Consensus analysis via integral quadratic constraints

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***Abstract*— This note proposes a unified approach to analyse linear time-invariant consensus problems via the use of integral quadratic constraints (IQCs) without recourse to loop transformations, which may cloud the inherent structural properties of the multi-agent networked systems. The main technical hindrance to using IQCs lies in the presence of the marginally stable integral action in consensus setups. It is shown that by working with conditions defined on modified signal spaces of interests and exploiting the graph structure underlying the connections between the dynamic systems, IQC methods can be applied directly to consensus analysis. A decentralised and scalable condition for consensus is proposed in this setting, which generalises some of the existing results in the literature.**

***Index Terms*— consensus, multi-agent systems, feedback systems, integral quadratic constraints**

***AMS Subject Classifications*—93A15, 93C05, 93C80**

I. Introduction

The problems of consensus, where multiple agents in a large-scale network are intended to collectively reach an agreement on object of interest, have been studied extensively in the literature; see the survey paper [1] for a list of references. Consensus protocols find applications in various areas such as flight formations, flocking behaviour, and distributed computing. Various consensus algorithms have been proposed for single and double integrator multi-agent systems in [1], [2].

The theory of integral quadratic constraints (IQCs) introduces a computationally attractive approach to encapsulating structural uncertainties of open-loop systems [3]. It presents itself as a useful tool in closed-loop stability/performance analysis. IQC stability conditions, in their simplest forms, are traditionally applied to open-loop stable components. The integrator inherent in consensus problems is therefore an impediment to the use of IQC analysis. One workaround is to employ loop transformations to the systems to yield a feedback interconnection whose stability implies that of the original one [4], [5] — a related idea is exploited in [6] to study systems with rate limiters. Specifically, the work [5] considers the more general problem of synchronisation of heterogenous linear time-invariant (LTI) systems perturbed by nonlinear uncertainties. [4, Thm. 4] proposes an scalable consensus certificate for heterogenous LTI systems interconnected on a possibly time-varying graph. In the case where the network interconnection matrix is normal, a certain factorisation can be exploited to transform the systems to a form to which IQC analysis is applicable to conclude higher-order consensus (multiple poles at the origin) [7].

A main aim of this paper is to establish that the theory of integral quadratic constraints (IQCs) [3] can be applied directly to the study of consensus of LTI systems without appealing to loop transformations to accommodate the marginally stable integral dynamics in the open-loop plants or exploiting structures of the interconnection matrices. This input-output approach serves as an alternative to the results in the literature, which are chiefly based on the generalised Nyquist criterion. The idea is to modify the definition of the standard frequency-domain $\mathbf{L}_2$ signal space with an integration contour that avoids the pole at the origin and apply IQC theory to the open-loop systems, which are stable with respect to the new space. The proof method differs from [3] in that graph-topological results of [8] are used to establish closed-loop well-posedness for LTI systems, thereby simplifying the IQC conditions for consensus. A scalable distributed consensus certificate for heterogenous networks that generalises a result in [4] is proposed within the IQC setting using ideas from recent works [9], [10].

The paper is organised as follows. Notation is defined in the following section. In Section III the problem of consensus is formulated. IQCs are reviewed in Section IV and consensus analysed. Section V proposes a decentralised scalable consensus certificate within the IQC framework. Finally, some concluding remarks are provided.

II. Notation and preliminaries

A. Matrices

Let $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex numbers respectively. $j\mathbb{R}$ denotes the imaginary axis, $\mathbb{C}_+$ (resp. $\bar{\mathbb{C}}_+$) the open (resp. closed) right half complex plane, and $|\cdot|$ the Euclidean norm. Given an $A \in \mathbb{C}^{m\times n}$ (resp. $\mathbb{R}^{m\times n}$), $A^* \in \mathbb{C}^{n\times m}$ (resp. $A^T \in \mathbb{R}^{n\times m}$) denotes its complex conjugate transpose (resp. transpose). A_{ij} denotes the (i,j) entry of A. The i^{th} row and j^{th} column of A are denoted respectively by $A_{i\bullet}$ and $A_{\bullet j}$. Given a vector $v \in \mathbb{C}^n$, $\mathrm{diag}(v) \in \mathbb{C}^{n\times n}$ denotes the diagonal matrix whose diagonal entires are $v_1, \dots, v_n$. Let $\otimes$ denote the Kronecker product and $\oplus$ the direct sum of matrices. Define $\bigoplus_{i=1}^n A_i := A_1 \oplus A_2 \oplus \dots \oplus A_n$. I_n denotes the identity matrix of dimensions $n \times n$.

