in \mathcal{X}^{\mathcal{V}}$ of P in the fraction of players adopting strategy 1 in each connected component. Given a strategy configuration $\underline{s}$ and $j \in \{1, \dots, d\}$ let $\underline{s}^j \in \mathcal{X}^M$ be the vector that keeps trace of the strategies of the players of the j-th connected component: $\forall i \in \{1, \dots, M\}$, $\underline{s}^j_i$ will be the strategy of the i-th player of the j-th group.
For every $\underline{s}^j \in \mathcal{X}^M$ we define

$$f_j = \frac{1}{M} \sum_{i=1}^{M} \underline{s}^j_i \in F_M.$$

Furthermore, consider a general positive integer $P > 0$. Given any $f \in F_P$, let

$$f_{-s} = \left(f - \frac{s}{P}\right) \frac{P}{P-1} \in F_{P-1}$$

be the fraction of strategy 1 players represented by f with the esclusion of a node with strategy s. Given the original Markov Chain process of strategies in $\mathcal{X}^{\mathcal{V}}$ one can consider the corresponding process $\underline{\rho}(t) \in F_M^d$

$$\underline{\rho}(t) = (f_1(t), \dots, f_d(t)).$$

This process is also a Markov Chain on F_M^d: given $\underline{f} = (f_1, \dots, f_d)$ and $\underline{f}' = (f_1', \dots, f_d') \in F_M^d$ we set

$$Q_{\underline{f},\underline{f}'} = 0 \quad \text{if } ||\underline{f} - \underline{f}'||_1 > \frac{1}{M}.$$

Otherwise, if $\underline{f}$ and $\underline{f}'$ differs of $\dfrac{1}{M}$ in the component j then

$$Q_{\underline{f},\underline{f}'} = \begin{cases} \dfrac{1}{d}(1 - f_j)\Phi_0\left(f_{j,-0}, f_{-0}\right) & \text{if } f_j' = f_j + \dfrac{1}{M} \\[2ex] \dfrac{1}{d} f_j \left(1 - \Phi_1\left(f_{j,-1}, f_{-1}\right)\right) & \text{if } f_j' = f_j - \dfrac{1}{M} \end{cases} \tag{1.10}$$

where $f = \dfrac{1}{d} \displaystyle\sum_{j=1}^{d} f_j \in F_N$. To conclude

$$Q_{\underline{f},\underline{f}} = 1 - \sum_{\underline{f}' \in F_M^d \setminus \{f\}} Q_{\underline{f},\underline{f}'}.$$

By construction we have that

$$\mathbb{P}\left(\underline{\rho}(t+1) = \underline{f}' \,\middle|\, \underline{\rho}(t) = \underline{f}\right) = Q_{\underline{f},\underline{f}'}. \tag{1.11}$$

This shows that $(\underline{\rho}(t))_{t \geq 0}$ is a Markov Chain with transition probability matrix given by Q. Furthermore we can easily observe that Q will inherit the

irreducibility and the aperiodicity of P: as in the original Markov Chain, its limiting time behavior will coincides with its invariant probability.
At a computational level Q is much simpler than P. Indeed there exists a positive scalar λ such that the state space of Q grows polynomially in N while the state space of P has cardinality 2^N. The advantage of working with Q is also theoretical: it allows us to study the behavior of its associated random process for large N with the tools presented in following section.

1.4 Kurtz's Approximation

In this section we will introduce a useful method to study the behavior of the Mean Field Markov Chains when the number N of players is large. We will then denote such Markov Chains by $(\underline{\rho}^{(N)}(t))_{t\geq 0}$, their conditional probability matrices as $Q^{(N)}$ and their invariant probability measures as $\mu^{(N)}$.
Consider $F^{(N)} : F_M \times F_N \to \mathbb{R}$

$$F^{(N)}(f_j, f) = f_j \Phi_1(f_{j,-1}, f_{-1}) + (1 - f_j)\Phi_0(f_{j,-0}, f_{-0}) - f_j$$

where $j \in \{1, \ldots, M\}$.
We have that

$$\mathbb{E}[\underline{\rho}^{(N)}(t+1)|\underline{\rho}^{(N)}(t)] = \underline{\rho}^{(N)}(t) + \frac{1}{N}\sum_{j=1}^{d} e_j F^{(N)}\left(\rho_j^{(N)}(t), \rho^{(N)}(t)\right)$$

where $\rho^{(N)}(t) = \frac{1}{d}\sum_{j=1}^{d}\rho_j^{(N)}(t)$ and $(e_j)_{j=1\ldots d}$ is the canonical base of $\mathbb{R}^d$.
This means that $\rho^{(N)}(t)$ evolves according to the following dynamic:

$$\underline{\rho}^{(N)}(t+1) = \underline{\rho}^{(N)}(t) + \frac{1}{N}\sum_{j=1}^{d} e_j F^{(N)}\left(\rho_j^{(N)}(t), \rho^{(N)}(t)\right) + \frac{1}{N}\sum_{j=1}^{d} e_j n_j(t) \quad (1.12)$$

where $n_j(t)$ are random variables such that $\mathbb{E}[n_j(t)|\underline{\rho}^{(N)}(t)] = 0$. They measure how much the real process departs from its conditioned average. If we consider a continuous time rescaling by defining $\underline{\tilde{\rho}}^{(N)}(\tau) = \underline{\rho}^{(N)}(\lfloor \tau N \rfloor)$ where $\tau \in [0, +\infty)$, dynamic (1.12) can be rewritten as

$$\frac{\underline{\tilde{\rho}}^{(N)}\left(\tau + N^{-1}\right) - \underline{\tilde{\rho}}^{(N)}\left(\tau\right)}{N^{-1}} = \sum_{j=1}^{d} e_j F^{(N)}\left(\tilde{\rho}_j^{(N)}(\tau), \tilde{\rho}^{(N)}(\tau)\right) + \sum_{j=1}^{d} e_j \tilde{n}_j(\tau)$$

where $\tilde{n}_j(\tau) = n_j(\lfloor \tau N \rfloor)$ and $\tilde{\rho}^{(N)}(\tau) = \frac{1}{d}\sum_{j=1}^{d}\tilde{\rho}_j^{(N)}(\tau)$.

Theorem 1.4.1. *(Kurtz's theorem)*
Suppose that $F^{(N)}$ converges uniformly to a Lipschitz continuous function $F : [0,1]^2 \to \mathbb{R}$ i.e. $\forall \varepsilon > 0\ \exists N_0 \in \mathbb{N}$ s.t.

$$|F^{(N)}(f_j, f) - F(f_j, f)| < \varepsilon \quad \forall N \geq N_0, \quad \forall (f_j, f) \in F_M \times F_N. \tag{1.13}$$

Assume that $\lim_{N\to+\infty} \underline{\rho}^{(N)}(0) = \underline{p}_{\,0} \in [0,1]^d$ and let $\underline{z} : \mathbb{R} \to [0,1]^d$ be the vector whose components z_j are the solutions of the Cauchy problem

$$\begin{cases} \dot{z}_j(\tau) = F(z_j(\tau), z(\tau)) \\ z_j(0) = \underline{p}_{\,0,j} \end{cases} \quad \forall j \in \{1, \dots, d\} \tag{1.14}$$

where $z(\tau) = \frac{1}{d}\sum_{j=1}^{d} z_j(\tau)$.
Then $\forall \varepsilon > 0\ \exists\ c_1, c_2 > 0$ such that $\forall N \in \mathbb{N}$ and $\forall T > 0$

$$\mathbb{P}\left(\sup_{0\leq t\leq T} ||\underline{\rho}^{(N)}(\tau) - \underline{z}(\tau)|| \geq \varepsilon\right) \leq c_1 e^{-c_2 N \frac{\varepsilon^2}{T}}.$$

Notice that, given $j \in \{1, \dots, d\}$, z_j represents the continuous fraction of players that adopts strategy 1 in the j-th connected component. In the sequel, we will refer to

$$\dot{z}_j(\tau) = F(z_j(\tau), z(\tau)) \quad \forall j \in \{1, \dots, d\} \tag{1.15}$$

as Kurtz's system of ODE.
Theorem (1.4.1) can be applied to our case. Indeed $F^{(N)}$ converges uniformly to $F : [0,1] \to \mathbb{R}$,

$$F(x_1, x_2) = x_1\Phi_1(x_1, x_2) + (1 - x_1)\Phi_0(x_1, x_2) - x_1$$

that, given the hypotesis on our interaction kernel, belongs to $C^1([0,1])$. This implies that for any initial condition in $\underline{p}_{\,0} \in [0,1]^d$ Kurtz's system of ODE admits a unique solution.
This continuous approximation is extremely effective since it allows us to study our discrete dynamics through differential equations on the compact set $[0,1]^d$. However, notice that Kurtz's theorem utility is restricted to small periods of time. To study the limiting time behavior of our dynamics we consider the result presented hereafter. When N is large, this provides us important informations about the invariant probability $\mu^{(N)}$ through Kurtz's system of ODE.

Lemma 1.4.2. *Let $(\mu^{(N)})_{N=N_0}^{+\infty}$ be the sequence of stationary distributions for the mean field Markov Chain $Q^{(N)}$. Furthemore, let $R \in [0,1]^d$ be the set of recurrent points of the related Kurtz's system of ODE. Given any open set $\mathcal{O} \subset \mathbb{R}^d$ containing $cl(R)$,*

$$\lim_{N\to+\infty} \mu^{(N)}(\mathcal{O}) = 1.$$

More informations regarding lemma (1.4.2) can be found in section 12B in Hofbauer et al. [9].
Since in this thesis we will treat large populations of players, continuous approximations will be fundamental. As we will see in next chapter lemma (1.4.2) can be refined when $\mathcal{G} = K_N$. Indeed, in this case, we have a closed formula for the invariant probability measure. This formula will allow us to restrict the set of recurrent points to the one of the locally stable equilibria of Kurtz's ODE. Furthermore, in case this set has a positive measure, we will see how $\mu^{(N)}$ will be distributed there in the particular case presented. On the other hand, lemma (1.4.2) will play a key role in chapter three, where more than one group of players will be taken into account.

Chapter 2

Voter Coordination

2.1 Interaction Kernels

In this chapter we will deal with interaction kernels that combine coordination and congestion effects in the following way: with a probability $p \in (0,1)$ the player acts in a coordination perspective, while, with a probability $1-p$, he will consider the congestion influence.
If he will take into account the coordination game he will adopt strategy 1 with a probability equal to the fraction of strategy 1 players in his neighborhood. This is the case of voter coordination: the interpretation is that the selected player chooses randomly an edge and copies the strategy of the linked node.
The probability to adopt strategy 1 when considering the congestion situation can be very general: it will depend not only on the related payoffs but also on the current strategy of the selected player. In the sequel we will see that in case our players are very 'lazy' (i.e. there are situations where no changes of strategy are possible for congestion effects) the invariant probability measures will assume particular distributions.
In this chapter we will consider a unique topology $\mathcal{G} = K_N$: this implies that concepts of locality and globality will not be present and, given a node v, $f^v_{loc} = f^v$. The interaction kernels (1.2.1) will then depend on the unique variable z. They will assume the form

$$\Phi_s(z) = pz + (1-p)g_s(z) \tag{2.1}$$

where

$$g_s : [0,1] \to [0,1], \quad g_s \in C^1([0,1])$$

is the congestion function related to strategy s. $g_s(z)$ represents the probability that a strategy s player adopts strategy 1 under congestion effects when a fraction z of players is assuming strategy 1. In particular we want

- $\dot{g}_s(z) \leq 0 \quad \forall z \in [0,1]$ (congestion effect);
- $g_s(0) > g_s(1)$ to avoid trivial cases.

This graph structure implies that our Mean Field Markov Chains will be defined on the state space F_N. In particular, since $Q^{(N)}_{f,f'} = 0$ if $|f - f'| > \dfrac{1}{N}$ we can use following notation for the only positive non diagonal elements of the transition matrix:

$$q^{+(N)}(f) = Q^{(N)}_{f,f+\frac{1}{N}}, \quad q^{-(N)}(f) = Q^{(N)}_{f,f-\frac{1}{N}}.$$

Substituting $Q^{(N)}$ with its expression (1.10), we have

$$q^{+(N)}(f) = (1-f)(pf_{-0} + (1-p)g_0(f_{-0})) \quad \forall f \in F_N \setminus \{1\} \tag{2.2}$$

and

$$q^{-(N)}(f) = f(p(1-f_{-1}) + (1-p)(1-g_1(f_{-1}))) \quad \forall f \in F_N \setminus \{0\}. \tag{2.3}$$

Such Markov Chains are called *birth and death processes* and are extensively studied in probabilistic literature (see Levin et al. [8]). In this case a closed formula for the invariant probability measure $\mu^{(N)}$ can be found. Indeed, for each $f \in F_N$ we define

$$\tilde{\mu}^{(N)}_f = \prod_{j=1}^{fN} \frac{q^{+(N)}\left(f - \frac{1}{N}\right)}{q^{-(N)}(f)}$$

(we set, conventionally, $\tilde{\mu}^{(N)}_0 = 1$).
The invariant probability measure will be

$$\mu^{(N)}_f = \frac{\tilde{\mu}^{(N)}_f}{\sum\limits_{f' \in F_N} \tilde{\mu}^{(N)}_{f'}}. \tag{2.4}$$

Notice that (2.4) implies that, $\forall f \in F_N \setminus \{1\}$,

$$\mu^{(N)}_f q^{+(N)}(f) = \mu^{(N)}_{f+\frac{1}{N}} q^{-(N)}\left(f + \frac{1}{N}\right). \tag{2.5}$$

To study the behavior of the invariant probability when N is large, a result that refines lemma (1.4.2) will be introduced in next section. In particular, it will tell us that $\mu^{(N)}$ can concentrate in a single point or in a interval with positive Lebesgue measure. These events occur depending on the type on congestion expressed by g_0 and g_1. In latter case, the distribution in such interval will be presented.

