Dynamics in network games with local coordination and global congestion effects

Gianluca Brero, Giacomo Como, and Fabio Fagnani

***Abstract*— Several strategic interactions over social networks display both negative and positive externalities at the same time. E.g., participation to a social media website with limited resources is more appealing the more of your friends participate, while a large total number of participants may slow down the website (because of congestion effects) thus making it less appealing. Similarly, while there are often incentives to choose the same telephone company as the friends and relatives with whom you interact the most frequently, concentration of the market share in the hands of a single firm typically leads to higher costs because of the lack of competition.**

In this work, we study evolutionary dynamics in network games where the payoff of each player is influenced both by the actions of her neighbors in the network, and by the aggregate of the actions of all the players in the network. In particular, we consider cases where the payoff increases in the number of neighbors who choose the same action (local coordination effect) and decreases in the total number of players choosing the same action (global congestion effect). We study noisy best-response dynamics in networks which are the union of two complete graphs, and prove that the asymptotic behavior of the invariant probability distribution is characterized by two phase transitions with respect to a parameter measuring the relative strength of the local coordination with respect to the global congestion effects. Extensions to random networks with strong community structure are studied through simulations.

***Index Terms*— network games, congestion games, coordination games, evolutionary dynamics.**

I. INTRODUCTION

Many social interactions exhibit a variety of distinct levels at which agents may influence each other. Typically, different network architectures can be introduced to model interactions happening through different social channels (friends, trust and reviewing web mechanisms, media, markets) and which may also be characterized by externalities of opposite sign. An interesting family of such behaviors are resource allocation problems where agents may have the need/wish to coordinate with their friends or geographically close partners, while, at the same time, they might be affected by congestion effects driven by the behavior of the global population. For example, participation to a social media website with limited resources is more appealing the more of your friends participate, while a large total number of participants may have a negative effect as congestion may determine an unpleasant slow down of the social resource. The choice of a company offering a given service is another instance of such problems: while there are often incentives to choose the same telephone company as the friends and relatives with whom you interact the most frequently, concentration of the market share in the hands of a single firm typically leads to higher costs because of the lack of competition.

In this paper we will concentrate on a model whereby a population of agents are simultaneously engaged in two distinct games. At one level, agents are connected through a given (friendship) network and are playing a classical coordination game. At another level, they are influenced by the choices of all agents (even those who are not directly connected to through the graph) and this is modeled by a congestion game (see [11]). In this paper we assume the option space of each agent to be binary $\mathcal{A} = \{0, 1\}$ and games to be semi-anonymous: each agent will be endowed with two payoff functions, one depending only on number of 0's and 1's in the agent neighborhood and one depending only on the aggregate number of 0's and 1's in the global population. Specifically, we consider cases where the local payoff increases in the number of neighbors who choose the same action (local coordination effect) while the global one decreases in the total number of players choosing the same action (global congestion effect).

Dynamics will be introduced in a classical way as a continuous-time Markov chain on the configuration space $\mathcal{A}^N$ driven by an interaction kernel: we assume each agent to be equipped with a Poisson clock whose click determines a possible revision of the agent's option. Revision is modeled as a (generalized) noisy best response action depending on a parameter $\alpha \in [0, 1]$ which interpolates between the pure global congestion case $\alpha = 0$ and the pure local coordination case $\alpha = 1$. This general revision model encompasses the situation of a noisy best response with respect to a payoff function which is the convex combination of the local and global ones, as well the situation where with probabilities α and $1-\alpha$, respectively, the agent makes a best response with respect to the global or to the local payoff function.

As the Markov chain is ergodic, its asymptotic behavior can be conveniently studied by analyzing the corresponding invariant probability distribution. A typical issue arising in this context concerns the possible concentration phenomena for such invariant probabilities in the large scale limit, i.e., when the number of agents n grows large, while the graph maintains some specific structure. In this paper we focus on the case when the network is the union of two complete

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graphs of size $n/2$ each. Analysis is carried on based on Kurtz's Theorem (see Draief and Massoulié [2] and Wormald [12]) which allows one to approximate the original Markov chain by a system of two ODE's in the variables ρ_1 and ρ_2 describing the fraction of 1's in the two populations. The two ODE's are coupled through the congestion term. A general result allows one to conclude that in the large scale limit, the invariant probabilities of the original chain concentrate into the Birkhoff center of the system of ODEs (see Sandholm [10], section 12B). A careful analysis of the Birkhoff center allows us to obtain the result presented in this paper which is the fact that the large scale behavior of the invariant probability distribution can encounter two phase transitions with respect to the parameter α. For small values of α corresponding to predomination of the congestion term, the system exhibits a globally attractive equilibrium in $(1/2, 1/2)$: 0's and 1's are uniformly distributed in the two populations and no coordination takes place. While α increases, if the first phase transition occurs, $(1/2, 1/2)$ becomes unstable and two new locally stable equilibria symmetrically placed with respect to the line $\rho_1 = \rho_2$ show up: the two populations show a bias towards coordination maintaining the congestion as low as possible. Finally, if also the second phase transition belongs to the domain where α varies, when α passes this second threshold the Birkhoff center gets larger including two further diagonal points and four nearby small rectangles: the congestion term has a small impact and also configuration with a global bias of 0's or of 1's will appear in the long range behavior.

