Throughput optimality and overload behavior of dynamical flow networks under monotone distributed routing

Giacomo Como, Enrico Lovisari, and Ketan Savla

***Abstract*—A class of distributed routing policies is shown to be throughput optimal for single-commodity dynamical flow networks. The latter are modeled as systems of ODEs based on mass conservation laws on directed graphs with maximum flow capacities on links and constant external inflow at some origin nodes. Distributed routing regulates the flow splitting at each node, as a function of information on the densities of the local links around the nodes. Under monotonicity properties of routing, it is proven that, if no cut capacity constraint is violated by the external inflow, then a globally asymptotically stable equilibrium exists and the network achieves maximal throughput. This holds for finite or infinite buffer capacities for the densities. The overload behavior, if any cut capacity constraint is violated, is also characterized: there exists a cut on which the link densities grow linearly in time for infinite buffer capacities, while they simultaneously reach their respective buffer capacities, when these are finite. The results rely on a novel l_1-contraction principle for monotone dynamical systems. Applications to dynamic traffic models and data networks are also discussed.**

I. Introduction

Rapid advancements in technologies are facilitating real-time control of infrastructure networks, such as transportation, in order to achieve the efficient utilization of these networks. Static network flows, e.g., see [1], have traditionally dominated the modeling framework for infrastructure networks. However, in order to realize the true potential of the emerging technologies, one needs to develop control design within a dynamical framework. In this paper, we study single-commodity dynamical flow networks, modeled as systems of ordinary differential equations derived from mass conservation laws on weighted directed graphs, possibly with cycles, and having constant external inflow at each of possibly multiple origins. The weights on the links are their maximum flow capacities. The flow of particles is regulated from a link to links downstream to it by deterministic rules, or routing policies, which depend on the state of the network, and the particles leave the network when they arrive at any of the possibly multiple destination nodes. Our first objective is to characterize routing policies that allow the network to achieve maximum throughput, i.e., the maximum possible external inflow at the origin nodes under which the link densities remain within the buffer capacities. Our secondary objective is the detailed characterization of the overload behavior of the network, when the the external inflow at the origin nodes is greater than the maximum throughput.

The first two authors were partially supported by the Swedish Research Council through the Junior Research Grant Information Dynamics in Large Scale Networks and the Linnaeus Center LCCC. G. Como and E. Lovisari are with the Department of Automatic Control, Lund University, SE-221 00 Lund, Sweden giacomo.como, enrico.lovisari@control.lth.se. K. Savla is with the Sonny Astani Department of Civil and Environmental Engineering, University of Southern California, Los Angeles, CA 90089-2531 ksavla@usc.edu

We focus on routing policies that are distributed: the routing at each link depends only on the local information consisting of density of itself and the links downstream to it. We propose a novel class of distributed routing policies, called *monotone distributed routing policies*, that are characterized by general monotonicity assumptions on the sensitivity of their action with respect to local information. We then establish throughput optimality of this routing policy, and give a detailed characterization of the overload behavior of the network operating under monotone distributed routing policies. Our main result is in the form of a dichotomy. If the the external inflow at the origin nodes does not violate any cut capacity constraints, then there exists a globally asymptotically stable equilibrium, and thus the network achieves maximal throughput. These results hold true for finite or infinite link-wise buffer capacities for the densities. When the external inflow at the origin nodes violates some cut capacity constraint, then the resulting overload behavior of the network exhibits the following feature: if the buffer capacities are infinite, then there exists a constraint-violating cut, independent of the initial condition, such that the particle densities on the origin side of the cut grow at most linearly in time; if the buffer capacities are finite, then there exists a constraint-violating cut, in general dependent on the initial condition, such that the links constituting the cut hit their buffer capacities simultaneously. The last case implies that a link reaches its buffer capacity only at the very moment at which it is unavoidable. We emphasize again that such an efficient utilization of the network is induced by a *distributed* routing policy relying only on local information.