2.2 Large Population Behavior

Let $z \in [0,1]$ be the continuous fraction of strategy 1 players. Consider the transition probabilities $q^{+(N)}$ and $q^{-(N)}$ as defined in (2.2) and (2.3) and assume that they uniformly converge (in the sense described in (1.13)) to two Lipschitz-continuous functions $q^+(z)$ and $q^-(z)$. Furthermore assume that $\exists A > 0$ and $s \in \mathbb{N}$ such that, for every $N \in \mathbb{N}$, given $f \in F_N$

$$q^{+(N)}(f) \geq A(1-f)^s, \quad q^{-(N)}(f) \geq A(f)^s. \tag{2.6}$$

Thanks to these observations $J : [0,1] \to \mathbb{R}$

$$J(z) = \int_0^z \ln \frac{q^+(\zeta)}{q^-(\zeta)} \, d\zeta \tag{2.7}$$

will be a continuous function on the compact set $[0,1]$.
Let M be the set of its absolute maxima. Next result says that the invariant measure concentrates around M when $N \to +\infty$. Given $\delta > 0$ we put

$$M_\delta \stackrel{def}{=} \{x \in \mathbb{R} : |x - y| \geq \delta \, \forall y \in M\}.$$

We have the following.

Lemma 2.2.1. *$\forall \delta > 0$, $\mu^{(N)}(M_\delta) \to 0$ for $N \to +\infty$.*

Proof. Consider the sequence of step size functions

$$\varphi^{(N)}(z) = \sum_{j=1}^{N} \ln \frac{q^{+(N)}(\frac{j-1}{N})}{q^{-(N)}(\frac{j}{N})} \, \mathbb{1}_{[\frac{j-1}{N}, \frac{j}{N})}(z)$$

and notice that it converges, for $N \to +\infty$, to the function $\ln \dfrac{q^+(z)}{q^-(z)}$ in the interval $(0,1)$. Given $f \in F_N$,

$$\frac{1}{N} \ln \frac{\mu_f^{(N)}}{\mu_0^{(N)}} = \frac{1}{N} \sum_{j=1}^{fN} \ln \frac{q^{+(N)}(\frac{j-1}{N})}{q^{-(N)}(\frac{j}{N})} = \int_0^f \varphi^{(N)}(\zeta) \, d\zeta.$$

Assumption (2.6) allows to use the dominated convergence theorem and to conclude that

$$\lim_{N \to +\infty} \frac{1}{N} \ln \frac{\mu_f^{(N)}}{\mu_0^{(N)}} = \int_0^f \ln \frac{q^+(\zeta)}{q^-(\zeta)} \, d\zeta \tag{2.8}$$

uniformly in $f \in [0,1]$.
Let $f^* \in F_N$ be any absolute maximum point of $\mu_f^{(N)}$ and assume that $f^* \to z^* \in [0,1]$ for $N \to +\infty$ (this assumption does not entail any loss of generality since we can eventually work with a subsequence). Because of the uniformity of convergence it is immediate to test that z^* is an absolute maximum point of $J(z)$. Let $\eta > 0$ be such that $J(z^*) - J(z) \geq \eta > 0$ for every $z \in M_\delta$ and notice that exists $N_0 \in \mathbb{N}$ such that

$$\frac{1}{N} \ln \frac{\mu_{f^*}^{(N)}}{\mu_0^{(N)}} - \frac{1}{N} \ln \frac{\mu_f^{(N)}}{\mu_0^{(N)}} \geq \frac{\eta}{2}$$

for every $N \geq N_0$ and for every $f \in M_\delta \cap F_N$. We can now complete the proof: for $N \geq N_0$ and $f \in M_\delta \cap F_N$ we obtain that

$$\frac{1}{N} \ln \mu_f^{(N)} \leq \frac{1}{N} \frac{\mu_f^{(N)}}{\mu_0^{(N)}} - \frac{1}{N} \frac{\mu_{f^*}^{(N)}}{\mu_0^{(N)}} \leq -\frac{\eta}{2}$$

which yelds $\mu_f^{(N)} \leq e^{-N\eta/2}$. Therefore, $\mu^{(N)}(M_\delta) \leq (N+1)e^{-N\eta/2}$ and the result is proven. ■

In our case we have that $q^{+(N)}$ and $q^{-(N)}$ uniformly converge to

$$q^+(z) = (1-z)(pz + (1-p)g_0(z)) \tag{2.9}$$

and

$$q^-(z) = z(p(1-z) + (1-p)(1-g_1(z))). \tag{2.10}$$

Notice that both (2.9) and (2.10) belong to $C^1([0,1])$. Furthermore, given the continuity of our congestion functions and the fact that

$$g_s(0) > 0 \text{ and } g_s(1) < 1 \quad \forall s \in \mathcal{X},$$

$\exists \alpha > 0$ s.t. $\forall z \in [0,1]$

$$q^+(z) > \alpha(1-z) \text{ and } q^-(z) > \alpha z.$$

Thank to these observations we can apply lemma (2.2.1) to our case. We can easily observe that the absolute maxima of J have to be searched among its stationary points in $(0,1)$. They will satisfy

$$h(z) = q^+(z) - q^-(z) = (1-z)g_0(z) - z(1-g_1(z)) = 0. \tag{2.11}$$

Furthermore,

$$\dot{h}(z) = (1-z)\dot{g}_0(z) + z\dot{g}_1(z) - (1-(g_1(z)-g_0(z))). \tag{2.12}$$

Considering the definition of congestion function and that

$$g_1(z) - g_0(z) \in [0,1] \quad \forall z \in (0,1),$$

we have that h is a continuous non increasing function in $(0,1)$: the solutions of (2.11) will form a closed interval in $(0,1)$ and will coincide with the absolute maxima of J. Given $0 < a \leq b < 1$ we define

$$I = [a,b] \text{ the interval of absolute maxima of } J. \tag{2.13}$$

Thanks to lemma (2.2.1) we have that $\forall \delta > 0$, when $N \to +\infty$,

$$\mu^{(N)}\left(F_N \cap ([0, a-\delta] \cup [b+\delta, 1])\right) \to 0. \tag{2.14}$$

In order to characterize I we notice that, if

$$g_1(z) - g_0(z) < 1 \quad \forall z \in (0,1),$$

then $h(z)$ will decrease in such domain. In this case $b = a$ and $I = \{a\}$.
Otherwise I coincides with the $z \in (0,1)$ where $g_1(z) - g_0(z) = 1$, i.e. $g_1(z) = 1$ and $g_0(z) = 0$. This latter case verifies when there are situations where no changes of strategies due to congestion are possible. In next section we will investigate how the invariant probability measure will be distributed in I when $b > a$ for then concluding with final results.

2.3 Final Results

Let

$$a_N = \frac{\lceil (N-1)a \rceil}{N} \text{ and } b_N = \frac{\lfloor (N-1)b \rfloor + 1}{N}.$$

Considering that $a > 0$ and $b < 1$ we will definitely have that $a_N > 0$ and $b_N < 1$.
Since $\forall z \in I$ $g_1(z) = 1$ and $g_0(z) = 0$, given $f \in F_N \cap I$, definitions (2.2) and (2.3) lead us to

$$q^{+(N)}(f) = q^{-(N)}(f) = pf(1-f)\frac{N}{N-1}.$$

Iterating (2.5) we have

$$\mu_f^{(N)} = \frac{(1-a_N)a_N}{(1-f)f}\mu_{a_N}^{(N)} \quad \forall f \in F_N \cap I. \tag{2.15}$$

Following lemmas derive from relation (2.15) and the fact that I has a positive measure: the first one states that, for every $f \in F_N \cap I$, the invariant probability measure $\mu_f^{(N)}$ has to decrease at least in the order of $\frac{1}{N}$ when N grows. In the second one we will show that, since

$$q^{+(N)}(f) \geq q^{-(N)}(f) \text{ if } f \leq b$$

and

$$q^{-(N)}(f) \geq q^{+(N)}(f) \text{ if } f \geq a,$$

the invariant probability measure of every fixed interval $I' \subset [0,1]$ is uniformely bounded by the length of such interval.

Lemma 2.3.1. *Suppose that the interval $I = [a,b]$ defined in (2.13) is such that $a < b$. Then $\exists\, c > 0$ such that, for sufficiently large N,*

$$\mu_f^{(N)} \leq \frac{c}{N} \qquad \forall f \in F_N \cap I.$$

Proof. Assume N large enough to have $F_N \cap I \neq \emptyset$. Fix $f \in F_N \cap I$. Given (2.15), $\forall f' \in F_N \cap I$,

$$\mu_{f'}^{(N)} = \frac{(1-f)f}{(1-f')f'}\mu_f^{(N)}.$$

On the other hand, the fact that $\mu^{(N)}$ is a probability measure implies

$$\mu_f^{(N)}(1-f)f \sum_{f'N=a_N N}^{b_N N} \frac{1}{(1-f')f'} \leq 1. \tag{2.16}$$

Let

$$e = \max\left\{\left|a-\frac{1}{2}\right|, \left|b-\frac{1}{2}\right|\right\} < \frac{1}{2}.$$

$\forall g \in F_N \cap I$ we have

$$\frac{1}{(1-g)g} \geq 4$$

and

$$(1-g)g \geq \left(1-\left(\frac{1}{2}+e\right)\right)\left(\frac{1}{2}+e\right) > 0.$$

Thesis follows by considering that, from (2.16), $\exists c' > 0$:

$$\mu_f^{(N)} c'(b_N - a_N + 1)N \leq 1.$$

■

Lemma 2.3.2. *Suppose that the interval $I = [a,b]$ defined in (2.13) is such that $a < b$. Then, for sufficiently large N, $\exists\, k > 0$:*

$$\mu^{(N)}(I') \leq k|I'| \quad \forall I' \subseteq [0,1].$$

Proof. Assume N large enough to have $F_N \cap I \neq \emptyset$. Let $f \in F_N$. Thanks to lemma (2.3.1) if $f \in F_N \cap I$ then $\exists c > 0 : \mu_f^{(N)} \leq \frac{c}{N}$.
If $f < a_N$ we define $q_m^+ = \min_{z \in [0,a]} q^+(z) > 0$ and

$$c_1 = \frac{q^-(a)+1}{q_m^+/2}\, c < +\infty.$$

Given a sufficiently large N, $q^{+(N)}(f) \geq \frac{q_m^+}{2}$ and $q^{-(N)}(a_N) \leq q^-(a) + 1$. Considering that

$$q^{-(N)}(g) \leq q^{+(N)}(g) \;\; \forall g \in [0,b] \cap F_N$$

we obtain

$$\mu_f^{(N)} = \left(\prod_{i=fN+1}^{a_N N-1} \frac{q^{-(N)}\left(\frac{i}{N}\right)}{q^{+(N)}\left(\frac{i}{N}\right)}\right) \frac{q^{-(N)}(a_N)}{q^{+(N)}(f)}\mu_{a_N}^{(N)} \leq \frac{q^{-(N)}(a_N)}{q^{+(N)}(f)}\mu_{a_N}^{(N)} \leq \frac{c_1}{N}.$$

With a similar reasoning we can see that if $f > b_N$ then $\mu_f^{(N)} < \dfrac{c_2}{N}$ where

$$c_2 = \frac{q^+(b)+1}{q_m^-/2}\, c < +\infty$$

and

$$q_m^- = \min_{f\in[b,1]} q^-(f) > 0.$$

Let $k = \max\{c_1, c_2, c\}$. $\forall I' = [l, r] \subset [0, 1]$

$$\mu^{(N)}(I') = \sum_{i=\lceil lN\rceil}^{\lfloor rN\rfloor} \mu_{i/N}^{(N)} \leq k|I'|.$$

■

Notice that result (2.2.1) and lemma (2.3.2) implies that

$$\sum_{fN=a_N N}^{b_N N} \mu_f^{(N)} \to 1 \tag{2.17}$$

when N grows to infinity. Indeed given $\varepsilon > 0$, k defined in lemma (2.3.2) and a positive $\delta < \dfrac{\varepsilon}{2k}$ we have following equality

$$\begin{aligned} 1 = \quad & \sum_{fN=0}^{\lfloor(a-\delta)N\rfloor} \mu_f^{(N)} + \sum_{fN=\lceil(a-\delta)N\rceil}^{a_N N-1} \mu_f^{(N)} + \\ & + \sum_{fN=a_N N}^{b_N N} \mu_f^{(N)} + \sum_{fN=b_N N+1}^{\lfloor(b+\delta)N\rfloor} \mu_f^{(N)} + \\ & + \sum_{fN=\lceil(b+\delta)N\rceil}^{1} \mu_f^{(N)}. \end{aligned}$$

From (2.2.1) we can state that there exists an N_0 such that the sum from 0 to $\lfloor(a-\delta)N\rfloor$ and the one from $\lceil(b+\delta)N\rceil$ to 1 will be lower than $\varepsilon - 2\delta k$. Lemma (2.3.2) leads us to (2.17).
We can now introduce the final result of this chapter.

Theorem 2.3.3. *Consider the Mean Field Markov Chain $Q^{(N)}$ on F_N with associated interaction kernels Φ_0 and Φ_1 of the form (2.1). Let I be their related interval defined in (2.13) and $\mu^{(N)}$ be the invariant probability measure of the Markov Chain.*
$(\mu^{(N)})_{N\in\mathbb{N}}$ weakly converges to μ^∞ on $([0,1], \mathcal{B}([0,1]))$ where μ^∞ is equal to δ_a if $b = a$ and to the absolutely continuous probability measure with density

$$p(x) = \mathbb{1}_I(x)\frac{1}{x(1-x)}\left(\int_I \frac{1}{y(1-y)}\,dy\right)^{-1}$$

if $a < b$.

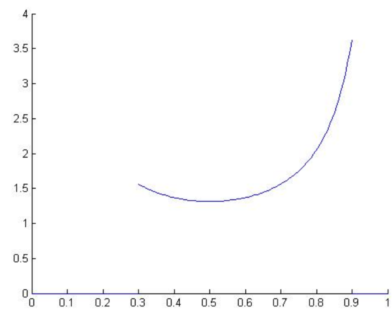


Figure 2.1: μ^{∞} when $I = [0.3, 0.9]$.