II. Model

A. Games on networks

We model networks as finite undirected graphs $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where nodes represent players. Throughout, $n = |\mathcal{V}|$ will denote the population size, d_i will stand for the degree of a node i in $\mathcal{G}$, and we will write $i \sim j$ to mean that two nodes i and j are neighbors in $\mathcal{G}$, i.e., $\{i, j\}$ is a link. In order to avoid trivial cases, we will always assume that $d_i \geq 1$ for all $i \in \mathcal{V}$. In our study every player is endowed with two games sharing the same binary action set $\mathcal{A} := \{0, 1\}$. The first one models the local coordination effect and its payoffs are functions of the actions of the neighbors of the player; the second one represents the global congestion trend with payoffs depending on the aggregate of the actions of the whole population of players.
We further restrict ourselves to the case where the dependence of the payoff on the player's neighbors' actions is only through their aggregate, and is homogeneous in the population.

In order to formalize this setting, let us denote by $\mathbf{a} \in \mathcal{A}^{\mathcal{V}}$ the action profile, whose i-th component a_i stands for player i's action. Let

$$\theta(\mathbf{a}) := \frac{1}{n}\sum_{i\in\mathcal{V}} a_i\,, \qquad \theta^i(\mathbf{a}) := \frac{1}{d_i}\sum_{j\sim i} a_j\,, \qquad i \in \mathcal{V}\,,$$

denote, respectively, the fractions of players in the whole population, and the fraction of neighbors of node i, choosing action 1. Finally, let

$$p^l_a : [0,1] \to \mathbb{R}\,, \qquad p^g_a : [0,1] \to \mathbb{R}\,, \qquad a \in \mathcal{A}\,,$$

be, respectively, the local and global payoff functions associated to players choosing action a. Throughout, we will refer to the game with players population $\mathcal{V}$, action space $\mathcal{A}^{\mathcal{V}}$, and payoffs

$$P^l_i(\mathbf{a}) = p^l_{a_i}(\theta^i(\mathbf{a}))\,, \qquad i \in \mathcal{V}\,,\, \mathbf{a} \in \mathcal{A}^{\mathcal{V}}\,,$$

as the (semi-anonymous) local coordination game on $\mathcal{G}$ with payoffs $\{p^l_a\}_{a\in\mathcal{A}}$. The game with players population $\mathcal{V}$, action space $\mathcal{A}^{\mathcal{V}}$, and payoffs

$$P^g_i(\mathbf{a}) = p^g_{a_i}(\theta(\mathbf{a}))\,, \qquad i \in \mathcal{V}\,,\, \mathbf{a} \in \mathcal{A}^{\mathcal{V}}\,,$$

will be referred as the (semi-anonymous) global congestion game on $\mathcal{G}$ with payoffs $\{p^l_a\}_{a\in\mathcal{A}}$.

In this work, we will focus on differentiable payoff functions; the local coordination and global congestion effects will be then modeled through the assumptions that

$$\frac{\mathrm{d}}{\mathrm{d}x_l} p^l_0(x_l) < 0\,, \qquad \frac{\mathrm{d}}{\mathrm{d}x_l} p^l_1(x_l) > 0\,, \qquad 0 \leq x_l \leq 1\,, \tag{1}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x_g} p^g_0(x_g) > 0\,, \qquad \frac{\mathrm{d}}{\mathrm{d}x_g} p^g_1(x_g) < 0\,, \qquad 0 \leq x_g \leq 1\,. \tag{2}$$

Example 1: Consider gocal coordination payoff functions

$$p^l_1(x_l) = 1 - p^l_0(x_l) = x_l$$

and global congestion payoff functions

$$p^g_1(x_g) = 1 - p^g_0(x_g) = 1 - x_g.$$

B. The dynamics

We now pass to describing the dynamics. We will consider continuous-time Markov chains $\mathbf{A}(t)$ with state space $\mathcal{A}^{\mathcal{V}}$. Here, the i-th component of $\mathbf{A}(t)$ denotes the action chosen by player i at any time $t \geq 0$. Throughout, we will assume that players are equipped with independent rate-1 Poisson clocks. If her clock ticks at time t, player i updates her action $A_i(t)$ to some $A_i(t^+)$ with conditional probability distribution

$$\begin{aligned} \mathbb{P}(A_i(t^+) = 1|\mathbf{A}(t)) &= 1 - \mathbb{P}(A_i(t^+) = 0|\mathbf{A}(t)) \\ &= \Phi(\theta^i(\mathbf{A}(t)), \theta(\mathbf{A}(t)))\,, \end{aligned} \tag{3}$$