B. Function spaces

Define the Lebesgue space

$$\mathbf{L}_\infty := \{\phi : j\mathbb{R} \to \mathbb{C} \,|\, \|\phi\|_\infty := \sup_{\omega\in\mathbb{R}} |\phi(j\omega)| < \infty\}$$

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and the Hardy space

$$\mathbf{H}_\infty := \left\{ \phi \in \mathbf{L}_\infty \;\middle|\; \begin{array}{l} \phi \text{ has analytic continuation into } \mathbb{C}_+ \\ \text{with } \sup_{s\in\mathbb{C}_+} |\phi(s)| = \|\phi\|_\infty < \infty \end{array} \right\}.$$

Let $\mathbf{C}$ be the class of functions continuous on $j\mathbb{R}\cup\{\infty\}$, and $\mathbf{S} := \mathbf{H}_\infty \cap \mathbf{C}$. Note that $\mathbf{C} \subset \mathbf{L}_\infty$. An $H \in \mathbf{C}^{n\times n}$ is said to be Hermitian if $H(j\omega) = H(j\omega)^*$ for all $\omega \in \mathbb{R}\cup\{\infty\}$ and positive definite if in addition, $H(j\omega) > 0$. Define a contour parameterised by $\epsilon \geq 0$ as

$$\mathcal{C}_\epsilon := j[\epsilon,\infty) \cup \{s \in \mathbb{C} : |s| = \epsilon, \Re(s) > 0\} \cup j(-\infty,-\epsilon],$$

that is, a straight line on the imaginary axis indented to the right of the origin by a semi-circle of radius ϵ. In particular, $C_0 = j\mathbb{R}$. Denote by $\mathcal{C}_\epsilon^+$ the open half plane that lies to the right of $\mathcal{C}_\epsilon$, i.e.

$$\mathcal{C}_\epsilon^+ := \{s = \sigma + j\omega \in \mathbb{C} \mid \bar{\sigma} + j\omega \in \mathcal{C}_\epsilon \implies \sigma > \bar{\sigma}\},$$

and $\bar{\mathcal{C}}_\epsilon^+$ its closure. Let $\mathbf{C}_\epsilon$ be the class of functions continuous on $\mathcal{C}_\epsilon \cup \{\infty\}$. Given $X \in \mathbf{C}_\epsilon^{n\times m}$, define $\|X\|_\infty := \sup_{s\in\mathcal{C}_\epsilon} \bar{\sigma}(X(s))$, where $\bar{\sigma}(\cdot)$ denotes the maximum singular value. An $H \in \mathbf{C}_\epsilon^{n\times n}$ is said to be Hermitian if $H(s) = H(s)^*$ for all $s \in \mathcal{C}_\epsilon \cup \{\infty\}$.

Let the Lebesgue space $\mathbf{L}_2^n$ denote the class of functions $f : [0,\infty) \to \mathbb{R}^n$ with finite energy, i.e. square-integrable $\|f\|_2^2 := \int_0^\infty |f(t)|^2\,dt < \infty$. The Fourier transform of $f \in \mathbf{L}_2^n$ is denoted $\hat{f}(j\omega) := \int_0^\infty e^{-j\omega t} f(t)\,dt$. Note that $\|\hat{f}\|_2 = \|f\|_2$ and $\hat{f}$ has analytic continuation into $\mathbb{C}_+$ and $\sup_{\sigma>0} \|\hat{f}(\sigma + \cdot)\|_2 = \|\hat{f}\|_2 < \infty$. The set of Fourier transforms of functions in $\mathbf{L}_2^n$ is denoted $\mathbf{H}_2^n$. A linear operator mapping between Banach spaces $X : \mathcal{X} \to \mathcal{Y}$ is said to be bounded if

$$\|X\|_{\mathcal{X}\to\mathcal{Y}} := \sup_{f\in\mathcal{X}:\|f\|_\mathcal{X}=1} \|Xf\|_\mathcal{Y} < \infty.$$

Note that multiplication by a transfer function in $\mathbf{S}$ as an operator on $\mathbf{H}_2$ defines a corresponding causal and bounded LTI operator on $\mathbf{L}_2$ in the time domain via the Laplace transform isomorphism [11].