Proof. To prove our theorem we show that

$$\mu^{(N)}(I') \to \mu^{\infty}(I') \quad \forall I' = [l, r] \subseteq [0, 1].$$

If $I = \{a\}$ the thesis is immediately obtained through (2.14).
If $a < b$ let

$$m_N = (1 - a_N) a_N \mu^{(N)}(a_N) N.$$

From (2.17) we have that

$$\frac{m_N}{N} \sum_{fN = a_N N}^{b_N N} \frac{1}{(1-f)f} \to 1$$

when N approaches infinity. Since the function $\gamma(x) = \dfrac{1}{(1-x)x}$ is Riemann integrable in $[a, b]$ we have that

$$m_N \to \left(\int_a^b \frac{1}{(1-x)x} \, dx \right)^{-1}.$$

Given an interval $I' \subseteq [0, 1]$, (2.17) implies that

$$\lim_{N \to +\infty} \mu^{(N)}(I' \setminus I) \to 0.$$

We can limit our study to $I' = [l, r] \subseteq [a, b]$. We have

$$\begin{aligned} \mu^{(N)}(I') &= \sum_{fN = \lceil lN \rceil}^{\lfloor rN \rfloor} \mu_f^{(N)} = m_N \left(\frac{1}{N} \sum_{fN = \lceil lN \rceil}^{\lfloor rN \rfloor} \frac{1}{(1-f)f} \right) \\ &\to \int_l^r \frac{1}{(1-x)x} \left(\int_a^b \frac{1}{(1-y)y} \, dy \right)^{-1}. \end{aligned}$$

The theorem is proven. ∎

Chapter 3

Symmetry of Strategies

3.1 Interaction Kernels

In this chapter we will deal with cases where every player that has the opportunity to change strategy considers the two option available without any a priori preference: the probability to adopt strategy 1 given certain fractions of local and global strategy 1 players is the same of adopting strategy 0 with equal fractions of local and global strategy 0 players. This implies that our interaction kernels will be equal to each other and will satisfy following symmetry condition

$$\Phi_s(1-x_1, 1-x_2) = 1 - \Phi_s(x_1, x_2) \quad \forall (x_1, x_2) \in [0,1]^2. \tag{3.1}$$

This leads us to situations where every player is inclined to coordinate with the local majority and to anticoordinate with the global one. Indeed, by considering (3.1) and the definition of Φ_s (1.2.1), we obtain

$$\Phi_s(x_1, x_2) \geq \frac{1}{2} \quad \text{if } (x_1, x_2) \in \left[\frac{1}{2}, 1\right] \times \left[0, \frac{1}{2}\right]$$

and

$$\Phi_s(x_1, x_2) \leq \frac{1}{2} \quad \text{if } (x_1, x_2) \in \left[0, \frac{1}{2}\right] \times \left[\frac{1}{2}, 1\right].$$

The behavior in $\left[0, \frac{1}{2}\right]^2$ and $\left[\frac{1}{2}, 1\right]^2$ depends on the relative weight of coordination with respect to congestion. To measure this weight we introduce a parameter α varying in an interval A contained in $\mathbb{R}_{\geq 0}$: the larger the α the more the coordination preveals on the congestion.
We denote our interaction kernels by Φ^α and we assume the following regularity condition

$$\Phi^\alpha(x_1, x_2) \in C^2([0,1]^2 \times A). \tag{3.2}$$

The α-effect translates into

$$\frac{\partial \Phi^\alpha}{\partial \alpha}(x_1, x_2) \leq 0 \quad \text{if } (x_1, x_2) \in \left[0, \frac{1}{2}\right]^2 \tag{3.3}$$

that, given the symmetry property (3.1), implies

$$\frac{\partial \Phi^\alpha}{\partial \alpha}(x_1, x_2) \geq 0 \quad \text{if } (x_1, x_2) \in \left[\frac{1}{2}, 1\right]^2.$$

Once defined an underlying graph for our dynamics, the purpose of our study will be to understand how the invariant probability measure of the resulting Mean Field Markov Chains will be distributed when varying α. In order to set up a starting point for the growth of α we assume that $0 \in A$ and that no coordination is present when $\alpha = 0$:

$$\frac{\partial \Phi^0}{\partial x_1}(x_1, x_2) = 0 \quad \forall (x_1, x_2)[0, 1]^2. \tag{3.4}$$

Our study will procede by referring to the following two examples.

Example 3.1.1. *In the first situation $A = [0, 1)$ and the two games of local coordination and global congestion will be perceived as a unique coupled game with payoffs*

$$p_1 = \alpha p_1^l + (1-\alpha) p_1^g = \alpha f_{loc}^v + (1-\alpha)(1 - f^v)$$

and

$$p_0 = \alpha p_0^l + (1-\alpha) p_0^g = \alpha(1 - f_{loc}^v) + (1-\alpha) f^v.$$

Given $\beta > 0$, the selected player will turn into strategy 1 according to the probability

$$\frac{1}{1 + e^{-\beta(p_1 - p_0)}}.$$

The interpretation is that $\frac{1}{\beta}$ quantifies the noise that affects our player in choosing his more profitable strategy: whatever the payoffs, if $\beta \to 0$, $\Phi^\alpha \to \frac{1}{2}$. Previous reasoning leads to the following interaction kernel

$$\Phi^\alpha(x_1, x_2) = \frac{1}{1 + e^{\beta(\alpha(1-2x_1) - (1-\alpha)(1-2x_2))}}. \tag{3.5}$$

Example 3.1.2. *The second example is inspired by chapter 2: again $A = [0, 1)$ but the two games of local coordination and global congestion stay separated. Here α represents the probability that the selected player updates its strategy according to the local coordination game. Otherwise he will consider the global congestion effect. Given the noise parameter $\beta > 0$ we have that the probability that the selected player will end up in strategy 1 will be*

$$\alpha \frac{1}{1 + e^{-\beta(p_1^l - p_0^l)}} + (1-\alpha) \frac{1}{1 + e^{-\beta(p_1^g - p_0^g)}}.$$

This leads to the following interaction kernel

$$\Phi^\alpha(x_1, x_2) = \alpha \frac{1}{1 + e^{\beta(1-2x_1)}} + (1-\alpha) \frac{1}{1 + e^{-\beta(1-2x_2)}}. \tag{3.6}$$

In the sequel we will study the dynamics expressed by interaction kernels (3.5) and (3.6) on different graphs.
We start from $\mathcal{G} = K_N$: in this case our Mean Field Markov Chains will be *birth and death processes*. This allows us to study the concentration regions of $\mu^{(N)}$ through the refined result (2.2.1).
The second topology considered will be $\mathcal{G} = K_{N/2} \cup K_{N/2}$. This case will be studied through its related Kurtz's system of ODE: given lemma (1.4.2) we know that $\mu^{(N)}$ concentrates in the set of its recurrent points in $[0, 1]^2$.
Before moving to the dynamic, some properties of the interaction kernels need to be highlighted. First we notice that the α-effect (3.3) leads to

$$\frac{\partial^2 \Phi^\alpha}{\partial \alpha \partial x_i}\left(\frac{1}{2}, \frac{1}{2}\right) \geq 0 \quad \text{for } i = 1, 2. \tag{3.7}$$

This implies that, as α increases, conditions

$$\frac{\partial \Phi^\alpha}{\partial x_1}\left(\frac{1}{2}, \frac{1}{2}\right) > 1 \tag{3.8}$$

and

$$\frac{\partial \Phi^\alpha}{\partial x_1}\left(\frac{1}{2}, \frac{1}{2}\right) + \frac{\partial \Phi^\alpha}{\partial x_2}\left(\frac{1}{2}, \frac{1}{2}\right) > 1 \tag{3.9}$$

will continue to hold once satisfied. We can then partition A in three susequent intervals referring to (3.8) and (3.9). In particular we define

$$\alpha^* = \inf\left\{\alpha \in A \text{ s.t. } \frac{\partial \Phi^\alpha}{\partial x_1}\left(\frac{1}{2}, \frac{1}{2}\right) > 1\right\} \tag{3.10}$$

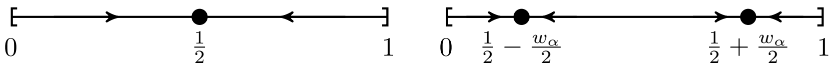
and

$$\alpha^{**} = \inf\left\{\alpha \in A \text{ s.t. } \frac{\partial \Phi^\alpha}{\partial x_1}\left(\frac{1}{2}, \frac{1}{2}\right) + \frac{\partial \Phi^\alpha}{\partial x_2}\left(\frac{1}{2}, \frac{1}{2}\right) > 1\right\} \tag{3.11}$$

where $\alpha^{**} \geq \alpha^*$ for congestion property presented in (1.2.1). Then, considering the regularity of Φ^α expressed in (3.2), we have that

- both (3.8) and (3.9) are not satisfied $\forall \alpha \in [0, \alpha^*] \cap A$;
- only (3.8) is satisfied $\forall \alpha \in (\alpha^*, \alpha^{**}] \cap A$;
- both (3.8) and (3.9) are satisfied $\forall \alpha \in (\alpha^{**}, +\infty) \cap A$.

Notice that, while second and third cases may not be encountered, the starting condition (3.4) implies that $\alpha^* > 0$. For example, concerning interaction kernels (3.5) and (3.6), we have that if $\beta \leq 2$ both α^* and α^{**} will not belong to A. Otherwise we have $\alpha^* = \dfrac{2}{\beta}$ and $\alpha^{**} = \dfrac{1}{\beta} + \dfrac{1}{2}$.



(a) $\alpha > \alpha^{**}$. (b) $\alpha \leq \alpha^{**}$.

Figure 3.1: Concentration points of $\mu^{(N)}$ when $\mathcal{G} = K_N$.

3.2 $\mathcal{G} = K_N$

As anticipated, we will start from the case $\mathcal{G} = K_N$. Let $z \in [0,1]$ be the continuous fraction of strategy 1 players when $N \to +\infty$.
Results are presented in following lemma. This states that, under certain conditions on Φ^α, the invariant probability measure distribution is characterized by a phase transition in α^{**} (see figure (3.1)). Both interaction kernel (3.5) and (3.6) fit in this result.

Lemma 3.2.1. *Let Φ^α be an interaction kernel with the properties presented in section 3.1. Consider the Mean Field Markov Chain obtained from the dynamic expressed by Φ^α on the graph $\mathcal{G} = K_N$. Assume that $\forall \alpha \in A$*

$$\left| \frac{d}{dz} \Phi^\alpha(z,z) \right| \text{ increasing for } z \in \left[0, \frac{1}{2}\right]. \tag{3.12}$$

Then, as N increases,

- *$\forall \alpha \in [0, \alpha^{**}] \cap A$ the invariant probability measure concentrates in the point $z = \dfrac{1}{2}$;*
- *$\forall \alpha \in (\alpha^{**}, +\infty) \cap A$ exists $w_\alpha \in (0,1)$ non decreasing in α s.t. the invariant probability measure concentrates in the two equilibria*

$$z = \frac{1}{2} \pm \frac{w_\alpha}{2}.$$

Proof. As in chapter 2, we define

$$q^{+(N)}(f) = (1-f)\Phi^\alpha(f_{-0}, f_{-0}) \quad \forall f \in F_N \setminus \{1\}$$

and

$$q^{-(N)}(f) = f(1 - \Phi^\alpha(f_{-1}, f_{-1})) \quad \forall f \in F_N \setminus \{0\}.$$

Notice that $q^{+(N)}(f)$ and $q^{-(N)}(f)$ uniformely converges in $[0,1]$ (in the sense described in (1.13)) to

$$q^+(z) = (1-z)\Phi^\alpha(z,z)$$

and

$$q^-(z) = z(1 - \Phi^\alpha(z,z)).$$

Since

$$0 < \Phi^\alpha(z,z) < 1 \quad \forall z \in [0,1]$$

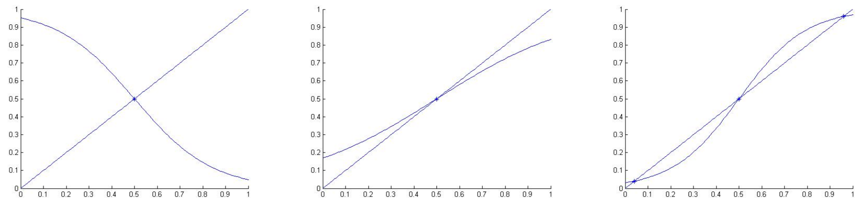


Figure 3.2: Solutions of $\Phi^{\alpha}(z,z) = z$.

we can use lemma (2.2.1) to state that the invariant probability measure of our Markov Chains will concentrate in the absolute maxima of the continuous function $J : [0,1] \to \mathbb{R}$

$$J(z) = \int_0^z \ln \frac{q^+(\zeta)}{q^-(\zeta)}\, d\zeta.$$

Substituting q^+ and q^- with their definitions we see that such absolute maxima belong to the interval $(0,1)$ and are stationary point of J. Notice that the derivative of J in $(0,1)$ has the sign of the function

$$\Phi^{\alpha}(z,z) - z. \tag{3.13}$$

Since $\Phi^{\alpha}(0,0) > 0$ (and, by symmetry, $\Phi^{\alpha}(1,1) < 1$), hypotesis (3.16) implies that $\forall \alpha \in A$ such that

$$\left.\frac{d}{dz}\Phi^{\alpha}(z,z)\right|_{\frac{1}{2},\frac{1}{2}} = \frac{\partial \Phi^{\alpha}}{\partial x_1}\left(\frac{1}{2},\frac{1}{2}\right) + \frac{\partial \Phi^{\alpha}}{\partial x_2}\left(\frac{1}{2},\frac{1}{2}\right) \leq 1$$

$z = \dfrac{1}{2}$ will be the unique stationary point of J and it will be an absolute maximum (see figure 3.2).
Otherwise it becomes a local minimum and $\exists w_{\alpha} \in (0,1)$ such that

$$z = \frac{1}{2} \pm \frac{w_{\alpha}}{2}$$

will be the only two local maxima of J in $(0,1)$. Considering symmetry property (3.1) we have that they will be two absolute maxima since

$$\Phi^{\alpha}\left(\frac{1}{2}+\frac{w_{\alpha}}{2},\frac{1}{2}+\frac{w_{\alpha}}{2}\right) = \Phi^{\alpha}\left(\frac{1}{2}-\frac{w_{\alpha}}{2},\frac{1}{2}-\frac{w_{\alpha}}{2}\right).$$

Furthermore, for (3.3), if $\alpha_2 > \alpha_1$

$$\Phi^{\alpha_1}\left(\frac{1}{2}-\frac{w_{\alpha_1}}{2},\frac{1}{2}-\frac{w_{\alpha_1}}{2}\right) \geq \Phi^{\alpha_2}\left(\frac{1}{2}-\frac{w_{\alpha_1}}{2},\frac{1}{2}-\frac{w_{\alpha_1}}{2}\right).$$

This implies that w_{α} will grow in α and the two absolute maxima will be further and further away from $\dfrac{1}{2}$. ∎

Notice that hypotesis (3.16) required in lemma (3.2.1) does not cover the situation of interaction kernel (3.5) with $\alpha = \frac{1}{2}$. Here

$$\Phi^{\alpha}(z,z) = \frac{1}{2} \quad \forall z \in [0,1].$$

However, the result can be easily extended to this particular case. Indeed here $\alpha < \alpha^{**}$ and from (3.13) we have that the unique maximum of J is $z = \frac{1}{2}$.