where

$$\Phi : [0,1]^2 \to [0,1]\,,$$

is referred to as the *interaction kernel*. Throughout this paper, we will refer to the continuous-time Markov chain described above as the *network dynamics* associated to the network $\mathcal{G}$ and the interaction kernel Φ. To study the different dynamical behaviors the emerge when varying the power balance between coordination and congestion we let the interaction kernel Φ depend in a regular way on a parameter $\alpha \in [0, 1]$ that measures the relative weight of coordination with respect to congestion. In particular we assume that $\alpha = 0$ models a situation where no coordination is present; on the other hand, $\alpha = 1$ represents a pure coordination case. This translates into the two conditions

$$\frac{\partial}{\partial x_l}\Phi^0_\beta(x_l,x_g) = 0\,, \qquad \forall x_l, x_g \in [0,1]\,, \tag{4}$$

$$\frac{\partial}{\partial x_g}\Phi^1_\beta(x_l,x_g) = 0\,, \qquad \forall x_l, x_g \in [0,1]\,. \tag{5}$$

Since we are interested in situations where both local coordination and global congestion are present, we will limit our results to the case $\alpha \in (0,1)$ and we will assume that, when α varies in this domain,

$$\frac{\partial}{\partial x_l}\Phi^\alpha_\beta(x_l,x_g) > 0\,,\ \frac{\partial}{\partial x_g}\Phi^\alpha_\beta(x_l,x_g) < 0\,,\ x_l,x_g \in [0,1]\,. \tag{6}$$

We still have to define the way α empowers the coordination with respect to the congestion. This is modeled with reference to the two examples presented hereafter.

Example 2: Consider the payoff functions

$$p_a(x_l,x_g) = p^\alpha_a(x_l,x_g) := \alpha p^l_a(x_l) + (1-\alpha)p^g_a(x_g)\,,\ a \in \mathcal{A}\,.$$

Throughout, we will refer to the game with players population $\mathcal{V}$, action space $\mathcal{A}^\mathcal{V}$, and payoffs

$$P_i(\mathbf{a}) = p_{a_i}(\theta^i(\mathbf{a}),\theta(\mathbf{a}))\,, \qquad i \in \mathcal{V}\,,\, \mathbf{a} \in \mathcal{A}^\mathcal{V}\,,$$

as the (semi-anonymous) local coordination-global congestion game on $\mathcal{G}$ with payoffs $\{p_a\}_{a\in\mathcal{A}}$.

Notice that, in the case where our local coordination-global congestion game is a pure congestion game ($\alpha = 0$), Nash equilibria $\mathbf{a}^* \in \mathcal{A}^\mathcal{V}$ can be proved to satisfy $\theta(\mathbf{a}^*) = x^* + O(1/n)$, where

$$x^* := \mathrm{argmin}\left\{|p^g_1(x_g) - p^g_0(x_g)| : x_g \in [0,1]\right\}\,.$$

On the other hand, when our local coordination-global congestion game is a pure coordination game ($\alpha = 1$), Nash equilibria $\mathbf{a}^* \in \mathcal{A}^\mathcal{V}$ includes all the action profiles $\mathbf{a}^* \in \mathcal{A}^\mathcal{V}$ such that i and j belong to the same connected component of $\mathcal{G}$ and $a^*_i = a^*_j$ (see Jackson and Zenou [6], section 3.3.2). For some $\beta > 0$, put

$$\Phi(x_l,x_g) = \Phi^\alpha_\beta(x_l,x_g) := \frac{e^{\beta p^\alpha_1(x_l,x_g)}}{e^{\beta p^\alpha_1(x_l,x_g)} + e^{\beta p^\alpha_0(x_l,x_g)}}\,.$$

The network dynamics associated to a network $\mathcal{G}$ and the interaction kernel Φ_β as above is known as the *noisy best response* dynamics. Here, $1/\beta$ is a measure of noise. In fact, observe that

$$\lim_{\beta\to\infty}\Phi^\alpha_\beta(x_l,x_g) = \begin{cases} 0 & \text{if} \quad p^\alpha_0(x_l,x_g) > p^\alpha_1(x_l,x_g) \\ 1/2 & \text{if} \quad p^\alpha_0(x_l,x_g) = p^\alpha_1(x_l,x_g) \\ 1 & \text{if} \quad p^\alpha_0(x_l,x_g) < p^\alpha_1(x_l,x_g)\,, \end{cases}$$

i.e., in the large β, i.e., small noise, limit the best response function is recovered.

Example 3: For some $\beta > 0$, put

$$\Phi(x_l,x_g) \;=\; \frac{\alpha e^{\beta p^l_1(x_l)}}{e^{\beta p^l_1(x_l)} + e^{\beta p^l_0(x_l)}} + \frac{(1-\alpha)e^{\beta p^g_1(x_g)}}{e^{\beta p^g_1(x_g)} + e^{\beta p^g_0(x_g)}}\,.$$

The interaction kernel above can be interpreted as the result of the superposition of a noisy best response to the local payoffs, and one to the global payoffs, with relative frequency α and $(1-\alpha)$, respectively.