The results presented in this paper rely on the ability of the routing policy to implicitly propagate congestion effects upstream, allowing the flow to be routed through the less congested parts of the network in a timely fashion. While algorithms for distributed computation of maximum network flow have long been known (e.g., see [2]) the novelty of our contribution consists in proving throughput optimality for flow dynamics naturally arising in physical networks. The proofs are based on a novel l_1 contraction principle for monotone conservation laws (Lemma 1), possibly of independent interest, and on a complete characterization of all possible combinations of limiting (as densities approach the buffer capacities) states of all the links around every node (Lemma 3).

The distributed routing architecture of this paper and the ensuing result on throughput optimality are reminiscent of the backpressure routing algorithm for data networks, first proposed in [3], and the maxweight-α policies for switched networks, e.g., see [4], [5]. Indeed, we elaborate on this connection in this paper, showing that, by defining an appropriate *dual* graph, we can transform our setup to fit within the setup of [3], [4], [5]. Moreover, modulo the discrete-time discretespace setting of [3], [4], [5], we point out that the backpressure and the maxweight-α policies satisfy the properties of monotone distributed routing policies proposed in this paper. This allows us to possibly extend the throughput optimality to the finite buffer capacity case. This throughput optimality for the finite buffer capacity under distributed routing has to be contrasted with existing work in [6], [7], where either throughput optimality is obtained under a centralized routing policy or there is a trade-off between throughput and buffer capacity under distributed routing policies. The dynamical formulation of this paper is also reminiscent of models of dynamic traffic flow over networks, e.g., see [8], [9], [10]. However, the key difference is that, unlike these existing works, routing policies in our framework depend on the (local) state of the network. We also mention the utility of our framework in analyzing a dynamic traffic model that is related to the well-known *cell transmission* model for traffic flow [11].

It is also imperative to highlight the difference between this paper and our previous work [12], [13], where we formulated the dynamical flow network framework when the routing policies can only control the splitting of incoming flow at a node among outgoing links. We proposed a class of *locally responsive policies* and established conditions for existence and stability of equilibrium when the buffer capacities are infinite for directed acyclic network topologies. We also studied resilience properties of the network under these routing policies and showed that the margin of resilience under locally responsive policies is maximal under the distributed architecture where the routing policies can not control the inflow at the nodes. In this paper, we extend and modify the framework from [12], [13] to allow for finite buffer capacities and cyclic network topologies, and also allow the routing policies to completely control (subject to capacity constraints) the flow transfer between links, i.e., the routing policies can also control the inflow arriving at nodes. Under this framework, we are able to establish global asymptotic stability of equilibrium, when the links have infinite or finite buffer capacities and for cyclic network topologies. Moreover, unlike [12], [13], we give a detailed characterization of the overload behavior of the network. Additionally, although we do not address resilience explicitly in this paper, we remark that the margin of resilience under distributed monotone routing policies is the maximum of all (not necessarily distributed) routing policies for a dynamical flow network.

The paper is organized as follows: in section II, we propose a general model for dynamical flow in networks, formulate the problem, and explain the connections between our framework and dynamic traffic models as well as routing in data networks. In section III, we state our main results. Section IV is devoted to the proofs of the main results. Finally, section V states conclusions and possible directions for future research.