For $\epsilon \geq 0$, define $\mathbf{H}_{2\epsilon}^n$ to be the set of functions $\hat{f} : \bar{\mathcal{C}}_\epsilon^+ \to \mathbb{C}^n$ that are analytic on $\mathcal{C}_\epsilon^+$ and square-integrable on $\mathcal{C}_\epsilon$, i.e. $\|\hat{f}\|_{\mathcal{C}_\epsilon}^2 := \int_{\mathcal{C}_\epsilon} |\hat{f}(s)|^2\,ds < \infty$. The $\hat{}$ notation is occasionally dropped when there is no need to distinguish between time and frequency-domain signals. Note that $\mathbf{H}_2^n = \mathbf{H}_{2\epsilon}^n$ when $\epsilon = 0$ and multiplication by a transfer function in $\mathbf{C}_\epsilon$ defines a bounded operator on $\mathbf{H}_{2\epsilon}$. Furthermore, $\mathbf{H}_{2\epsilon}$ is a Hilbert space with inner product $\langle u, v\rangle_{\mathcal{C}_\epsilon} := \int_{\mathcal{C}_\epsilon} u(s)^* v(s)\,ds$. It can be seen that multiplication by an $X \in \mathcal{S}$ is bounded on $\mathbf{H}_{2\epsilon}$ for all $\epsilon \geq 0$. One the other hand, multiplication by $\frac{1}{s}$ is bounded on $\mathbf{H}_{2\epsilon}$ for $\epsilon > 0$ but not on $\mathbf{H}_2$.

C. Graph theory

A graph is denoted by $\mathcal{G} = (V, E)$, where $V = \{v_1, \ldots, v_n\}$ is the set of nodes and $E \subset V \times V$, $E = \{e_1, \ldots, e_m\}$ is the set of edges such that $e_k = \{v_i, v_j\} \in E$ if node i is connected to node j. A graph is undirected if $\{v_i, v_j\} \in E$ then $\{v_j, v_i\} \in E$. A path on $\mathcal{G}$ of length N is an ordered set of distinct vertices $\{v_0, v_1, \ldots, v_N\}$ such that $\{v_i, v_{i+1}\} \in E$ for all $i \in \{0, 1, \ldots, N-1\}$. An undirected graph is said to be *connected* if any two nodes in V is connected by a path. The adjacency matrix $A = [A_{ij}] \in \mathbb{R}^{n\times n}$ is defined by $A_{ij} = 1$ if $\{v_i, v_j\} \in E$ and $A_{ij} = 0$ otherwise. Note that A is symmetric for an undirected graph. In an undirected graph, let the neighbours of node $v_i \in V$ be defined as $N_i := \{v_j \in V : \{v_i, v_j\} \in E\}$ and denote its degree by $|N_i|$. The graph Laplacian is defined as $L := \mathrm{diag}(|N_i|) - A$. L has a zero eigenvalue corresponding to the vector of ones $1_n \in \mathbb{R}^n$. The multiplicity of the zero eigenvalue is one if the graph is connected [12]. The Laplacian matrix can be factorised as $L = DD^T$, where $D = [D_{ik}] \in \mathbb{R}^{n\times m}$ is the incidence matrix. It is defined by associating an orientation to every edge of the graph: for each $e_k = \{v_i, v_j\} = \{v_j, v_i\}$, one of v_i, v_j is defined to be the head and the other tail of the edge.

$$D_{ik} := \begin{cases} +1 & \text{if } v_i \text{ is the head of } e_k \\ -1 & \text{if } v_i \text{ is the tail of } e_k \\ 0 & \text{otherwise.} \end{cases}$$

Note that the Laplacian matrix is invariant to the choice of orientation.