If we consider the local and global payoffs of Example 1, we then obtain that the interaction kernels of Examples 2 and 3 satisfy the following symmetry property:

$$\Phi^\alpha_\beta(1-x_l,1-x_g) = 1 - \Phi^\alpha_\beta(x_l,x_g)\,, \tag{7}$$

fo rall $x_l,x_g \in [0,1]$, $\alpha \in [0,1)$. In this particular situation, the way α empowers the coordination with respect to the congestion can be represented through the following assumption

$$\frac{\partial}{\partial\alpha}\Phi^\alpha_\beta(x_l,x_g) \le 0\,, \qquad \forall x_l,x_g \in [0,1/2],\, \forall\alpha \in [0,1). \tag{8}$$

Moreover, thanks to the convexity properties of the exponential function, one can prove that the equations

$$\Phi^\alpha_\beta\left(x,\frac{1}{2}\right) = x\,, \qquad \Phi^\alpha_\beta(x,x) = x \tag{9}$$

admit no more than one solution $x \in [0,1/2)$. In the next section we will see how those properties play an important role in determining the behavior of the Markov chain $\mathbf{A}(t)$.

III. Results

In this section, we present our results. These concern the behavior of stationary probability distributions of the Markov chain $\mathbf{A}(t)$. Throughout this section, we will restrict ourselves to the case where the network is the disjoint union of two fully connected components of equal size, as for the following

Assumption 1: For positive even n, the network $\mathcal{G}_n = (\mathcal{V},\mathcal{E})$ has node set $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$, with $|\mathcal{V}_1| = |\mathcal{V}_2| = n/2$, and link set $\mathcal{E}$ such that $\{i,j\} \in \mathcal{E}$ if and only if $i \neq j \in \mathcal{V}_1$ or $i \neq j \in \mathcal{V}_2$.

The key simplification implied by Assumption 1 along with the restriction to semi-anonymous games, is that the pair $\rho^n(t) = (\rho^n_1(t),\rho^n_2(t))$, where

$$\rho^n_k(t) := \frac{2}{n}\sum_{i\in\mathcal{V}_k} A_i(t)\,, \qquad k = 1,2\,, \tag{10}$$

forms a Markov chain. This implies a dramatic reduction in the size of the state space, from $|\mathcal{A}^\mathcal{V}| = 2^n$ to $(n/2+1)^2$. Our focus will be on the stationary probability distributions μ_n of $\rho^n(t)$, and on their behavior in the limit as the population size n grows large. We will interpret μ_n as a probability measure on $[0,1]^2$ and use the following terminology

Definition 1: Let $\{\mu_n\}$ be a sequence of probability measures on $[0,1]^2$ and $E \subseteq [0,1]^2$ a subset. Then, μ_n *concentrate on E as n grows large*, if $\lim_n \mu_n(\mathcal{O}) = 1$ for every open $\mathcal{O} \subseteq [0,1]^2$ such that $E \subseteq \mathcal{O}$.

We can now introduce the result of this paper. It states the existence of two threshold values

$$\alpha^* = \inf\left\{\alpha \in [0,1) :\ \frac{\partial\Phi^\alpha}{\partial x_l}\left(\frac{1}{2},\frac{1}{2}\right) > 1\right\} \tag{11}$$

$$\alpha^{**} = \inf\left\{\alpha \in [0,1) : \frac{\partial \Phi^\alpha}{\partial x_l}\left(\frac{1}{2},\frac{1}{2}\right) + \frac{\partial \Phi^\alpha}{\partial x_g}\left(\frac{1}{2},\frac{1}{2}\right) > 1\right\} \tag{12}$$

such that $0 < \alpha^* \leq \alpha^{**}$ and the asymptotic behaviors of the stationary probability distributions of $\mathbf{A}(t)$ in the limit as n grows large are qualitatively different, depending on whether $\alpha \in [0, \alpha^*]$, $\alpha \in (\alpha^*, \alpha^{**}]$, or $\alpha \in (\alpha^{**}, 1)$.

Theorem 1: Let $\Phi^\alpha : [0,1]^2 \to [0,1]$, $0 \leq \alpha < 1$ be a family of interaction kernels satisfying properties (4), (6)-(9), and let α^* and α^{**} be defined as in (11) and (12), respectively. Let $\mathbf{A}(t)$ be the network dynamics associated to Φ^α and a network $\mathcal{G}_n$ as in Assumption 1, and let μ_n be an invariant probability distribution for $\rho^n(t)$ defined as in (10). Furthermore, let

$$a : [\alpha^*, 1) \to [0,1)\,, \qquad d : [\alpha^{**}, 1) \to [0,1)$$

be two continuous nondecreasing functions such that $a(\alpha^*) = d(\alpha^{**}) = 0$, and $d(\alpha) \leq a(\alpha)$ for $\alpha \in [\alpha^{**}, 1)$, and R_α be the union of interiors of the four rectangles (see the gray-shaded areas in Figure 1 (c)) with vertices