We conclude this section by introducing some notational conventions to be used throughout the paper. Let $\mathbb{R}$ be the set of real numbers and $\mathbb{R}_+ := \{x \in \mathbb{R} : x \geq 0\}$ be the set of nonnegative real numbers. Let $\mathcal{A}$ and $\mathcal{B}$ be finite sets. Then $|\mathcal{A}|$ denotes the cardinality of $\mathcal{A}$, $\mathbb{R}^{\mathcal{A}}$ (respectively, $\mathbb{R}_+^{\mathcal{A}}$) the space of real-valued (nonnegative-real-valued) vectors whose components are indexed by elements of $\mathcal{A}$, and $\mathbb{R}^{\mathcal{A}\times\mathcal{B}}$ the space of matrices whose real entries are indexed by pairs in $\mathcal{A}\times\mathcal{B}$. If $\mathcal{B} \subseteq \mathcal{A}$ and $x \in \mathbb{R}^{\mathcal{A}}$, then $x_{\mathcal{B}} \in \mathbb{R}^{\mathcal{B}}$ stands for the projection of x on $\mathcal{B}$. The transpose of a matrix $M \in \mathbb{R}^{\mathcal{A}\times\mathcal{B}}$ is denoted by $M' \in \mathbb{R}^{\mathcal{B}\times\mathcal{A}}$, while $\mathbb{1}$ stands for an all-one vector of suitable dimension. The natural partial ordering of $\mathbb{R}^{\mathcal{A}}$ will be denoted by $x \preceq y$ for two vectors $x, y \in \mathbb{R}^{\mathcal{A}}$ such that $x_a \leq y_a$ for all $a \in \mathcal{A}$.

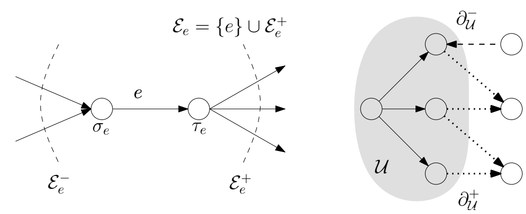

Figure 1. Graphical depiction of some key notations. In the right, links comprising $\partial^-_{\mathcal{U}}$ and $\partial^+_{\mathcal{U}}$ are shown by dashed and dotted arrows, respectively; links comprising $\mathcal{E}^+_{\mathcal{U}} \setminus \partial^+_{\mathcal{U}}$ are shown in solid arrows.

A weighted directed multi-graph is a triple $\mathcal{G} = (\mathcal{V}, \mathcal{E}, C)$, where $\mathcal{V}$ and $\mathcal{E}$ stand for the node set and the link set, respectively, and are both finite. They are endowed with three vectors: $\sigma, \tau \in \mathcal{V}^{\mathcal{E}}$, and $C \in (0, +\infty]^{\mathcal{E}}$. For every $e \in \mathcal{E}$, σ_e and τ_e stand for the tail and head nodes respectively of link e and C_e for the positive (and possibly infinite) flow capacity of link e. We shall always assume that there are no self-loops, i.e., $\tau_e \neq \sigma_e, \forall e \in \mathcal{E}$. On the other hand, we allow for parallel links. For a node $v \in \mathcal{V}$, let $\mathcal{E}^+_v := \{e : \sigma_e = v\}$ and $\mathcal{E}^-_v := \{e : \tau_e = v\}$. For a link $e \in \mathcal{E}$, let $\mathcal{E}^+_e := \mathcal{E}^+_{\tau_e}$ be the set of links downstream to e and $\mathcal{E}^-_e := \mathcal{E}^-_{\sigma_e}$ be the set of links upstream to e. Put $\mathcal{E}_e := \{e\} \cup \mathcal{E}^+_e$. For a vector $x \in \mathbb{R}^{\mathcal{E}}$, we shall denote by $x^e := \{x_j : j \in \mathcal{E}_e\}$ its projection on $\mathcal{E}_e$. For a node subset $\mathcal{U} \subseteq \mathcal{V}$, define $\mathcal{E}^+_{\mathcal{U}} := \cup_{u\in\mathcal{U}} \mathcal{E}^+_u$ and $\mathcal{E}^-_{\mathcal{U}} := \cup_{u\in\mathcal{U}} \mathcal{E}^-_u$. Let $\partial^+_{\mathcal{U}} := \{e \in \mathcal{E} : \sigma_e \in \mathcal{U}, \tau_e \notin \mathcal{U}\}$ and $\partial^-_{\mathcal{U}} := \{e \in \mathcal{E} : \sigma_e \in \mathcal{V} \setminus \mathcal{U}, \tau_e \in \mathcal{U}\}$ be the set of links from $\mathcal{U}$ to $\mathcal{V} \setminus \mathcal{U}$ and from $\mathcal{V} \setminus \mathcal{U}$ to $\mathcal{U}$, respectively. See Figure 1 for an illustration of some of these notations.