III. Problem Formulation

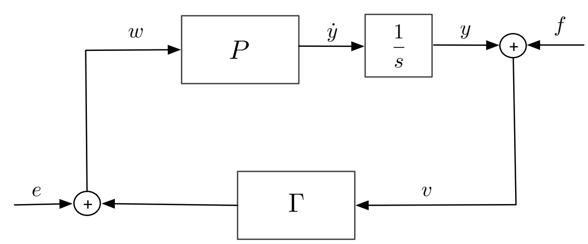

Fig. 1. Feedback interconnection for consensus.

Consider the feedback interconnection in Figure 1. There, $P := \bigoplus_{i=1}^n P_i$ with the dynamical agents $P_i \in \mathbf{S}$ and $\Gamma \in \mathbf{S}^{n\times n}$ denotes the interconnection matrix. The interactions between the agents is determined by an underlying undirected and connected graph $\mathcal{G} = (V, E)$, where each node $v_i \in V$ is associated with a corresponding P_i and the edges describe the communication/connections between the agents. The following standing assumption is made throughout the paper.

Assumption 3.1: $\Gamma(0)$ has a simple zero eigenvalue corresponding to the eigenvector 1_n.

In the simplest case, Γ can be equal to L, the graph Laplacian matrix for the graph $\mathcal{G}$. Dynamics can be included via the expression $\Gamma = D\,\mathrm{diag}(\Gamma_i)\,D^T$, where D denotes the incidence matrix and $\Gamma_i \in \mathcal{S}$ for $i = 1, \ldots, m$; see Figure 2. Note that for both cases Γ satisfies Assumption 3.1 by the connectedness of the graph $\mathcal{G}$.

Definition 3.2: The interconnection in Figure 1 is said to reach consensus if $|y_i(t) - y_j(t)| \to 0$ as $t \to \infty$ for all $i, j \in \{1, 2, \ldots, n\}$. That is, the agents asymptotically reach an agreement in their output y_i. In other words, $\lim_{t\to\infty} y(t)$ lies in the subspace spanned by 1_n, i.e. $\mathrm{span}\{1_n\}$.

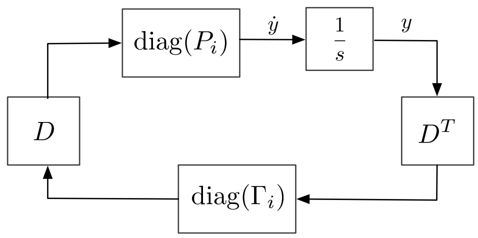

Fig. 2. A consensus setup with dynamical interconnection matrix.

IV. Integral quadratic constraint based analysis of consensus

This section introduces a unified framework within which to analyse the problem of consensus using integral quadratic constraints (IQCs) [3]. First an IQC-based stability result is in order.

Definition 4.1: Given $\epsilon \geq 0$, $P : \mathbf{H}^n_{2\epsilon} \to \mathbf{H}^n_{2\epsilon}$ and $\Gamma : \mathbf{H}^n_{2\epsilon} \to \mathbf{H}^n_{2\epsilon}$ in $\mathbf{S}^{n\times n}$, the feedback interconnection in Figure 1:

$$\begin{cases} v = \frac{1}{s}Pw + f \\ w = \Gamma v + e \end{cases} \tag{1}$$

is said to be $\mathbf{H}_{2\epsilon}$-*stable* if the map $(v, w) \mapsto (f, e)$ has a bounded inverse on $\mathbf{H}^{2n}_{2\epsilon}$.

Given an $\epsilon \geq 0$, define the graph of $X \in \mathbf{C}^{n\times m}_\epsilon$ to be

$$\mathcal{G}_\epsilon(X) := \begin{bmatrix} I_m \\ X \end{bmatrix} \mathbf{H}^m_{2\epsilon} = \left\{ \begin{bmatrix} u \\ y \end{bmatrix} \in \mathbf{H}^{n+m}_{2\epsilon} : y = Xu \right\}.$$

Similarly, define $\mathcal{G}'_\epsilon(X) := \left[\begin{smallmatrix} X \\ I_m \end{smallmatrix}\right] \mathbf{H}^m_{2\epsilon}$.