- $(\frac{1}{2} + \frac{d(\alpha)}{2}, \frac{1}{2}), (\frac{1}{2} + \frac{a(\alpha)}{2}, \frac{1}{2})$,
  $(\frac{1}{2} + \frac{a(\alpha)}{2}, \frac{1}{2} + \frac{d(\alpha)}{2}), (\frac{1}{2} + \frac{d(\alpha)}{2}, \frac{1}{2} + \frac{d(\alpha)}{2})$;
- $(\frac{1}{2}, \frac{1}{2} + \frac{d(\alpha)}{2}), (\frac{1}{2}, \frac{1}{2} + \frac{a(\alpha)}{2})$,
  $(\frac{1}{2} + \frac{d(\alpha)}{2}, \frac{1}{2} + \frac{a(\alpha)}{2}), (\frac{1}{2} + \frac{d(\alpha)}{2}, \frac{1}{2} + \frac{d(\alpha)}{2})$;
- $(\frac{1}{2} - \frac{d(\alpha)}{2}, \frac{1}{2}), (\frac{1}{2} - \frac{a(\alpha)}{2}, \frac{1}{2})$,
  $(\frac{1}{2} - \frac{a(\alpha)}{2}, \frac{1}{2} - \frac{d(\alpha)}{2}), (\frac{1}{2} - \frac{d(\alpha)}{2}, \frac{1}{2} - \frac{d(\alpha)}{2})$;
- $(\frac{1}{2}, \frac{1}{2} - \frac{d(\alpha)}{2}), (\frac{1}{2}, \frac{1}{2} - \frac{a(\alpha)}{2})$,
  $(\frac{1}{2} - \frac{d(\alpha)}{2}, \frac{1}{2} - \frac{a(\alpha)}{2}), (\frac{1}{2} - \frac{d(\alpha)}{2}, \frac{1}{2} - \frac{d(\alpha)}{2})$.

Then, as n grows large μ_n concentrate on a set $E_\alpha \subseteq [0,1]^2$, where

(i) for $\alpha \in [0, \alpha^*]$,

$$E_\alpha = \{(1/2, 1/2)\}\,;$$

(ii) for $\alpha \in (\alpha^*, \alpha^{**}]$,

$$E_\alpha = \{(1/2, 1/2)\} \cup \{(1/2 \mp a(\alpha), 1/2 \pm a(\alpha))\}\,;$$

(iii) for $\alpha \in (\alpha^{**}, 1)$,

$$E_\alpha = \{(1/2, 1/2)\} \cup \{(1/2 \mp a(\alpha), 1/2 \pm a(\alpha))\} \cup \{(1/2 \pm d(\alpha), 1/2 \pm d(\alpha))\} \cup \mathcal{R}_\alpha\,.$$

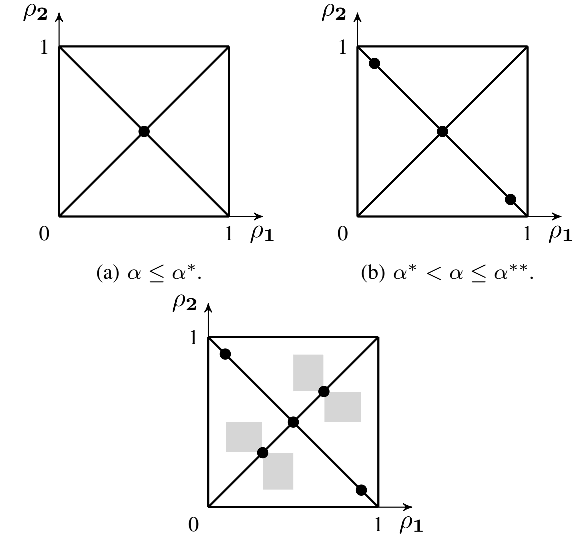
The three cases of Theorem 1 are illustrated in Figure 1.

A. (Sketch of the) proof of Theorem 1

The proof of Theorem 1 is based on the study of the asymptotic properties of the following mean-field dynamical system:

$$\begin{aligned} \dot{\rho}_1 &= \Phi^\alpha\left(\rho_1, \tfrac{\rho_1(t)+\rho_2(t)}{2}\right) - \rho_1(t) \\ \dot{\rho}_2 &= \Phi^\alpha\left(\rho_2, \tfrac{\rho_1(t)+\rho_2(t)}{2}\right) - \rho_2(t). \end{aligned} \tag{13}$$

Standard results allow one to relate both the transient and the asymptotic behaviors of $\rho^n(t)$ and $\rho(t)$. In particular, Kurtz's Theorem presented in [2] and [12], and results in [10, Sect. 12B] imply that μ_n concentrates in the Birkhoff center of the dynamical system (13). Then, in order to prove



(a) $\alpha \leq \alpha^*$. (b) $\alpha^* < \alpha \leq \alpha^{**}$. (c) $\alpha > \alpha^{**}$.