II. Problem statement

A. Static single-commodity network flows and the max-flow min-cut theorem

We shall identify a network with a weighted directed multi-graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, C)$ and denote its set of destinations by

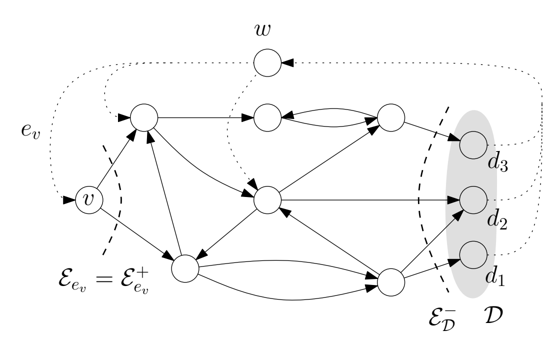

Figure 2. An example of multi-destination network with cycles and parallel edges. The links added in the augmented graph $\mathcal{G}_\lambda$ are shown in dotted line.

$\mathcal{D} := \{v \in \mathcal{V} : \mathcal{E}_v^+ = \emptyset\}$ and the set of its feasible flows by

$$\mathcal{F}^* := \left\{ x \in \prod_{e\in\mathcal{E}}[0, C_e] \,:\, \sum_{e\in\mathcal{E}_v^+} x_e - \sum_{e\in\mathcal{E}_v^-} x_e \geq 0\,,\ \forall v \in \mathcal{V}\backslash\mathcal{D} \right\}.$$

For $f^* \in \mathcal{F}^*$, the vector $\lambda(f^*) \in \mathbb{R}_+^{\mathcal{V}\backslash\mathcal{D}}$ with components $\lambda_v(f^*) := \sum_{e\in\mathcal{E}_v^+} f_e^* - \sum_{e\in\mathcal{E}_v^-} f_e^*$ will be referred to as the value of f^*.[1] For $\lambda \in \mathbb{R}_+^{\mathcal{V}\backslash\mathcal{D}}$, we introduce the *augmented* network $\mathcal{G}_\lambda = (\mathcal{V}_\lambda, \mathcal{E}_\lambda, C)$ (see Figure 2) with node and link sets $\mathcal{V}_\lambda = \mathcal{V} \cup \{w\}$ and $\mathcal{E}_\lambda := \mathcal{E} \cup \mathcal{O}_\lambda \cup \mathcal{E}_\mathcal{D}^+$, respectively, where $\mathcal{O}_\lambda := \{e_v := (w, v) : \lambda_v > 0\}$, $\mathcal{E}_\mathcal{D}^+ := \{e_d := (d, w) : d \in \mathcal{D}\}$, and $C_{e_v} = C_{e_d} = +\infty$ for all $v \in \mathcal{V} \setminus \mathcal{D}$ and $d \in \mathcal{D}$. The extra node w may be thought of as representing an external world, playing the double role of source of the flow for nodes with positive value of flow, and sink of the flow exiting from the destination nodes, respectively. We shall refer to links in $\mathcal{O}_\lambda$ as *origin links* and adopt the notation $\mathcal{E}_{e_v} = \mathcal{E}_{e_v}^+ := \mathcal{E}_v^+$, for all $v \in \mathcal{V} \setminus \mathcal{D}$.

Throughout this paper, we shall make the following assumptions on the network topology.