3.3 $\mathcal{G} = K_{N/2} \cup K_{N/2}$

We now move to the case $\mathcal{G} = K_{N/2} \cup K_{N/2}$. Let z_1 and z_2 be the continuous fraction of strategy 1 players in the first and the second connected component. The global fraction of them will be represented by variable z, i.e. $z = \frac{z_1 + z_2}{2}$. Again, we start with introducing a theorem that collects the results that will be proven in the sequel. This theorem states that, under certain conditions, the invariant probability measure distribution is characterized by two phase transitions in α^* and α^{**} (see figure (3.3)). Again, these results are fitted by the two interaction kernels (3.5) and (3.6).

Theorem 3.3.1. *Let Φ^{α} be an interaction kernel with the properties presented in section 3.1. Consider the Mean Field Markov Chain obtained from the dynamic expressed by Φ^{α} on the graph $\mathcal{G} = K_{N/2} \cup K_{N/2}$. Assume that Φ^{α} represents strong congestion, i.e.*

$$\frac{\partial \Phi^{\alpha}}{\partial x_2}(x_1, x_2) < 0 \textit{ for } (x_1, x_2) \in [0,1]^2. \tag{3.14}$$

Furthermore suppose that

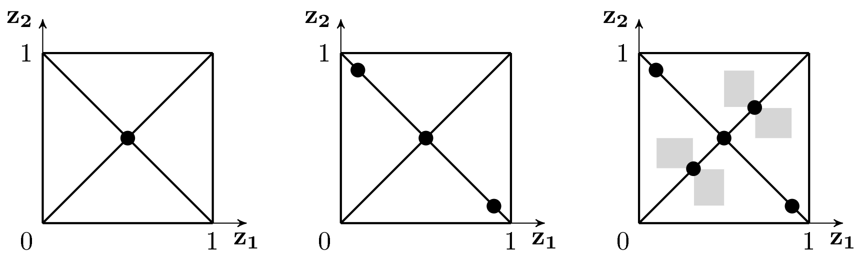
$$\Phi^{\alpha}\left(x_1, \frac{1}{2}\right) \textit{ strictly convex in } \left[0, \frac{1}{2}\right] \tag{3.15}$$

and

$$\left|\frac{d}{dz}\Phi^{\alpha}(z,z)\right| \textit{ increasing for } z \in \left[0, \frac{1}{2}\right]. \tag{3.16}$$

Then, as N grows,

- *$\forall \alpha \in [0, \alpha^*] \cap A$ the invariant probability measure concentrates in the unique point $(z_1, z_2) = \left(\frac{1}{2}, \frac{1}{2}\right)$;*
- *$\forall \alpha \in (\alpha^*, \alpha^{**}] \cap A$, in addition to the point presented in previous step, $\exists y_{\alpha} \in (0,1)$ non decreasing in α s.t. the invariant probability measure can also concentrate into two antidiagonal equilibria $(z_1, z_2) = \left(\frac{1}{2} \pm \frac{y_{\alpha}}{2}, \frac{1}{2} \mp \frac{y_{\alpha}}{2}\right)$;*



(a) $\alpha \le \alpha^*$. (b) $\alpha^* < \alpha \le \alpha^{**}$. (c) $\alpha > \alpha^{**}$.

Figure 3.3: Concentration points of $\mu^{(N)}$ when $\mathcal{G} = K_{N/2} \cup K_{N/2}$.

- *$\forall \alpha \in A$, $\alpha > \alpha^{**}$, in addition to the points presented in previous step, $\exists w_\alpha \in (0, y_\alpha]$ non decreasing in α such that the invariant probability measure can also concentrate in $(z_1, z_2) = \left(\frac{1}{2} \pm \frac{w_\alpha}{2}, \frac{1}{2} \pm \frac{w_\alpha}{2}\right)$ and in the region*

$$\left(\frac{1}{2} + \frac{w_\alpha}{2}, \frac{1}{2} + \frac{y_\alpha}{2}\right) \times \left(\frac{1}{2}, \frac{1}{2} + \frac{w_\alpha}{2}\right) \cup \left(\frac{1}{2} - \frac{y_\alpha}{2}, \frac{1}{2} - \frac{w_\alpha}{2}\right) \times \left(\frac{1}{2}, \frac{1}{2} - \frac{w_\alpha}{2}\right) \cup$$

$$\left(\frac{1}{2}, \frac{1}{2} + \frac{w_\alpha}{2}\right) \times \left(\frac{1}{2} + \frac{w_\alpha}{2}, \frac{1}{2} + \frac{y_\alpha}{2}\right) \cup \left(\frac{1}{2}, \frac{1}{2} - \frac{w_\alpha}{2}\right) \times \left(\frac{1}{2} - \frac{y_\alpha}{2}, \frac{1}{2} - \frac{w_\alpha}{2}\right).$$

3.3.1 Mean Field System

Given an interaction kernel Φ^α the Kurtz's system of ODE (1.15) associated to our dynamic will be the following

$$\begin{cases} \dot{z}_1 = \Phi^\alpha(z_1, z) - z_1 \\ \dot{z}_2 = \Phi^\alpha(z_2, z) - z_2. \end{cases} \tag{3.17}$$

Results presented in theorem (3.3.1) are derived by considering lemma (1.4.2) introduced in the first chapter. We will be then interested in studyng the set of recurrent points of (3.17) in $[0,1]^2$. Congestion property introduced in the interaction kernels definition (1.2.1) will be fundamental in this study since it makes our system competitive in its domain. This implies that every trajectory in $[0,1]^2$ will converge to an equilibrium: the set of recurrent points will then be the union of such equilibria (see theorem 3.22 by Cañada et al. [6]). Furthermore, given two initial conditions p_1 and $p_2 \in [0,1]^2$ such that

$$p_{1,z_1} \le p_{2,z_1} \text{ and } p_{1,z_2} \ge p_{2,z_2}$$

and their associated solution trajectories $(\varphi_t(p_1))_{t\ge0}$ and $(\varphi_t(p_2))_{t\ge0}$ it follows

$$\varphi_{t,z_1}(p_1) \le \varphi_{t,z_1}(p_2) \text{ and } \varphi_{t,z_2}(p_1) \ge \varphi_{t,z_2}(p_2) \quad \forall t \ge 0 \tag{3.18}$$

(see Muller [1] and Kamke [2]).
The other fundamental property of system (3.17) is that it has two invariant subsets coinciding with the two diagonals of the square $[0,1]^2$. Indeed, if we consider the variable $z_1 - z_2$, we obtain the equation

$$\dot{z}_1 - \dot{z}_2 = \Phi^\alpha(z_1, z) - \Phi^\alpha(z_2, z) - (z_1 - z_2)$$

that implies that the diagonal $z_1 - z_2 = 0$ is an invariant subset for system (3.17).
On the other hand, considering the variable $z_1 + z_2$, we obtain

$$\dot{z}_1 + \dot{z}_2 = \Phi^\alpha(z_1, z) + \Phi^\alpha(z_2, z) - (z_1 + z_2).$$

Since symmetry property (3.1) implies that

$$\Phi^\alpha\left(1 - z_1, \frac{1}{2}\right) = 1 - \Phi^\alpha\left(z_1, \frac{1}{2}\right),$$

we have that the antidiagonal $z_1 + z_2 = 1$ is another invariant subset for system (3.17) in $[0,1]^2$.
Let $w = z_1 + z_2 - 1$ be the diagonal variable and $y = z_1 - z_2$ the antidiagonal one. If we introduce

$$R = \left\{(w, y) \in \mathbb{R}^2 \text{ s.t. } |w + y| \leq 1\right\} \tag{3.19}$$

system (3.17) can be rewritten on this domain as follows:

$$\begin{cases} \dot{w} = f_w(w, y) \\ \dot{y} = f_y(w, y) \end{cases} \tag{3.20}$$

where

$$f_w(w, y) = \Phi^\alpha\left(\frac{1}{2} + \frac{w + y}{2}, \frac{1}{2} + \frac{w}{2}\right) + \Phi^\alpha\left(\frac{1}{2} + \frac{w - y}{2}, \frac{1}{2} + \frac{w}{2}\right) - w - 1$$

and

$$f_y(w, y) = \Phi^\alpha\left(\frac{1}{2} + \frac{w + y}{2}, \frac{1}{2} + \frac{w}{2}\right) - \Phi^\alpha\left(\frac{1}{2} + \frac{w - y}{2}, \frac{1}{2} + \frac{w}{2}\right) - y.$$

Notice that, for every value of $w \in [-1 + y, 1 - y]$, f_w is an even function with respect to y. Furthermore, since symmetry property (3.1) implies

$$\Phi^\alpha\left(\frac{1}{2} - \frac{w + y}{2}, \frac{1}{2} - \frac{w}{2}\right) = 1 - \Phi^\alpha\left(\frac{1}{2} + \frac{w + y}{2}, \frac{1}{2} + \frac{w}{2}\right)$$

and

$$\Phi^\alpha\left(\frac{1}{2} - \frac{w - y}{2}, \frac{1}{2} - \frac{w}{2}\right) = 1 - \Phi^\alpha\left(\frac{1}{2} + \frac{w - y}{2}, \frac{1}{2} + \frac{w}{2}\right),$$

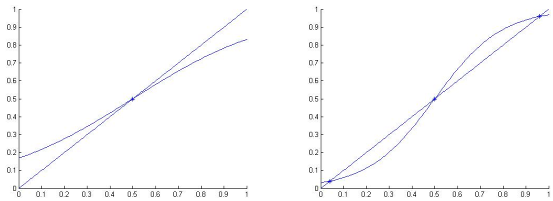


Figure 3.4: Solutions of $\Phi^\alpha\left(x_1, \dfrac{1}{2}\right) = x_1$.

we have that, for every $y \in [-1+w, 1-w]$, f_w is an odd function with respect to w.
With a similar reasoning we obtain that, for every value of $w \in [-1+y, 1-y]$, f_y is an odd function in y and an even function in w for every value of $y \in [-1+w, 1-w]$.
We procede with considering the equilibria of our system in the two diagonal invariant subsets. A natural conjecture emerging from simulations at the end of next chapter states that the invariant probability measure will concentrate only in the locally stable equilibria: stability aspects will be then highlighted. Results obtained in these two cases will be fundamental for studying the general situation in $[0,1]^2$ presented at the end of the chapter.

3.3.2 Antidiagonal Invariant Subset

Consider our system in the form (3.20). The antidiagonal equilibria characterized by $w = 0$ will satisfy the equation

$$\Phi^\alpha\left(\frac{1}{2} + \frac{y}{2}, \frac{1}{2}\right) = \frac{1}{2} + \frac{y}{2} \text{ for } y \in [-1, 1]. \tag{3.21}$$

We will be then interested in considering how $\Phi^\alpha\left(x_1, \dfrac{1}{2}\right)$ grows in x_1.

Since $\Phi^\alpha\left(0, \dfrac{1}{2}\right) > 0$ (and, by symmetry, $\Phi^\alpha\left(1, \dfrac{1}{2}\right) < 1$), hypothesis (3.15) implies that $\forall \alpha \in A$ such that

$$\frac{\partial \Phi^\alpha}{\partial x_1}\left(\frac{1}{2}, \frac{1}{2}\right) \leq 1$$

equation (3.21) will have the unique solution $y = 0$ (see figure (3.4)). On the other hand, if $\dfrac{\partial \Phi^\alpha}{\partial x_1}\left(\dfrac{1}{2}, \dfrac{1}{2}\right) > 1$, $\exists y_\alpha \in (0,1)$ such that

$$(w, y) = \left(0, \frac{1}{2} \pm \frac{y_\alpha}{2}\right)$$

will be the only two others antidiagonal equilibria for our system. Symmetry property (3.1) implies that

$$\frac{\partial \Phi^\alpha}{\partial x_1}\left(\frac{1}{2}+\frac{y_\alpha}{2},\frac{1}{2}\right)=\frac{\partial \Phi^\alpha}{\partial x_1}\left(\frac{1}{2}-\frac{y_\alpha}{2},\frac{1}{2}\right)$$

and concavity condition (3.15) yelds to

$$\frac{\partial \Phi^\alpha}{\partial x_1}\left(\frac{1}{2}+\frac{y_\alpha}{2},\frac{1}{2}\right)<1. \tag{3.22}$$

Furthermore, considering that for (3.3) if $\alpha_2>\alpha_1$ then

$$\Phi^{\alpha_1}\left(\frac{1}{2}-\frac{y_{\alpha_1}}{2},\frac{1}{2}\right)\geq \Phi^{\alpha_2}\left(\frac{1}{2}-\frac{y_{\alpha_1}}{2},\frac{1}{2}\right),$$

y_α will not decrease in α.
As anticipated, we will be interested in studying the stability of these antidiagonal equilibria: since $w=0$ is an invariant subset for our system,

$$f_w(0,y)=0 \quad \forall y\in[-1,1].$$

This means that the term $\dfrac{\partial f_w}{\partial y}$ is null when $w=0$. On the other hand, given the symmetries of f_y, we have that $\dfrac{\partial f_y}{\partial w}$ is an odd function with respect to w. This implies

$$\frac{\partial f_y}{\partial w}(0,y)=0 \quad \forall y\in[-1,1].$$

Thank to these observations and to the fact that

$$\frac{\partial \Phi^\alpha}{\partial x_i}\left(\frac{1}{2}-\frac{y}{2},\frac{1}{2}\right)=\frac{\partial \Phi^\alpha}{\partial x_i}\left(\frac{1}{2}+\frac{y}{2},\frac{1}{2}\right)\ \text{ for } i=1,2$$

we have the the jacobian matrix of system (3.20) in every antidiagonal point $(0,y)$ has the form

$$J_f(0,y)=\begin{pmatrix}\frac{\partial \Phi^\alpha}{\partial x_1}\left(\frac{1}{2}+\frac{y}{2},\frac{1}{2}\right)+\frac{\partial \Phi^\alpha}{\partial x_2}\left(\frac{1}{2}+\frac{y}{2},\frac{1}{2}\right)-1 & 0\\ 0 & \frac{\partial \Phi^\alpha}{\partial x_1}\left(\frac{1}{2}+\frac{y}{2},\frac{1}{2}\right)-1\end{pmatrix}.$$

The whole situation for the original system (3.17) is presented in following lemma.