Fig. 1: Set E_α

Theorem 1, it is sufficient to show that the Birkhoff center of (13) is contained in E_α.

Proposition 1: For every $\alpha \in [0,1)$, and every $\rho^0 \in [0,1]^2$, the solution of the dynamical system (13) with initial condition $\rho(0) = \rho^0$ converges to an equilibrium $\rho^* \in E_\alpha$.

The key observation is that (13) is a competitive dynamical system in the sense of Hirsch (see Hirsch [4]). In turn, this implies the following monotonicity property: let ϕ^t be the semiflow associated to the dynamical system (13). Then, for every ρ^1 and $\rho^2 \in [0,1]^2$ such that

$$\rho_1^1 \leq \rho_1^2 \text{ and } \rho_2^1 \geq \rho_2^2$$

one has that

$$\phi_1^t(\rho_1) \leq \phi_1^t(\rho_2) \text{ and } \phi_2^t(\rho_1) \geq \phi_2^t(\rho_2) \quad \forall t \geq 0. \tag{14}$$

It is a standard result (see Cañada, Drabek and Fonda [1], Theorem 3.22) that all trajectories of a 2-dimensional monotone system have each component eventually non increasing or nondecreasing and hence converging to a limit point, which, in our case, is necessarily an equilibrium.

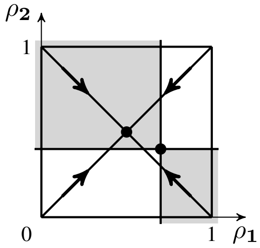
Furthermore we can easily notice that property (14) implies that every equilibrium $(\rho_1^e, \rho_2^e) \in [0,1]^2$ determines the two invariant sets

$$\left\{(\rho_1, \rho_2) \in [0,1]^2 : \rho_1 \leq \rho_1^e \text{ and } \rho_2 \geq \rho_2^e\right\}$$

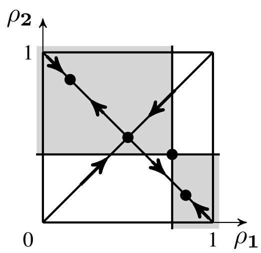
and

$$\left\{(\rho_1, \rho_2) \in [0,1]^2 : \rho_1 \geq \rho_1^e \text{ and } \rho_2 \leq \rho_2^e\right\}.$$

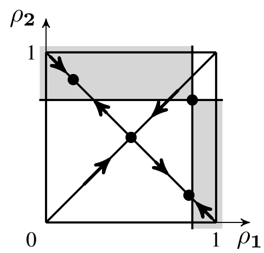
From the symmetry property (7) one can notice that the diagonal $\rho_1 = \rho_2$ and the anti-diagonal $\rho_1 = 1 - \rho_2$ of the square $[0,1]^2$ are invariant subsets for our system (13). It follows that the center $\rho_1 = \rho_2 = \frac{1}{2}$ will always be an equilibrium of our system for every α in $[0,1)$. Furthermore, thanks to properties (4), (6) and (8), the two values α^* and α^{**} defined in (11) and (12) characterize the transition of



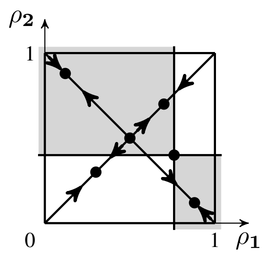
(a) $\alpha \leq \alpha^*$.



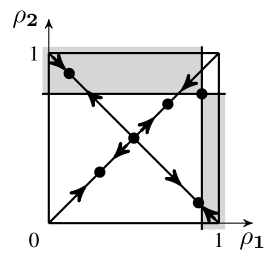
(b) $\alpha^* < \alpha \leq \alpha^{**}$.



(c) $\alpha^* < \alpha \leq \alpha^{**}$.



(d) $\alpha^{**} < \alpha$.



(e) $\alpha^{**} < \alpha$.

Fig. 2: Using monotonicity to prove that there cannot exist equilibria outside E_α. In (a), the existence of an equilibrium outside $(1/2, 1/2)$ would prevent the trajectory starting from some diagonal or some anti-diagonal point to converge to $(1/2, 1/2)$. In (b) and (c) the existence of an equilibrium outside the anti-diagonal set would prevent either the trajectory starting from some anti-diagonal point to converge to the two stable equilibria or some the trajectory starting from some diagonal point to converge to $(1/2, 1/2)$. In (d) and (e), the analogous argument allows one to exclude the existence equilibria outside the union of the three anti-diagonal equilibria and the four rectangles.

this equilibrium respectively from a locally asymptotically stable equilibrium to a saddle point and from a saddle point to an unstable focus. Indeed, from assumptions (4) and (6) and given the regularity of Φ with respect to α, we notice that the equilibrium $\rho_1 = \rho_2 = \frac{1}{2}$ will be locally stable when α belongs to a certain neighborhood of 0. Furthermore property (8) implies that both $\frac{\partial \Phi^\alpha}{\partial x_l}\left(\frac{1}{2}, \frac{1}{2}\right)$ and $\frac{\partial \Phi^\alpha}{\partial x_g}\left(\frac{1}{2}, \frac{1}{2}\right)$ are non decreasing.