Assumption 1. *The set of destinations $\mathcal{D}$ is nonempty, and the augmented network $\mathcal{G}_\lambda$ is strongly connected.*

Assumption 1 is equivalent to the properties that, in $\mathcal{G}$, from every $v \in \mathcal{V} \backslash \mathcal{D}$ there exists at least one directed path to some destination node $d \in \mathcal{D}$, and there exists at least one directed path from some u with $\lambda_u > 0$ to every $v \in \mathcal{V}$.

A cut is a non-empty subset of non-destination nodes $\mathcal{U} \subseteq \mathcal{V} \setminus \mathcal{D}$. For a cut $\mathcal{U}$, we shall denote its capacity by $C_\mathcal{U} := \sum_{e\in\partial_\mathcal{U}^+} C_e$ and put $\lambda_\mathcal{U} := \sum_{v\in\mathcal{U}} \lambda_v$. The definition of $\mathcal{O}_\lambda$ implies that, under Assumption 1, there is no subset $\mathcal{A} \subseteq \mathcal{V}$ that is unreachable in $\mathcal{G}_\lambda$, i.e., it is not possible in $\mathcal{G}$ to have $\partial_\mathcal{A}^- = \emptyset$, and $\lambda_\mathcal{A} = 0$. Cut capacities determine potential bottlenecks for network flows. This is formalized in the celebrated max-flow min-cut theorem [14], [15], which states that, for $\mathcal{G} = (\mathcal{V}, \mathcal{E}, C)$ satisfying Assumption 1, it holds

$$\max_{f^*} \max_{\mathcal{U}} \{\lambda_\mathcal{U}(f^*) - C_\mathcal{U}\} = 0\,, \tag{1}$$

where the maximizations run over all feasible flows $f^* \in \mathcal{F}^*$, and cuts $\mathcal{U}$. Consider the special case when we restrict feasible $f^* \in \mathcal{F}^*$ such that $\lambda_o(f^*) > 0$ only for a single node $o \in \mathcal{V} \setminus \mathcal{D}$. In this case, one has $\lambda_\mathcal{U} = \lambda_o$ whenever $o \in \mathcal{U}$, so that (1) reduces to the better-known formulation of the max-flow min-cut theorem: $\max_{f^*} \lambda_o(f^*) = \min_\mathcal{U} C_\mathcal{U}$, where the maximization runs over the feasible flows f^* such that $\lambda_v(f^*) = 0, \forall v \in \mathcal{V} \setminus (\mathcal{D} \cup \{o\})$. For given $\mathcal{G} = (\mathcal{V}, \mathcal{E}, C)$ and λ, (1) gives a necessary and sufficient condition for the existence of a feasible flow with value λ, namely, $\sum_{v\in\mathcal{U}} \lambda_v \leq C_\mathcal{U}$ for every cut $\mathcal{U}$.

B. Dynamical flow networks and monotone distributed routing

We now introduce dynamics over a network $\mathcal{G} = (\mathcal{V}, \mathcal{E}, C)$. We associate, to each link $e \in \mathcal{E}$, a positive, and possibly infinite, buffer capacity $B_e \in (0, +\infty]$. Let $\mathcal{R} := \prod_{e\in\mathcal{E}} [0, B_e)$. For $e \in \mathcal{E} \cup \mathcal{O}_\lambda$, let $\overline{\rho}^e := \{B_j : j \in \mathcal{E}_e\}$, $\mathcal{R}_e := \prod_{j\in\mathcal{E}_e} [0, B_j)$, and $\mathcal{R}_e^\circ := \prod_{j\in\mathcal{E}_e} [0, B_j]$. Let $\mathcal{R}_e^\bullet$ be defined as $\mathcal{R}_e^\bullet = \mathcal{R}_e^\circ$ if $e \in \mathcal{E}_\mathcal{D}^-$, and $\mathcal{R}_e^\bullet = \mathcal{R}_e^\circ \setminus \{\overline{\rho}^e\}$ if $e \in (\mathcal{E} \cup \mathcal{O}_\lambda) \setminus \mathcal{E}_\mathcal{D}^-$. Finally, let the set of feasible flows on the outgoing links of e under capacity constraint be defined as $\mathcal{F}_e := [0, C_e]$ if $e \in \mathcal{E}_\mathcal{D}^-$ and $\mathcal{F}_e := \{x \in \mathbb{R}_+^{\mathcal{E}_e^+} : \sum_{j\in\mathcal{E}_e^+} x_j \leq C_e\}$ if $e \in (\mathcal{E} \cup \mathcal{O}_\lambda) \setminus \mathcal{E}_\mathcal{D}^-$.