Theorem 4.2: Given $\epsilon > 0$, $P, \Gamma \in \mathbf{S}^{n\times n}$, the feedback interconnection of $\frac{1}{s}P$ and Γ in Figure 1 is $\mathbf{H}_{2\epsilon}$-stable if there exists a $\Pi \in \mathbf{C}^{2n\times 2n}_\epsilon$ such that the following IQC conditions hold:

(i) $\langle v, \Pi v\rangle_{\mathcal{C}_\epsilon} \geq 0$ for all $v \in \mathcal{G}_\epsilon\left(\frac{1}{s}P\right)$;
(ii) there exists a $\gamma > 0$ for which $\langle w, \Pi w\rangle_{\mathcal{C}_\epsilon} \leq -\gamma\|w\|^2_{\mathcal{C}_\epsilon}$ for all $w \in \mathcal{G}'_\epsilon(\tau\Gamma)$ and $\tau \in [0, 1]$.

Proof: Using an argument in [13], let $\Psi := 2\Pi + \gamma I$, the IQC conditions become

$$\langle v, \Psi v\rangle_{\mathcal{C}_\epsilon} \geq \gamma\|v\|^2_{\mathcal{C}_\epsilon}\ \forall v \in \mathcal{G}_\epsilon\left(\frac{1}{s}P\right)$$

and

$$\langle w, \Psi w\rangle_{\mathcal{C}_\epsilon} \leq -\gamma\|w\|^2_{\mathcal{C}_\epsilon}\ \forall w \in \mathcal{G}'_\epsilon(\tau\Gamma), \tau \in [0, 1].$$

It follows that for any $v \in \mathcal{G}_\epsilon\left(\frac{1}{s}P\right)$, $w \in \mathcal{G}'_\epsilon(\tau\Gamma)$ and $\tau \in [0, 1]$,

$$\begin{aligned}
&\gamma(\|v\|^2_{\mathcal{C}_\epsilon} + \|w\|^2_{\mathcal{C}_\epsilon}) \\
&\leq \langle v, \Psi v\rangle_{\mathcal{C}_\epsilon} - \langle w, \Psi w\rangle_{\mathcal{C}_\epsilon} \\
&= \langle v + w, \Psi(v + w)\rangle_{\mathcal{C}_\epsilon} - 2\langle w, \Psi(v + w)\rangle_{\mathcal{C}_\epsilon} \\
&\leq \|\Psi\|_\infty\|v + w\|^2_{\mathcal{C}_\epsilon} + 2\|\Psi\|_\infty\|w\|_{\mathcal{C}_\epsilon}\|v + w\|_{\mathcal{C}_\epsilon} \\
&\leq \|\Psi\|_\infty\|v + w\|^2_{\mathcal{C}_\epsilon} + \frac{2\|\Psi\|^2_\infty\|v + w\|^2_{\mathcal{C}_\epsilon}}{\gamma} + \frac{\gamma}{2}\|w\|^2_{\mathcal{C}_\epsilon},
\end{aligned}$$

where the last inequality holds since $2xy \leq \frac{x^2}{\beta} + \beta y^2$ for any $x, y, \beta \in \mathbb{R}$. This implies

$$\begin{aligned}
\left(1 + \frac{2}{\gamma}\|\Psi\|_\infty\right)\|\Psi\|_\infty\|v + w\|^2_{\mathcal{C}_\epsilon} &\geq \gamma\|v\|^2_{\mathcal{C}_\epsilon} + \frac{\gamma}{2}\|w\|^2_{\mathcal{C}_\epsilon} \\
&\geq \frac{\gamma}{2}\|w\|^2_{\mathcal{C}_\epsilon} \\
\implies \|v + w\|^2_{\mathcal{C}_\epsilon} &\geq \eta^2\|w\|^2_{\mathcal{C}_\epsilon},
\end{aligned} \tag{2}$$

for any positive $\eta \leq \frac{\gamma}{\sqrt{2\|\Psi\|_\infty(\gamma + 2\|\Psi\|_\infty)}}$.