We proceed by considering how these two phase transitions affect the global situation. We then study separately the two monodimensional systems

$$\dot{x} = \Phi^\alpha\left(x, \frac{1}{2}\right) - x \tag{15}$$

related to the anti-diagonal and

$$\dot{x} = \Phi^\alpha\left(x, x\right) - x \tag{16}$$

related to the diagonal. Thanks to the symmetry property (7), to the effect of parameter α described in (8) and to the considerations made in (9), one can derive the following results:

Lemma 1: consider the threshold α^* defined in (11). There exists a non decreasing function $a : [\alpha^*, 1) \to [0, 1)$ such that (15) has the unique equilibrium $x = \frac{1}{2}$ if $\alpha \leq \alpha^*$ and the two additional equilibria $x = \frac{1}{2} \pm \frac{a(\alpha)}{2}$ if $\alpha > \alpha^*$.

Lemma 2: Consider the threshold α^{**} defined in (12). There exists a non decreasing function $d : [\alpha^{**}, 1) \to [0, 1)$ such that (16) has the unique equilibrium $x = \frac{1}{2}$ if $\alpha < \alpha^{**}$ and the two additional equilibria $x = \frac{1}{2} \pm \frac{d(\alpha)}{2}$ if $\alpha \geq \alpha^{**}$.

Notice that when the two additional anti-diagonal equilibria described in Lemma 1 are present they are also locally stable. On the other hand, the two additional diagonal equilibria described in Lemma 2 can be unstable in case of perturbations on the direction of the anti-diagonal.

The proof of Theorem 1 then follows from the monotonicity property (14) as illustrated in Figure 2.

If we apply our Theorem to the network dynamics related to the interaction kernels of Examples 2 and 3 we obtain that $\alpha^* = \alpha^*_\beta = \frac{2}{\beta}$ and $\alpha^{**} = \alpha^{**}_\beta = \frac{1}{2} + \frac{1}{\beta}$. This implies that if $\beta \leq 4$ our agents' behavior is affected by too much noise and no phase transitions are present when α varies in $[0, 1)$.

IV. Numerical simulations

In this section we report some simulations related to our theoretical results. Our goal is to verify their stability when the dynamics are associated to undirected networks $\hat{\mathcal{G}}_n = (\mathcal{V}, \mathcal{E})$ that have node set $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$, with $|\mathcal{V}_1| = |\mathcal{V}_2| = n/2$, and where the number of connections between a node belonging to $\mathcal{V}_1$ and a node belonging to $\mathcal{V}_2$ is relatively low with respect to the number of connections within each of these sets. In particular, we consider a stochastic block model random network $\hat{\mathcal{G}}_n$ obtained as follows:

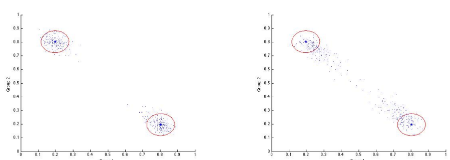
- we first generate two Erdős Rényi random graphs each with $n/2$ nodes and with a probability p of existence of an edge between two nodes;
- we connect every couple of nodes belonging to different Erdős Rényi random graphs according to a probability $q << p$.

The simulations will be performed on the network dynamics associated with the interaction kernel of Example 2 and both the network $\mathcal{G}_n$ described in Assumption 1 and the random network $\hat{\mathcal{G}}_n$ with $p = 10^{-1}$ and $q = 10^{-3}$, each of them with $n = 500$ vertices. We will set parameter β to 4: this implies that, in the case expressed in Theorem 1, we will obtain $\alpha^* = 0.5$ and $\alpha^{**} = 0.75$. Three situations will be considered:

- $\alpha = 0.58$, where the dynamical system (13) has two locally stable anti-diagonal equilibria in $(1/2 \mp a(0.58), 1/2 \pm a(0.58))$ and a saddle point in $(1/2, 1/2)$;

- $\alpha = 0.77$ where the dynamical system (13) has two locally stable anti-diagonal equilibria in $(1/2 \mp a(0.77), 1/2 \pm a(0.77))$, two diagonal saddle point in $(1/2 \pm d(0.77), 1/2 \pm d(0.77))$, an unstable equilibrium in $(1/2, 1/2)$ and other possible equilibria in $\mathcal{R}_{0.77}$;
- $\alpha = 0.93$ where the dynamical system (13) has four locally stable equilibria in $(1/2 \mp a(0.93), 1/2 \pm a(0.93))$ and in $(1/2 \pm d(0.93), 1/2 \pm d(0.93))$, an unstable equilibrium in $(1/2, 1/2)$ and other possible equilibria in $\mathcal{R}_{0.93}$.