Lemma 3.3.2. *Consider the α^* defined in (3.10).*
$\forall\alpha\in A$ s.t. $\alpha\leq\alpha^*$ the only antidiagonal equilibrium of system (3.17) will be $(z_1,z_2)=\left(\frac{1}{2},\frac{1}{2}\right)$ and it will be locally stable.
$\forall\alpha\in A$ s.t. $\alpha>\alpha^*$ $\exists y_\alpha\in(0,1)$ non decreasing in α such that

$$(z_1,z_2)=\left(\frac{1}{2}\pm\frac{y_\alpha}{2},\frac{1}{2}\mp\frac{y_\alpha}{2}\right)$$

will be the only two other antidiagonal equilibria and they will be locally stable. For these α the center $(z_1, z_2) = \left(\frac{1}{2}, \frac{1}{2}\right)$ will be unstable under antidiagonal perturbations.

Proof. Since $(w, y) = (0, 0)$ implies $(z_1, z_2) = \left(\frac{1}{2}, \frac{1}{2}\right)$ this will be the only antidiagonal equilibrium for system (3.17) when $\alpha \leq \alpha^*$. The jacobian matrix of system (3.17) in the center $\left(\frac{1}{2}, \frac{1}{2}\right)$ will be equivalent to

$$\begin{pmatrix} \frac{\partial \Phi^\alpha}{\partial x_1}\left(\frac{1}{2}, \frac{1}{2}\right) + \frac{\partial \Phi^\alpha}{\partial x_2}\left(\frac{1}{2}, \frac{1}{2}\right) - 1 & 0 \\ 0 & \frac{\partial \Phi^\alpha}{\partial x_1}\left(\frac{1}{2}, \frac{1}{2}\right) - 1 \end{pmatrix}.$$

Considering that congestion property implies $\frac{\partial \Phi^\alpha}{\partial x_2}\left(\frac{1}{2}, \frac{1}{2}\right) \leq 0$, the center will be locally stable for $\alpha \leq \alpha^*$ and unstable at least under antidiagonal perturbations if $\alpha^* > \alpha$. In this case, as previously discussed, system (3.20) will have the two other antidiagonal equilibria

$$(w, y) = \left(0, \frac{1}{2} \pm \frac{y_\alpha}{2}\right)$$

i.e.

$$(z_1, z_2) = \left(\frac{1}{2} \pm \frac{y_\alpha}{2}, \frac{1}{2} \mp \frac{y_\alpha}{2}\right).$$

The jacobian matrix of the system computed in these two points will be

$$\begin{pmatrix} \frac{\partial \Phi^\alpha}{\partial x_1}\left(\frac{1}{2} + \frac{y_\alpha}{2}, \frac{1}{2}\right) + \frac{\partial \Phi^\alpha}{\partial x_2}\left(\frac{1}{2} + \frac{y_\alpha}{2}, \frac{1}{2}\right) - 1 & 0 \\ 0 & \frac{\partial \Phi^\alpha}{\partial x_1}\left(\frac{1}{2} + \frac{y_\alpha}{2}, \frac{1}{2}\right) - 1 \end{pmatrix}.$$

Inequality (3.22) and congestion property of Φ^α lead us to the thesis. ∎

3.3.3 Diagonal Invariant Subset

The equilibria on the diagonal $y = 0$ have to satisfy the equation

$$\Phi^\alpha\left(\frac{1}{2} + \frac{w}{2}, \frac{1}{2} + \frac{w}{2}\right) = \frac{1}{2} + \frac{w}{2} \text{ for } w \in [-1, 1]. \tag{3.23}$$

Notice that this condition is exactly the one related to the stationary points in $(0, 1)$ of the function J in the monodimensional case. Again, if we consider condition (3.16), we obtain that equation (3.23) will have the unique equilibrium $w = 0$ $\forall \alpha \in A$ such that

$$\frac{\partial \Phi^\alpha}{\partial x_1}\left(\frac{1}{2},\frac{1}{2}\right) + \frac{\partial \Phi^\alpha}{\partial x_2}\left(\frac{1}{2},\frac{1}{2}\right) \leq 1.$$

Otherwise $\exists w_\alpha \in (0,1)$ such that $(w,y) = \left(\frac{1}{2} \pm \frac{w_\alpha}{2}, 0\right)$ will be two other diagonal equilibria with

$$\frac{\partial \Phi^\alpha}{\partial x_1}\left(\frac{1}{2} \pm \frac{w_\alpha}{2}, \frac{1}{2} \pm \frac{w_\alpha}{2}\right) + \frac{\partial \Phi^\alpha}{\partial x_2}\left(\frac{1}{2} \pm \frac{w_\alpha}{2}, \frac{1}{2} \pm \frac{w_\alpha}{2}\right) < 1. \tag{3.24}$$

Considering that for property (3.3) if $\alpha_2 > \alpha_1$ then

$$\Phi^{\alpha_1}\left(\frac{1}{2} - \frac{w_{\alpha_1}}{2}, \frac{1}{2} - \frac{w_{\alpha_1}}{2}\right) \geq \Phi^{\alpha_2}\left(\frac{1}{2} - \frac{w_{\alpha_1}}{2}, \frac{1}{2} - \frac{w_{\alpha_1}}{2}\right),$$

w_α will grow in α so that these two equilibria will be further and further away from $\frac{1}{2}$.

To study their stability we notice that, for a reasoning similar to the one made for the antidiagonal case, the jacobian matrix of the system (3.20) in any diagonal point $(w, 0)$ will be

$$\begin{pmatrix} \frac{\partial \Phi^\alpha}{\partial x_1}\left(\frac{1}{2} + \frac{w}{2}, \frac{1}{2} + \frac{w}{2}\right) + \frac{\partial \Phi^\alpha}{\partial x_2}\left(\frac{1}{2} + \frac{w}{2}, \frac{1}{2} + \frac{w}{2}\right) - 1 & 0 \\ 0 & \frac{\partial \Phi^\alpha}{\partial x_1}\left(\frac{1}{2} + \frac{w}{2}, \frac{1}{2} + \frac{w}{2}\right) - 1 \end{pmatrix}.$$

The whole situation for the original system (3.17) is presented in following lemma.

Lemma 3.3.3. *Consider the α^{**} defined in (3.11). $\forall \alpha \in A$ s.t. $\alpha \leq \alpha^{**}$ the only equilibrium on the diagonal of system (3.17) will be $(z_1, z_2) = \left(\frac{1}{2}, \frac{1}{2}\right)$. Otherwise, $\forall \alpha \in A$ s.t. $\alpha > \alpha^{**}$, $\exists w_\alpha \in (0,1)$ non decreasing in α such that*

$$(z_1, z_2) = \left(\frac{1}{2} \pm \frac{w_\alpha}{2}, \frac{1}{2} \pm \frac{w_\alpha}{2}\right)$$

will be the only two other diagonal equilibria. They will be always stable under diagonal perturbations; locally stable if

$$\frac{\partial \Phi^\alpha}{\partial x_1}\left(\frac{1}{2} + \frac{w_\alpha}{2}, \frac{1}{2} + \frac{w_\alpha}{2}\right) < 1$$

and saddle points if

$$\frac{\partial \Phi^\alpha}{\partial x_1}\left(\frac{1}{2} + \frac{w_\alpha}{2}, \frac{1}{2} + \frac{w_\alpha}{2}\right) > 1.$$

Finally, if $\alpha > \alpha^{**}$ the center $(z_1, z_2) = \left(\frac{1}{2}, \frac{1}{2}\right)$ will be unstable.

Proof. For the properties of Φ^α discussed above, if $\alpha \leq \alpha^{**}$ $(w, y) = (0, 0)$ i.e. $(z_1, z_2) = \left(\frac{1}{2}, \frac{1}{2}\right)$ will be the only solution of (3.23). Since in this point the jacobian matrix of system (3.17) will be equivalent to

$$\begin{pmatrix} \frac{\partial \Phi^\alpha}{\partial x_1}\left(\frac{1}{2}, \frac{1}{2}\right) + \frac{\partial \Phi^\alpha}{\partial x_2}\left(\frac{1}{2}, \frac{1}{2}\right) - 1 & 0 \\ 0 & \frac{\partial \Phi^\alpha}{\partial x_1}\left(\frac{1}{2}, \frac{1}{2}\right) - 1 \end{pmatrix}.$$

the center wll be stable under diagonal perturbations for $\alpha \leq \alpha^{**}$ and an unstable equilibrium for $\alpha > \alpha^{**}$. In latter case, as previously discussed, equation (3.23) will have the two others antidiagonal equilibria $(w, y) = \left(\frac{1}{2} \pm \frac{w_\alpha}{2}, 0\right)$ i.e.

$$(z_1, z_2) = \left(\frac{1}{2} \pm \frac{w_\alpha}{2}, \frac{1}{2} \pm \frac{w_\alpha}{2}\right).$$

The jacobian matrix for the system computed in these two points will be equivalent to

$$\begin{pmatrix} \frac{\partial \Phi^\alpha}{\partial x_1}\left(\frac{1}{2} + \frac{w_\alpha}{2}, \frac{1}{2} + \frac{w_\alpha}{2}\right) + \frac{\partial \Phi^\alpha}{\partial x_2}\left(\frac{1}{2} + \frac{w_\alpha}{2}, \frac{1}{2} + \frac{w_\alpha}{2}\right) - 1 & 0 \\ 0 & \frac{\partial \Phi^\alpha}{\partial x_1}\left(\frac{1}{2} + \frac{w_\alpha}{2}, \frac{1}{2} + \frac{w_\alpha}{2}\right) - 1 \end{pmatrix}.$$

Thanks to inequality (3.24) they will be always stable under diagonal perturbations. Furthermore, they will be locally stable if

$$\frac{\partial \Phi^\alpha}{\partial x_1}\left(\frac{1}{2} + \frac{w_\alpha}{2}, \frac{1}{2} + \frac{w_\alpha}{2}\right) < 1$$

and saddle points if

$$\frac{\partial \Phi^\alpha}{\partial x_1}\left(\frac{1}{2} + \frac{w_\alpha}{2}, \frac{1}{2} + \frac{w_\alpha}{2}\right) > 1.$$

■

Notice that lemmas (3.3.2) and (3.3.3) imply that $(z_1, z_2) = \left(\frac{1}{2}, \frac{1}{2}\right)$ will be locally stable $\forall \alpha \leq \alpha^*$, a saddle point unstable under antidiagonal perturbations if $\alpha^* < \alpha \leq \alpha^{**}$ and unstable when $\alpha > \alpha^{**}$.

3.3.4 Out Diagonals Equilibria

We now deal with the equilibria out of the two diagonals. As in previous cases we start with considering Kurtz's system in its diagonal form (3.20). Given the symmetries of f_w and f_y and the set R defined in (3.19), every equilibrium

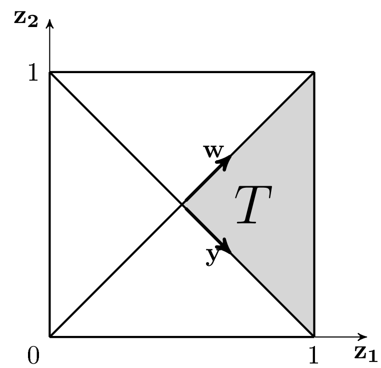


Figure 3.5: Set T.

$(w^e, y^e) \in R$ implies that $(-w^e, y^e)$, $(w^e, -y^e)$, $(-w^e, -y^e) \in R$ will be three other equilibria. We can then limit our study to the region

$$T = \{(w, y) \in R \;\; s.t. \;\; w \geq 0, \; y \geq 0\} \tag{3.25}$$

(see figure (3.5)) and extend the results in R according to these simmetries.
Given the initial condition $p = (p_w, p_y) \in T$, let $(\varphi_t(p))_{t\geq 0}$ be the corresponding solution of system (3.20). Considering an equilibrium $(w^e, y^e) \in T$ the competitive system condition (3.18) implies that for every $t \geq 0$

- if $w^e - y^e \leq p_w \leq w^e + y^e$ then $w^e - y^e \leq \varphi_{t,w}(p) \leq w^e + y^e$;
- if $p_y \leq y^e - w^e$ then $\varphi_{t,y}(p) \leq y^e - w^e$; otherwise if $p_y \geq y^e + w^e$ then $\varphi_{t,y}(p) \geq y^e + w^e$.

Every equilibrium $(w^e, y^e) \in T$ will then determine the following invariant subsets of T (see figure (3.6)):

- $I_1^e = \{(w, 0) \in T$ such that $w^e - y^e \leq w \leq w^e + y^e\}$ always non empty;
- $I_2^e = \{(0, y) \in T$ such that $y \geq y^e + w^e\}$ always non empty;
- $I_3^e = \{(0, y) \in T$ such that $y \leq y^e - w^e\}$ empty if $y^e < w^e$.