Now observe that $\tau \in [0, 1] \mapsto \tau\Gamma$ is continuous in the graph topology induced by the gap metric with the ambient space taken to be $\mathbf{H}_{2\epsilon}$ [8]. Since the feedback interconnection is $\mathbf{H}_{2\epsilon}$-stable for $\tau = 0$, inequality (2) shows that the corresponding robust stability margin $b_{\mathcal{M},\mathcal{N}}$ in [8, Section 5] is bounded away from zero with $\mathcal{N} = \mathcal{G}_\epsilon\left(\frac{1}{s}P\right)$ and $\mathcal{M} = \mathcal{G}'_\epsilon(0)$; see [14, Lem. 3.2.8]. Application of [8, Thm. 3] then leads to the feedback connection being stable for $\tau \in [0, \zeta]$ for some non-zero ζ. Repetitively using the aforementioned arguments yields feedback stability for $\tau \in [\zeta, 2\zeta], [2\zeta, 3\zeta], \ldots$, and eventually for $\tau = 1$, as required. ■

The main result on consensus is stated next.

Theorem 4.3: The feedback configuration in Figure 1 with $P := \bigoplus_{i=1}^n P_i : P_i \in \mathbf{S}$ and $\Gamma \in \mathbf{S}^{n\times n}$ that satisfies Assumption 3.1 reaches consensus if there exists a $\Pi \in \mathbf{C}^{2n\times 2n}$ such that

(i) $\begin{bmatrix} I_n \\ \frac{1}{j\omega}P(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} I_n \\ \frac{1}{j\omega}P(j\omega) \end{bmatrix} \geq 0\ \forall\omega \in (0, \infty]$;
(ii) $\begin{bmatrix} \tau\Gamma(j\omega) \\ I_n \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \tau\Gamma(j\omega) \\ I_n \end{bmatrix} \leq -\gamma\ \forall\omega \in (0, \infty], \tau \in [0, 1]$, where γ is some positive constant.

Proof: The conditions of the theorem imply that the IQC conditions in Theorem 4.2 are satisfied for arbitrarily small $\epsilon > 0$, whereby the feedback configuration is $\mathbf{H}_{2\epsilon}$-stable. In turn, this implies that

$$\frac{1}{s}P(s)\left(I - \Gamma(s)\frac{1}{s}P(s)\right)^{-1} = P(s)(sI - \Gamma(s)P(s))^{-1}$$

has no poles on $\bar{\mathbb{C}}_+\backslash\{0\}$, i.e. $\det(sI - \Gamma(s)P(s))$ has no zeros on $\bar{\mathbb{C}}_+\backslash\{0\}$. Moreover, by Assumption 3.1, $\det(sI - \Gamma(s)P(s))$ has a simple zero at the origin corresponding to the null space N satisfying $P(0)N \subset \operatorname{span}\{1_n\}$.

Now note that for any $e, f \in \mathbf{L}_2$, it can be derived from (1) and Figure 1 that

$$\begin{aligned}
\hat{y} &= \frac{1}{s}P(I - \Gamma\frac{1}{s}P)^{-1}(\hat{e} + \Gamma\hat{f}) \\
&= P(sI - \Gamma P)^{-1}(\hat{e} + \Gamma\hat{f}),
\end{aligned}$$

which has a simple pole at the origin. As such, it follows from the above that $\hat{y} = \hat{v} + \hat{w}$, for some $\hat{v} \in \mathbf{H}_2$ and $\hat{w} \in \frac{1}{s}\operatorname{span}\{1_n\}$. Taking the inverse Laplace transform yields that $y = v + w$, where $\lim_{t\to\infty} v(t) = 0$ because $v \in \mathbf{L}_2$ and $w(t)$ is a constant vector with equal entries for all $t \geq 0$. In other words, the feedback interconnection reaches consensus as time approaches infinity. ■

Remark 4.4: It can be seen from the proof of Theorem 4.2 that the conditions of Theorem 4.3 may also be written as

(i) $\begin{bmatrix} I_n \\ \frac{1}{j\omega}P(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} I_n \\ \frac{1}{j\omega}P(j\omega) \end{bmatrix} \geq \gamma > 0;$

(ii) $\begin{bmatrix} \tau\Gamma(j\omega) \\ I_n \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \tau\Gamma(j\omega) \\ I_n \end{bmatrix} \leq 0 \; \forall\tau \in [0, 1],$

for all $\omega \in (0, \infty]$.