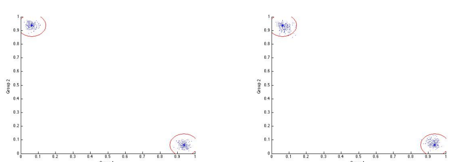
For each network and each value of α we will perform 400 simulations for $T = 30000$ time steps starting from a random state. Each final configuration will be represented by a point in the square $[0, 1]^2$. Our conjecture is that every random dynamic can be attracted only by the locally stable equilibria of system (13) and that no equilibria are present in the region $\mathcal{R}_\alpha$. To verify this conjecture we highlight the neighborhoods (radius 0.08) of locally stable equilibria belonging to the two diagonals of the square $[0, 1]^2$ and we measure the frequency $\hat{f}_{400}$ according to which a dynamic will end up in one of them.



(a) *Network* $\mathcal{G}_{500}$, $\hat{f}_{400} = 0.8575$.

(b) *Network* $\hat{\mathcal{G}}_{500}$, $\hat{f}_{400} = 0.5$.

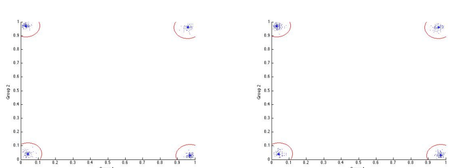
Fig. 3: Interaction kernel of Example 2 with $\alpha = 0.58$.



(a) *Network* $\mathcal{G}_{500}$, $\hat{f}_{400} = 1$.

(b) *Network* $\hat{\mathcal{G}}_{500}$, $\hat{f}_{400} = 0.9875$.

Fig. 4: Interaction kernel of Example 2 with $\alpha = 0.77$.



(a) *Network* $\mathcal{G}_{500}$, $\hat{f}_{400} = 1$.

(b) *Network* $\hat{\mathcal{G}}_{500}$, $\hat{f}_{400} = 1$.

Fig. 5: Interaction kernel of Example 2 with $\alpha = 0.93$.

V. Conclusion

In this paper we considered network games where players are represented as nodes of a graph and their payoffs depend both on the local aggregate of the their neighbors' actions and of the global aggregate of the whole population's actions. We focused on payoffs exhibiting local coordination and global congestion effects, and on graph topologies that are the disjoint union of two complete components. We proved concentration results, as the population size grows large, for the stationary distribution of the Markov chain associated to the noisy best response dynamics. In particular, we showed that the asymptotic behavior of such stationary distribution is characterized by two phase transitions with respect to a parameter measuring the relative strength of the local coordination and the global congestion effects. Our results are presented with reference to particular examples of payoffs functions and noisy best responses. However, since only hypotheses (1)-(2), (4), (6)-(8) are fundamental for our results, we believe that many more cases can be covered. Furthermore, as we can notice from the presented simulations, we conjecture that our results can be generalized to every network with strong community structure. Ongoing work includes extensions of the results to more general topologies both analytically and via simulations.

References

[1] Cañada, A., Pavel Drabek, Alessandro Fonda *Handbook of Differential Equations, Ordinary Differential Equations, vol. 2.* Elsevier, 2005.

[2] Draief, Moez, Laurent Massoulié. *Epidemics and Rumors in Complex Networks.* London Mathematical Society Lecture Note Series, vol. 369. Cambridge University Press, Cambridge (2010).

[3] Easley, David, and Jon Kleinberg. *Networks, crowds, and markets.* Cambridge University Press, 2010.

[4] Hirsch, Morris W. *Systems of differential equations that are competitive or cooperative. I: Limit sets.* SIAM Journal on Mathematical Analysis, vol. 13 (1982): 167-179.

[5] Jackson, Matthew O. *Social and economic networks.* Princeton University Press, 2010.

[6] Jackson, Matthew O., and Yves Zenou. *Games on networks.* CEPR, 2012.

[7] Kamke, E. *Zur Theorie der Systeme gewöhnlicher Differentialgleichungen. II.* Acta Mathematica 58.1 (1932): 57-85.

[8] Levin, David Asher, Yuval Peres, and Elizabeth Lee Wilmer. *Markov chains and mixing times.* American Mathematical Soc., 2009.

[9] Müller, Max. *Über das Fundamentaltheorem in der Theorie der gewöhnlichen Differentialgleichungen.* Mathematische Zeitschrift 26.1 (1927): 619-645.

[10] Sandholm, William H. *Population games and evolutionary dynamics.* MIT press, 2010.

[11] Szabó, György, and Gabor Fath. *Evolutionary games on graphs.* Physics Reports 446.4 (2007): 97-216.

[12] Wormald, Nicholas C. *Differential equations for random processes and random graphs.* The annals of applied probability (1995): 1217-1235.