I_1^e, I_2^e and I_3^e provide us constraints that have to be satisfied by (w^e, y^e) and that will be fundamental for our final result. Indeed, as already mentioned, given the competitive form of our system, theorem 3.22 by Cañada et al. (see [6]) implies that every solution trajectory of system (3.20) in R has to converge to an equilibrium. In particular, if the initial condition belongs to an invariant subset, it has to be attracted by an equilibrium in the same invariant subset.
Before presenting the results for the original system (3.17), given $i \in \{1, 2\}$, notice that strong congestion hypothesis (3.14) implies that there cannot be two equilibria equal in the component z_i. Indeed, if we consider the ODE related

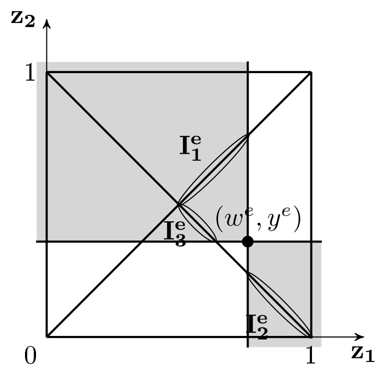


Figure 3.6: Equilibrium (w^e, y^e) with related invariant subsets I_1^e, I_2^e and I_3^e.

to z_i in (3.17), strong congestion property implies that exists at maximum one $\hat{z} \in [0,1]$ such that

$$z_i = \Phi^\alpha(z_i, \hat{z}). \tag{3.26}$$

Lemma 3.3.4. *Consider system (3.17) and assume that the interaction kernel Φ^α represents strong congestion (3.14). Let α^* and α^{**} be the threshold defined in (3.10) and (3.11). Then*

- *$\forall \alpha \in [0, \alpha^*] \cap A$ system (3.17) has the unique equilibrium*

$$(z_1, z_2) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

that is globally stable;

- *$\forall \alpha \in (\alpha^*, \alpha^{**}] \cap A$ our system has three equilibria: the center $(z_1, z_2) = \left(\frac{1}{2}, \frac{1}{2}\right)$ and the two antidiagonal ones*

$$(z_1, z_2) = \left(\frac{1}{2} \pm \frac{y_\alpha}{2}, \frac{1}{2} \mp \frac{y_\alpha}{2}\right)$$

with properties discussed in lemma (3.3.2);

- *$\forall \alpha \in (\alpha^{**}, +\infty) \cap A$ system (3.17) has at least five equilibria: the center $(z_1, z_2) = \left(\frac{1}{2}, \frac{1}{2}\right)$, the two antidiagonal ones $(z_1, z_2) = \left(\frac{1}{2} \pm \frac{y_\alpha}{2}, \frac{1}{2} \mp \frac{y_\alpha}{2}\right)$ with properties discussed in lemma (3.3.2) and the two diagonal ones*

$$(z_1, z_2) = \left(\frac{1}{2} \pm \frac{w_\alpha}{2}, \frac{1}{2} \pm \frac{w_\alpha}{2}\right)$$

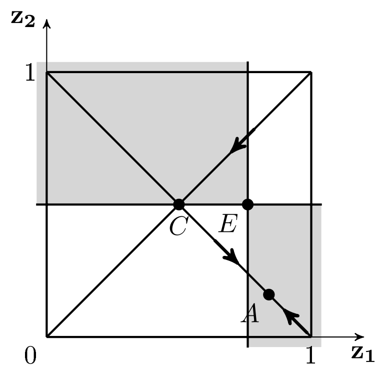


Figure 3.7: $\alpha \in [\alpha^*, \alpha^{**})$.

of lemma (3.3.3). These two diagonal equilibria will be characterized by $w_\alpha \leq y_\alpha$. *All the other equilibria can only belong to the set*

$$\left(\frac{1}{2}+\frac{w_\alpha}{2},\frac{1}{2}+\frac{y_\alpha}{2}\right)\times\left(\frac{1}{2},\frac{1}{2}+\frac{w_\alpha}{2}\right)\cup\left(\frac{1}{2}-\frac{y_\alpha}{2},\frac{1}{2}-\frac{w_\alpha}{2}\right)\times\left(\frac{1}{2},\frac{1}{2}-\frac{w_\alpha}{2}\right)\cup$$

$$\left(\frac{1}{2},\frac{1}{2}+\frac{w_\alpha}{2}\right)\times\left(\frac{1}{2}+\frac{w_\alpha}{2},\frac{1}{2}+\frac{y_\alpha}{2}\right)\cup\left(\frac{1}{2},\frac{1}{2}-\frac{w_\alpha}{2}\right)\times\left(\frac{1}{2}-\frac{y_\alpha}{2},\frac{1}{2}-\frac{w_\alpha}{2}\right).$$

Proof. Consider the system in the diagonal coordinates (w, y).
As anticipated, symmetries of f_y and f_w allow us to limit our study to the region T. The results are extended to the domain R by considering that for every equilibrium $(w^e, y^e) \in T$ we have the three others equilibria $(w^e, -y^e)$ $(-w^e, y^e)$ and $(-w^e, -y^e)$ in R.

If $\alpha \in [0, \alpha^*]$ lemma (3.3.2) implies that $(w, y) = (0, 0)$ is the only equilibrium on the antidiagonal. Every equilibrium $(w^e, y^e) \in T$ must ensure that $(0, 0)$ belongs to its related invariant subset I_2^e. This yields to $y^e + w^e \leq 0$, i.e. $(w^e, y^e) = (0, 0)$.
$(z_1, z_2) = \left(\frac{1}{2}, \frac{1}{2}\right)$ will then be the only equilibrum for the original system (3.17) and, since it has to attract every solution trajectory in $[0, 1]^2$, it will be globally stable.

If $\alpha \in (\alpha^*, \alpha^{**}]$ we have the three antidiagonal equilibria described in lemma (3.3.2). The whole situation in T is presented in figure (3.7).
Every equilibrium $(w^e, y^e) \in T$ must ensure that I_2^e contains the antidiagonal equilibrium $(w, y) = (0, y_\alpha)$. It will then satisfy

$$y_\alpha \geq y^e + w^e. \tag{3.27}$$

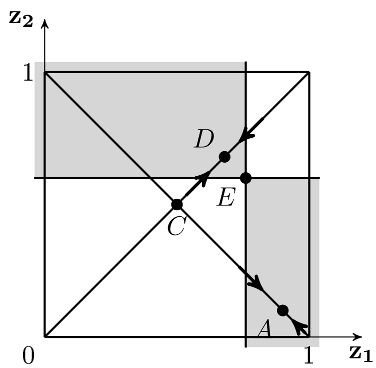


Figure 3.8: $\alpha > \alpha^{**}$.

Furthermore, if $y^e > w^e$ then $(w^e, y^e) = (0, y_\alpha)$. Indeed $y^e > w^e$ implies that $(0, y_\alpha)$ has to belong also to I_3^e since $(0,0)$ cannot attract all the trajectories starting there. Every equilibrium out of the two diagonals will then satisfy

$$y^e \leq w^e. \tag{3.28}$$

On the other hand, since $\alpha \leq \alpha^{**}$ lemma (3.3.3) states that $(0,0)$ is the unique equilibrium on the diagonal. Every equilibrium $(w^e, y^e) \in T$ must then ensure $(0,0) \in I_1^e$ i.e.

$$y^e \geq w^e. \tag{3.29}$$

From conditions (3.27), (3.28) and (3.29) we have that every equilibrium out of the two diagonals must satisfy

$$w^e = y^e \leq \frac{y_\alpha}{2}.$$

In original reference $z_1 \times z_2$ we have that every equilibrium out of the two diagonal has to belong to the region

$$\left(\frac{1}{2}, \frac{1}{2} + \frac{y_\alpha}{2}\right] \times \left\{\frac{1}{2}\right\}$$

(see figure(3.7)). Strong congestion property (3.14) and the fact that $(z_1, z_2) = \left(\frac{1}{2}, \frac{1}{2}\right)$ is an equilibrium implies that no equilibria out of the two diagonals can be present when $\alpha \in (\alpha^*, \alpha^{**}] \cap A$.

If $\alpha > \alpha^{**}$ we have the diagonal bifurcation described in lemma (3.3.3). The whole situation is presented in figure (3.8). Condition (3.27) continues to hold for every equilibrium in T and, in particular, for the diagonal one $(w_\alpha, 0)$. It follows that

$$y_\alpha \geq w_\alpha.$$

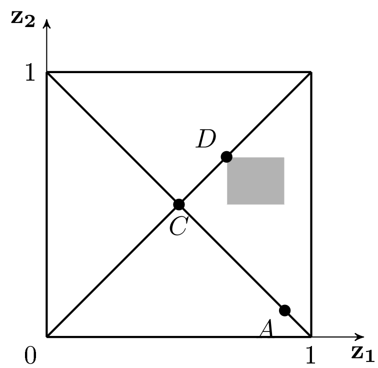


Figure 3.9: Region of possible out diagonals equilibria in T.

On the other hand, with a reasoning analogous to the one made in previous step, we have that every equilibrium in T out of the two diagonals has to satisfy (3.28).
Furthermore, the diagonal bifurcation implies that every equilibrium in T must guarantee $(w_\alpha, 0) \in I_1^e$, i.e.

$$w^e - y^e \leq w_\alpha \leq w^e + y^e. \tag{3.30}$$

In the original reference $z_1 \times z_2$, conditions (3.27), (3.28) and (3.30) imply that every equilibrium out of the two diagonals must belong to the region

$$\left[\frac{1}{2} + \frac{w_\alpha}{2}, \frac{1}{2} + \frac{y_\alpha}{2}\right] \times \left[\frac{1}{2}, \frac{1}{2} + \frac{w_\alpha}{2}\right] \tag{3.31}$$

(see figure (3.9)). The strong congestion property (3.14) allows us to discard the sides of rectangle (3.31).

Chapter 4

Simulations

In this chapter we will perform some simulations to have an empirical confirmation of the results previously presented. Let $\mu^{(N)}$ be the invariant probability measure of the Mean Field Markov Chains associated to our dynamics. Since our results regard the concentration points of $\mu^{(N)}$ when N approaches infinity, they are valid when our dynamics procede for an infinite amount of time and when an infinite number of players is involved.
As already mentioned, in case the underlying graph $\mathcal{G}$ is made of a unique complete connected component, formula (2.4) allows us to compute the invariant probability measure of the related Mean Field Markov Chain. Considering that $\mu^{(N)}$ already represents the limiting time behavior of our dynamic, we can focuse only on the size of the underlying graphs. In particular, unless specified, we will fix

$$N = 500.$$

However, to deal with real empirical results, time has to be involved. In this situation, unless specified, we will simulate the real dynamic described in chapter 1 starting from a random vector for

$$T = 30000$$

time steps.
We will procede as follows: first we fix a region L made of the unions of the neighborhoods of concetration points of $\mu^{(N)}$. Then we consider the value of $\mu^{(N)}(F_N \cap L)$ and the probability q that our dynamic ends in a state belonging to L. To estimate q we will perform 400 trials and we consider their mean success $\hat{q}_{400}$. Thanks to the central limit theorem we have that

$$q \in [\hat{q}_{400} - 0.05, \hat{q}_{400} + 0.05]$$

accordingly to a probability of around 95%.

4.1 Voter Coordination

In this section simulations will be performed on a complete graph with interaction kernels of the form

$$\Phi_s(z) = pz + (1-p)g_s(z)$$

where z represents the continuos fraction of strategy one players in the graph. As proven in chapter 2, the invariant probability measure will concentrate in a region $[a, b]$ such that $0 < a \leq b < 1$. This region is defined by the equation

$$(1-z)g_0(z) = z(1-g_1(z)).$$

To obtain $[a, b]$ we define the congestion functions g_0 and g_1 as follows:

$$g_0(z) = \begin{cases} \frac{1}{a^2}(z-a)^2 & z \in [0, a] \\ 0 & z \in [a, 1] \end{cases} \tag{4.1}$$

and

$$g_1(z) = \begin{cases} 1 & z \in [0, b] \\ 1 - \frac{1}{(1-b)^2}(z-b)^2 & x \in [b, 1]. \end{cases} \tag{4.2}$$

The probability p that our player will behave by taking into account the coordination effect will be fixed to 0.5. Both cases where $a = b$ and $b > a$ will be considered.
To deal with single point situation, we set $a = b = 0.3$ and we consider an interval

$$L = [0.3 - \varepsilon, 0.3 + \varepsilon].$$

In the strict case $\varepsilon = 0.01$ we obtain

- $\mu^{(N)}(F_N \cap L) = 0.1202$;
- $\hat{q}_{400} = 0.1375$.

If we increase the tolerance by assuming $\varepsilon = 0.1$ we have

- $\mu^{(N)}(F_N \cap L) = 0.879$;
- $\hat{q}_{400} = 0.88$.

The situation is presented in figure (4.2) at the end of the chapter. On the other hand, we study the behavior of the Mean Field Markov Chain when the concentration region is $[a, b]$ with $a = 0.3$ and $b = 0.9$. Here we fix

$$L = [0.3, 0.9].$$

In this case we have

- $\mu^{(N)}(F_N \cap L) = 0.8734$;

- $\hat{q}_{400} = 0.881$.

The situation is presented in figure (4.3) at the end of the chapter. From the histogram we notice that the distribution of the ending states in $[a, b]$ is very far from the expectations. This situation is caused by the fact that expected number of times that each player consider the voter coordination is too small. This voter coordination is what determines the distribution in $[a, b]$. We then repeat the simulation in time assuming $p = 0.7$, $N = 150$ and $T = 50000$. As we can see from figure (4.4) we have a distribution in $[a, b]$ that is more coherent with the expectations.

4.2 Symmetry of Strategies

In this section we will present the simulations related to the third chapter. We will then consider the interaction kernels of the form (3.5) and (3.6) with noise parameter $\beta = 4$. This value of β implies that both α^* and α^{**} will belong to the interval $A = [0, 1)$. In particular, in both cases we have $\alpha^* = 0.5$ and $\alpha^{**} = 0.75$.
We will start from $\mathcal{G} = K_N$. In this case we will deal with *birth and death processes*. As in previous section, invariant probability measures will be computed and related simulations in time will be performed.
We will then move to the case $\mathcal{G} = K_{N/2} \cup K_{N/2}$. In this situation, only simulations in time will be considered.