V. Scalable consensus conditions

Consider the consensus setup in Figure 2, where $P := \bigoplus_{i=1}^{n} P_i : P_i \in \mathbf{S}$, $\Gamma := \bigoplus_{i=1}^{m} \Gamma_i : \Gamma_i \in \mathbf{S}$, and D denotes the incidence matrix of a connected graph $\mathcal{G}$. Given a $B \in \mathbb{C}^{m\times n}$ such that $|B_{\bullet j}| = 1$ for $j = 1, 2, \ldots, m$, i.e. columns of B are normalised, let

$$C_{ij} := \begin{cases} 0 & \text{if } B_{ij} = 0 \\ B_{ij}^{-1} & \text{otherwise.} \end{cases} \tag{3}$$

Theorem 5.1: Suppose there exist $B \in \mathbb{C}^{m\times n}$ with normalised columns, $H := \bigoplus_{i=1}^{n} H_i$, $J := \bigoplus_{i=1}^{n} J_i$ with $H_i, J_i \in \mathbf{C}$ and $K \in \mathbf{C}^{n\times n}$ such that $H_i + H_i^*$ is positive definite, J_i, K are positive semidefinite, for $i = 1, \ldots, n$, and

(i) $[D\Gamma(j\omega)^* D^T](\tau J(j\omega) - K(j\omega))[D\Gamma(j\omega)D^T] \leq 0$ for all $\tau \in [0, 1]$ and $\omega \in (0, \infty]$;

(ii) for all $i = 1, \ldots, m$ and $\omega \in (0, \infty]$,

$$\begin{bmatrix} I_n \\ I_n \end{bmatrix}^* \Pi_i(j\omega) \begin{bmatrix} I_n \\ I_n \end{bmatrix} \geq \gamma > 0,$$

where

$$\Pi_i = \begin{bmatrix} H + H^* + J & \Phi_i \\ \Phi_i^* & \Omega_i \end{bmatrix},$$

with

$$\Phi_i := -H(\text{diag}(C_{i\bullet}^*)D_{\bullet i})\Gamma_i(D_{i\bullet}^T \text{diag}(C_{i\bullet}))\frac{1}{s}P, \tag{4}$$

$$\Omega_i := -\left(\frac{1}{s}P\right)^* \text{diag}(C_{i\bullet}^*)D_{\bullet i}\Gamma_i^* D_{i\bullet}^T K \; D_{\bullet i}\Gamma_i(D_{i\bullet}^T \text{diag}(C_{i\bullet}))\frac{1}{s}P,$$

and C is as defined in (3). Then the feedback connection in Figure 2 reaches consensus.

Proof: It can be shown as in [9, Thm. 3.1] that the hypothesis implies

$$\begin{bmatrix} \tau D\Gamma(j\omega)D^T \\ I_n \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \tau D\Gamma(j\omega)D^T \\ I_n \end{bmatrix} \leq 0$$

and

$$\begin{bmatrix} I_n \\ \frac{1}{j\omega}P(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} I_n \\ \frac{1}{j\omega}P(j\omega) \end{bmatrix} \geq \gamma > 0$$

for all $\tau \in [0, 1]$ and $\omega \in (0, \infty]$, where

$$\Pi := \begin{bmatrix} H + H^* + J & -HD\Gamma D^T \\ -D\Gamma^* D^T H^* & -D\Gamma^* D^T K D\Gamma D^T \end{bmatrix}.$$

Consensus then follows from Theorem 4.3. ∎

The consensus certificate in Theorem 5.1 is distributed and scalable in that it involves only each Γ_i and the associated agents P_i's as manifested by the incidence matrix; see (4). A 'complementary' distributed certificate can also be stated in terms of each individual P_i.

Theorem 5.1 generalises [4, Thm. 1] in the following manner. The original versions of these results are derived to establish closed-loop stability. In particular, the sibling of Theorem 5.1, [9, Thm 3.1], generalises the stability certificate in [10]. Via a specific choice of the parameter H, the latter reduces to a Nyquist graphical test involving ellipses across frequency. It is shown to be stronger than [15, Thm. 1], the sibling of [4, Thm. 1], in that the ellipses are subsets of the S-hulls employed in [15].

VI. Conclusions

The paper establishes a way for integral quadratic constraint based analysis to be applied directly to the investigation of consensus problems without the need of loop transformations. It also proposes a scalable and distributed test for consensus, which extends previous results in this direction. The more general problem of synchronisation may be examined along similar lines as future research.

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