4.2.1 $\mathcal{G} = K_N$

In case $\mathcal{G} = K_N$ from the lemma (3.2.1) we know that if $\alpha \leq \alpha^{**}$ the invariant probability measure concentrates in $z = \frac{1}{2}$. Otherwise, if $\alpha > \alpha^{**}$, exists $w_\alpha \in (0, 1)$ such that the invariant probability measure concentrates in the two points $z = \frac{1}{2} \pm \frac{w_\alpha}{2}$. Since both interaction kernels (3.5) and (3.6) are characterized by the same $\alpha^{**} = 0.75$ we will consider each of them in the two situations where $\alpha = 0.33$ and $\alpha = 0.93$.

Coupled payoffs

We start from interaction kernel (3.5) with $\alpha = 0.33$. Results are presented in figure (4.5).
Assume that

$$L = \left[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right].$$

In the strict case with $\varepsilon = 0.01$ we obtain

- $\mu^{(N)}(F_N \cap L) = 0.3983$;

- $\hat{q}_{400} = 0.395$.

Comparing these values with the ones of the voter coordination with $a = b = 0.3$ we notice that we have a faster convergence of $\mu^{(N)}$. If we increase the tolerance by assuming $\varepsilon = 0.05$ we obtain

- $\mu^{(N)}(F_N \cap L) = 0.9955$;
- $\hat{q}_{400} = 1$.

We now move to the case where $\alpha = 0.93$. From lemma (3.2.1) we know that the invariant probability measure concentrates in the two points $\frac{1}{2} \pm \frac{w_\alpha}{2}$ where $w_\alpha \in (0, 1)$. We will then set

$$L = \left[\frac{1}{2} + \frac{w_\alpha}{2} - \varepsilon, \frac{1}{2} + \frac{w_\alpha}{2} + \varepsilon\right] \cup \left[\frac{1}{2} - \frac{w_\alpha}{2} - \varepsilon, \frac{1}{2} - \frac{w_\alpha}{2} + \varepsilon\right].$$

In the strict case $\varepsilon = 0.01$ we obtain

- $\mu^{(N)}(F_N \cap L) = 0.6666$;
- $\hat{q}_{400} = 0.6525$.

while, if we increase the tolerance with $\varepsilon = 0.05$, we obtain

- $\mu^{(N)}(F_N \cap L) = 1$;
- $\hat{q}_{400} = 1$.

These results are presented in figure (4.6).

Uncoupled payoffs

We will now repeat the same experiments for the interaction kernel (3.6): when $\alpha = 0.33$ (see figure (4.7)) we consider the interval

$$L = \left[\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon\right].$$

In the strict case with $\varepsilon = 0.01$ we obtain

- $\mu^{(N)}(F_N \cap L) = 0.3981$;
- $\hat{q}_{400} = 0.3675$.

If we increase the tolerance by assuming $\varepsilon = 0.05$ we obtain

- $\mu^{(N)}(F_N \cap L) = 0.9955$;
- $\hat{q}_{400} = 995$.

In the case $\alpha = 0.93$ (figure (4.8)) we have that the invariant probability measure concentrates in the two points $\frac{1}{2} \pm \frac{w_\alpha}{2}$ where $w_\alpha \in (0,1)$. We will then set

$$L = \left[\frac{1}{2} + \frac{w_\alpha}{2} - \varepsilon, \frac{1}{2} + \frac{w_\alpha}{2} + \varepsilon\right] \cup \left[\frac{1}{2} - \frac{w_\alpha}{2} - \varepsilon, \frac{1}{2} - \frac{w_\alpha}{2} + \varepsilon\right].$$

In the strict case $\varepsilon = 0.01$ we obtain

- $\mu^{(N)}(F_N \cap L) = 0.4668$;
- $\hat{q}_{400} = 0.4375$,

while, if we increase the tolerance with $\varepsilon = 0.05$, we obtain

- $\mu^{(N)}(F_N \cap L) = 0.9974$;
- $\hat{q}_{400} = 0.9975$.

4.2.2 $\mathcal{G} = K_{N/2} \cup K_{N/2}$

We now move to situations where $\mathcal{G} = K_{N/2} \cup K_{N/2}$.
In short, from theorem (3.3.1), we have that if $\alpha \leq \alpha^*$ the invariant probability measure concentrates in the center of the square $[0,1]^2$. If $\alpha^* < \alpha \leq \alpha^{**}$ there are two other possible points of concentration on the antidiagonal of the square. If $\alpha > \alpha^{**}$ the invariant probability measure can concentrate in two other points on the diagonal and in some regions out of the two diagonals (see figure (3.6.))
The simulations of this subsection aim to support following conjectures:

- the invariant probability measure can concentrate only in the equilibria of the Kurtz's System (3.17) that are locally stable;
- in our particular cases, no equilibria out of the diagonals will be present;
- our results are topologically stable: we verify that the addition of a small percentage of edges that connect the two connected components has a bounded impact on the limiting behavior of our dynamics. This percentage is computed with respect to the total number of edges. These new edges will be called 'crossing edges'.

To verify the first two suppositions we consider the stable diagonal equilibria of Kurtz's system (3.17). We then compute $\hat{q}_{400}$ on a region obtained through the union of their neighborhoods of radius 0.08 (approximately 2% of the total area). These equilibria will be

- the center $\left(\frac{1}{2}, \frac{1}{2}\right)$ if $\alpha \leq \alpha^*$;
- the two antidiagonal equilibria $\left(\frac{1}{2} \pm \frac{y_\alpha}{2}, \frac{1}{2} \mp \frac{y_\alpha}{2}\right)$ described in lemma (3.3.2) if $\alpha^* \leq \alpha < \alpha^{**}$;

- the two antidiagonal equilibria $\left(\frac{1}{2} \pm \frac{y_\alpha}{2}, \frac{1}{2} \mp \frac{y_\alpha}{2}\right)$ if $\alpha > \alpha^{**}$ and if the two diagonal equilibria $\left(\frac{1}{2} \pm \frac{w_\alpha}{2}, \frac{1}{2} \pm \frac{w_\alpha}{2}\right)$ described in lemma (3.3.3) are saddle points;
- the two antidiagonal equilibria $\left(\frac{1}{2} \pm \frac{y_\alpha}{2}, \frac{1}{2} \mp \frac{y_\alpha}{2}\right)$ and the two diagonal ones $\left(\frac{1}{2} \pm \frac{w_\alpha}{2}, \frac{1}{2} \pm \frac{w_\alpha}{2}\right)$ in the remaining situation.

From lemma (3.3.3) we know that the two diagonal equilibria, when present, will be surely saddle points if

$$\frac{\partial \Phi^\alpha}{\partial x_1}\left(\frac{1}{2} + \frac{w_\alpha}{2}, \frac{1}{2} + \frac{w_\alpha}{2}\right) > 1 \tag{4.3}$$

and will be surely locally stable if

$$\frac{\partial \Phi^\alpha}{\partial x_1}\left(\frac{1}{2} + \frac{w_\alpha}{2}, \frac{1}{2} + \frac{w_\alpha}{2}\right) < 1. \tag{4.4}$$

In the sequel we verify that our two cases both satisfy condition (4.3) if $\alpha = 0.77$ and condition (4.4) $\forall \alpha \geq 0.83$.
Since $\alpha^* = 0.5$ and $\alpha^{**} = 0.75$, the four situations described above will be respectively tested for $\alpha = 0.33$, $\alpha = 0.58$, $\alpha = 0.77$ and $\alpha = 0.93$.
To verify the topological stability of our results we add 630 crossing edges to our graph and we consider the mean success $\hat{q}^p_{400}$ of 400 trials on our perturbed graph on the same region of the disconnected case. This amount of edges is chosen in order to have a connected graph with bottleneck ratio equal to 0.01.

Coupled Payoffs

Before presenting the results of the simulations we prove that, given $\alpha > \alpha^{**}$, the two diagonal equilibria $\left(\frac{1}{2} + \frac{w_\alpha}{2}, \frac{1}{2} + \frac{w_\alpha}{2}\right)$ will be saddle points when $\alpha = \alpha_1 = 0.77$ and locally stable equilibria when $\alpha \geq \alpha_2 = 0.83$.
We first notice that

$$\frac{\partial \Phi^\alpha}{\partial x_1}(z, z) = \frac{\beta\alpha}{1 + \cosh(\beta(2\alpha - 1)(1 - 2z))} \tag{4.5}$$

is decreasing when z belongs to $\left[\frac{1}{2}, 1\right]$.
Furthermore the concavity of Φ^α expressed in property (3.16) implies that, given $\bar{z} \in \left[\frac{1}{2}, 1\right]$ and $\alpha > \alpha^{**}$, if $\Phi^\alpha(\bar{z}, \bar{z}) < \bar{z}$ then $\frac{1}{2} + \frac{w_\alpha}{2} < \bar{z}$ and, viceversa, if $\Phi^\alpha(\bar{z}, \bar{z}) > \bar{z}$ then $\frac{1}{2} + \frac{w_\alpha}{2} > \bar{z}$ (see figure (4.1)).

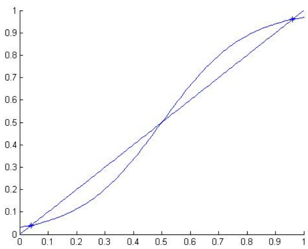


Figure 4.1: Interaction Kernel (3.5): $z = \pm\frac{w_\alpha}{2}$ when $\alpha = 0.93$.

To prove the non stability in $\alpha = \alpha_1$ we use this observation and property (4.5). In particular we notice that $\Phi^{\alpha_1}(0.8, 0.8) < 0.8$ and $\frac{\partial \Phi^{\alpha_1}}{\partial x_1}(0.8, 0.8) > 1$.

On the other hand, to prove the local stability when $\alpha \geq \alpha_2$ we notice that since $\Phi^{\alpha_2}(0.75, 0.75) > 0.75$ and w_α never decreases in α we have that

$$\frac{1}{2} + \frac{w_\alpha}{2} > 0.75 \quad \forall \alpha \geq \alpha_2.$$

We conclude by considering (4.5) and that

$$\frac{\beta\alpha_2}{1 + \cosh(\beta(2\alpha_2 - 1)(1 - 2 \cdot 0.75))} < \frac{\beta}{1 + \cosh(\beta(2\alpha_2 - 1)(1 - 2 \cdot 0.75))} < 1.$$

The results of the simulations are presented in table (4.1). Images with related phase plots of the mean field system (3.17) can be found at the end of current chapter. Notice that the mean of the trials ending up in the neighborhood of a local stable equilibrium on the two diagonals is always high. This happens expecially when α is large and also equilibria out of the two diagonals can be present. In the last two cases we can notice that we have trials that end up in the neighborhood of diagonals equilibria only when these are stable. Finally we notice that the results related to the perturbed graph are very closed to the ones related to the disconnected case. We can conclude that the outcome is consistent with our conjectures.

α	Antidiagonal equilibria	Diagonal equilibria	$\hat{q}_{400}$	$\hat{q}_{400}$ % diagonal	$\hat{q}^p_{400}$	$\hat{q}^p_{400}$ % diagonal
0.33	-	-	0.8325	-	0.8325	-
0.58	Stable	-	0.8675	-	0.7050	-
0.77	Stable	Not Stable	1	0%	0.9975	0%
0.93	Stable	Stable	1	36.25%	1	36.5%

Table 4.1: Interaction kernel (3.5): simulation results.

α	Antidiagonal equilibria	Diagonal equilibria	$\hat{q}_{400}$	$\hat{q}_{400}$ % diagonal	$\hat{q}^p_{400}$	$\hat{q}^p_{400}$ % diagonal
0.33	-	-	0.8450	-	0.8375	-
0.58	Stable	-	0.6675	-	0.6425	-
0.77	Stable	Not Stable	0.9850	0%	0.9825	0%
0.93	Stable	Stable	0.9975	32.83%	0.995	12.31%

Table 4.2: Interaction kernel (3.5): simulation results.

Uncoupled Payoffs

As in previuos case we prove that, given $\alpha > \alpha^{**}$, the two diagonal equilibria $\left(\frac{1}{2}+\frac{w_\alpha}{2}, \frac{1}{2}+\frac{w_\alpha}{2}\right)$ will be saddle points if $\alpha = \alpha_1 = 0.77$ and will be locally stable when $\alpha \geq \alpha_2 = 0.83$.
The procedure will be analogous of the previous case by considering the concavity property of Φ^α (3.16) and the fact that

$$\frac{\partial \Phi^\alpha}{\partial x_1}(z,z) = \frac{\beta\alpha}{1+\cosh(\beta(1-2z))}$$

is decreasing in $\left[\frac{1}{2}, 1\right]$.
In particular, to prove the non stability in $\alpha = \alpha_1$ we notice that since $\Phi^{\alpha_1}(0.65, 0.65) < 0.65$ then $\frac{1}{2}+\frac{w_{\alpha_1}}{2} < 0.65$. The result follows by considering that

$$\frac{\partial \Phi^{\alpha_1}}{\partial x_1}(0.65, 0.65) > 1.$$

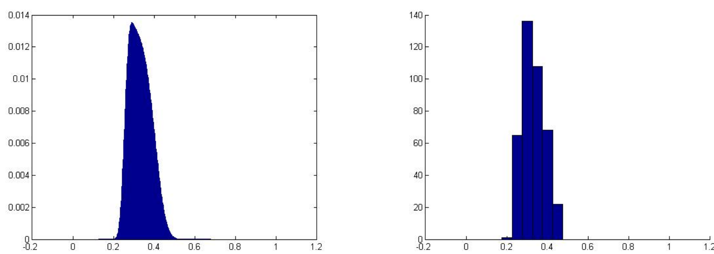
To prove the local stability when $\alpha \geq \alpha_2$ we consider that $\Phi^{\alpha_2}(0.75, 0.75) > 0.75$ and w_α never decreases in α; we then have that

$$\frac{1}{2}+\frac{w_\alpha}{2} > 0.75 \quad \forall \alpha \geq \alpha_2.$$

We can conclude by considering that

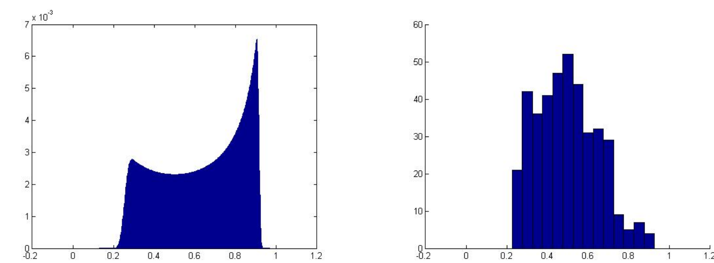
$$\frac{\beta\alpha_2}{1+\cosh(\beta(1-2\cdot 0.75))} < \frac{\beta}{1+\cosh(\beta(1-2\cdot 0.75))} < 1.$$

The results of the simulations are presented in table (4.2). Images with related phase plots of the mean field system (3.17) can be found at the end of current chapter. Again, we notice that the results confirm our conjectures.



(a) *Invariant probability measure.* (b) *Time simulation: histogram of 400 trials.*

Figure 4.2: Voter coordination with concentration point $a = 0.3$.



(a) *Invariant probability measure.* (b) *Time simulation: histogram of 400 trials.*

Figure 4.3: Voter coordination with concentration interval $[0.3, 0.9]$.

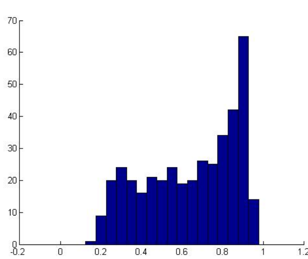
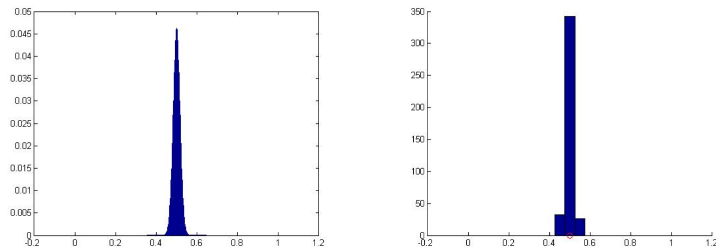
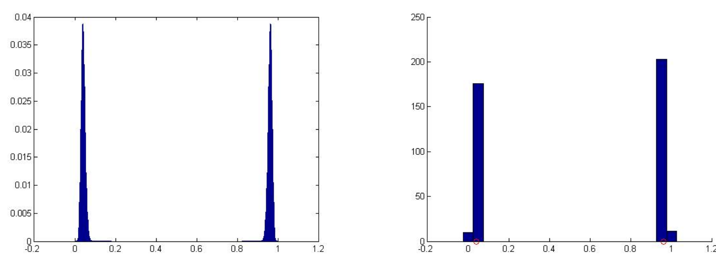


Figure 4.4: Voter coordination on $N = 150$ players with concentration interval $[0.3, 0.9]$ and $p = 0.7$: histogram of 400 trials after $T = 50000$ time steps.



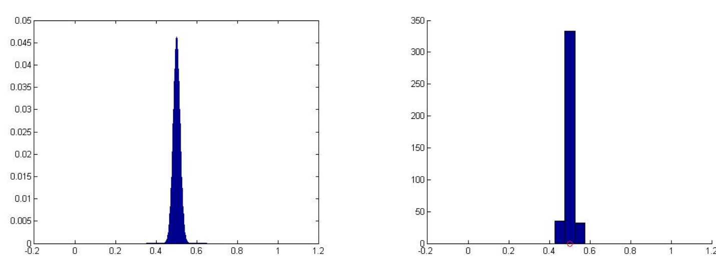
(a) *Invariant probability measure.* (b) *Time simulation: histogram of 400 trials.*

Figure 4.5: Interaction kernel (3.5) on $\mathcal{G} = K_N$ when $\beta = 4$ and $\alpha = 0.33$.



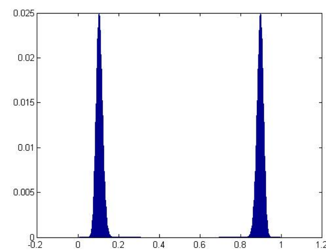
(a) *Invariant probability measure.* (b) *Time simulation: histogram of 400 trials.*

Figure 4.6: Interaction kernel (3.5) on $\mathcal{G} = K_N$ when $\beta = 4$ and $\alpha = 0.93$.

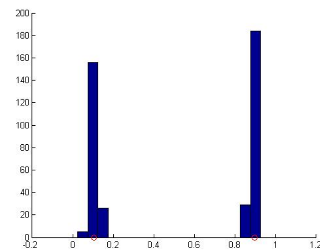


(a) *Invariant probability measure.* (b) *Time simulation: histogram of 400 trials.*

Figure 4.7: Interaction kernel (3.6) on $\mathcal{G} = K_N$ when $\beta = 4$ and $\alpha = 0.33$.

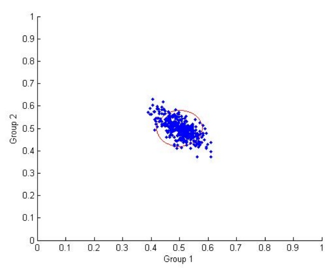


(a) *Invariant probability measure.*

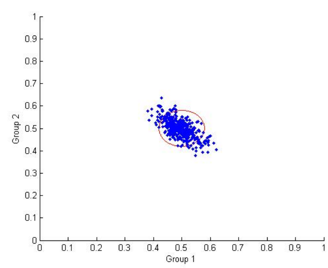


(b) *Time simulation: histogram of 400 trials.*

Figure 4.8: Interaction kernel (3.6) on $\mathcal{G} = K_N$ when $\beta = 4$ and $\alpha = 0.93$.



(a) *Disconnected graph* $\hat{q} = 0.8325$.



(b) *Perturbed graph* $\hat{q}^p = 0.8325$.

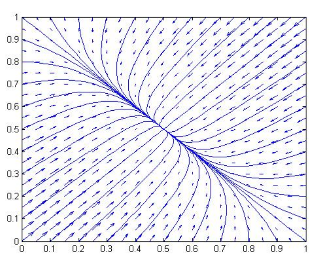
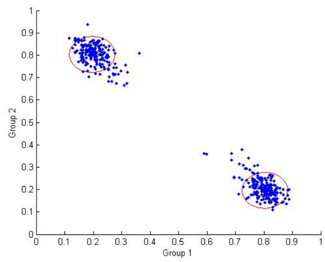
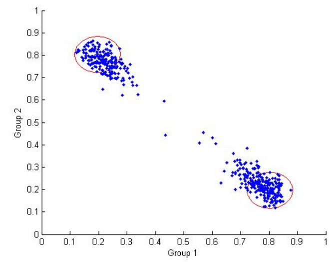


Figure 4.9: Interaction kernel (3.5) with $\alpha = 0.33$: central equilibrium.



(a) *Disconnected graph* $\hat{q} = 0.8375$.



(b) *Perturbed graph* $\hat{q}^p = 0.7050$.

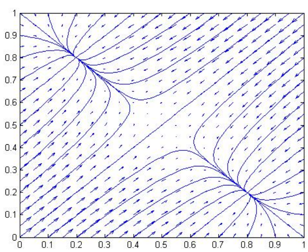
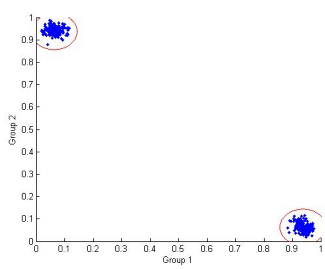
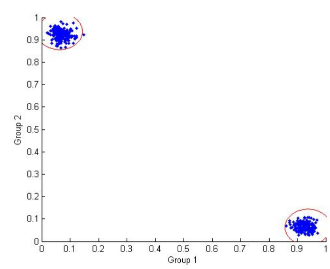


Figure 4.10: Interaction kernel (3.5) with $\alpha = 0.58$: antidiagonal equilibria locally stable.



(a) *Disconnected graph* $\hat{q} = 1$.



(b) *Perturbed graph* $\hat{q}^p = 0.9975$.

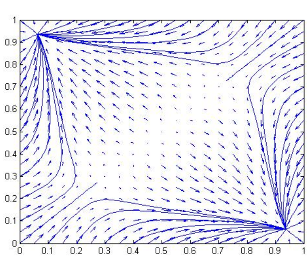
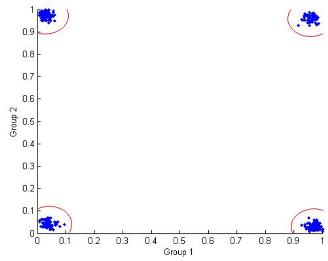
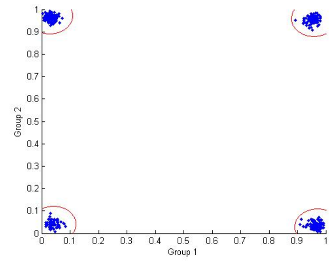


Figure 4.11: Interaction kernel (3.5) with $\alpha = 0.77$: diagonal equilibria not stable.



(a) *Disconnected graph* $\hat{q} = 1$. *Diagonal percentage of success* 36.5%.



(b) *Perturbed graph* $\hat{q}^p = 0.995$. *Diagonal percentage of success* 36.25%.

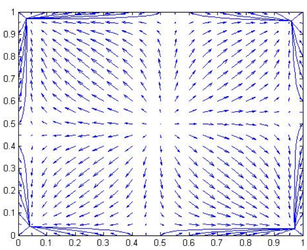
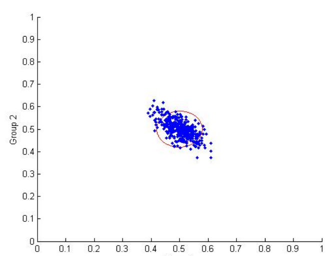
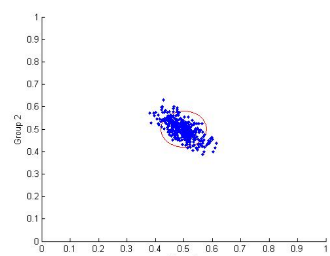


Figure 4.12: Interaction kernel (3.5) with $\alpha = 0.93$: diagonal equilibria locally stable.



(a) *Disconnected graph* $\hat{q} = 0.7925$.



(b) *Perturbed graph* $\hat{q}^p = 0.7975$.

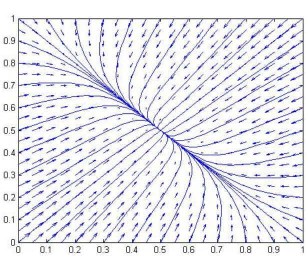
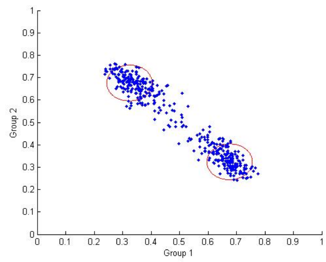
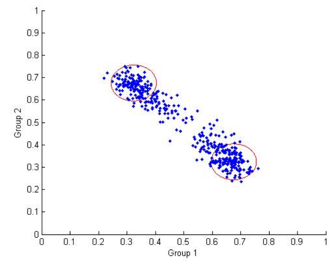


Figure 4.13: Interaction kernel (3.6) with $\alpha = 0.33$: central equilibrium.



(a) *Disconnected graph* $\hat{q} = 0.8675$.



(b) *Perturbed graph* $\hat{q}^p = 0.71$.

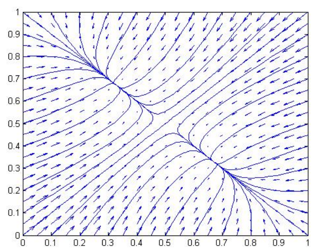
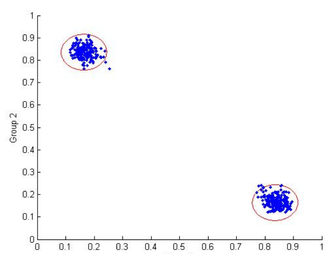
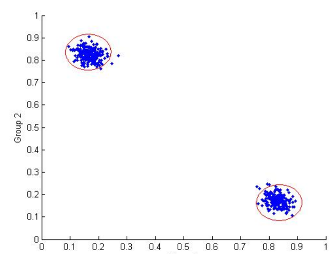


Figure 4.14: Interaction kernel (3.6) with $\alpha = 0.58$: antidiagonal equilibria locally stable.



(a) *Disconnected graph* $\hat{q} = 0.9675$.



(b) *Perturbed graph* $\hat{q}^p = 0.92$.

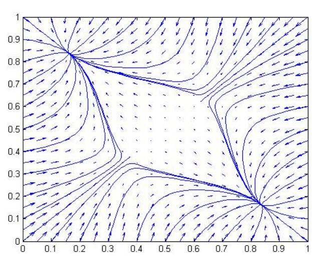
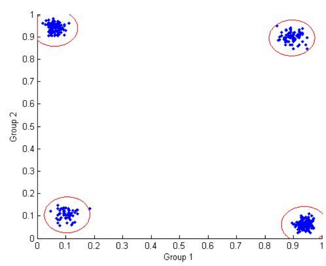
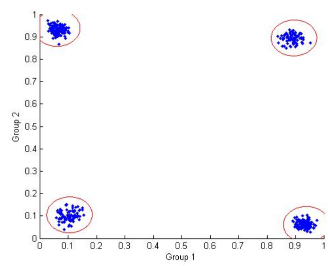


Figure 4.15: Interaction kernel (3.6) with $\alpha = 0.77$: diagonal equilibria not stable.



(a) *Disconnected graph* $\hat{q} = 0.9975$. *Diagonal percentage of success* 32.83%.



(b) *Perturbed graph* $\hat{q}^p = 1$. *Diagonal percentage of success* 39, 75%.

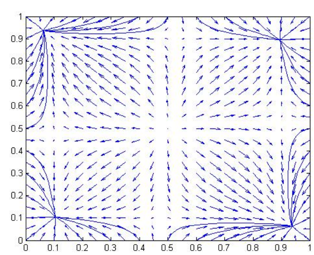


Figure 4.16: Interaction kernel (3.6) with $\alpha = 0.93$: diagonal equilibria locally stable.

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