We shall consider a dynamical system with state vector $\rho(t) \in \mathcal{R}$ whose e-th component, $\rho_e(t) \in [0, B_e)$, represents the time-varying density on link $e \in \mathcal{E}$. Dynamics is driven by conservation of mass and by a distributed routing policy, which determines how the outflow from each link depends on the current density and how it gets split among its following links. We shall loosely use the phrase *a set of links getting congested* to refer to the fact that the densities on those links approach their respective buffer capacities.

Definition 1. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, C)$ be a network satisfying Assumption 1. A distributed routing policy f with value $\lambda \in \mathbb{R}_+^{\mathcal{V}\backslash\mathcal{D}}$ and buffer capacities $\{B_e \in (0, +\infty] : e \in \mathcal{E}\}$ is a family of Lipschitz-continuous maps*

$$f^e : \mathcal{R}_e^\bullet \to \mathcal{F}_e\,, \qquad e \in \mathcal{E} \cup \mathcal{O}_\lambda\,, \tag{2}$$

such that

$$f^e(\rho^e) = \begin{cases} \{f_{e\to j}(\rho^e)\}_{j\in\mathcal{E}_e^+} & \text{if} \quad e \notin \mathcal{E}_\mathcal{D}^- \\ f_{e\to e_d}(\rho_e) & \text{if} \quad e \in \mathcal{E}_d^-\,,\, d \in \mathcal{D} \end{cases}$$

$$f_e^{\text{out}}(\rho^e) := \sum_{j\in\mathcal{E}_e^+} f_{e\to j}(\rho^e)$$

satisfy

$$f_{e_v}^{\text{out}}(\rho^{e_v}) = \lambda_v\,, \qquad \rho^{e_v} \in \mathcal{R}_{e_v}^\bullet\,,\ v \in \mathcal{V} \setminus \mathcal{D}\,, \tag{3}$$

and, for all $e \in \mathcal{E}$ and $\rho^e \in \mathcal{R}_e^\bullet$,

$$\rho_e = 0 \qquad \Longrightarrow \qquad f_e^{\text{out}}(\rho^e) = 0\,, \tag{4}$$

$$\rho_e = B_e \qquad \Longrightarrow \qquad f_e^{\text{out}}(\rho^e) = C_e\,, \tag{5}$$

and, for all $e \in \mathcal{E} \cup \mathcal{O}_\lambda$, $k \in \mathcal{E}_e^+$, $\rho^e \in \mathcal{R}_e^\bullet$

$$\rho_k = B_k \qquad \Longrightarrow \qquad f_{e\to k}(\rho^e) = 0\,. \tag{6}$$

The functions $f_{e\to j}(\rho^e)$ specify both how the outflow f_e^{out} depends on the local density and how it gets split into the outgoing links of τ_e. Notice that the domain of f^e is $\mathcal{R}_e^\bullet$, thus if $e \notin \mathcal{E}_\mathcal{D}^-$ it is not defined at the point $\overline{\rho}^e = \{B_j : j \in \mathcal{E}_e\}$, where (6) and (5) cannot hold simultaneously. On the other hand, f^e is well defined when the density is strictly less than its buffer capacity at least on one link in $\mathcal{E}_e$. Also notice that, because of the structure imposed by (2), the functions $\{f_{e\to j}(\rho^e)\}$ depend on the local density only, and in particular for $e \in \mathcal{E} \setminus \mathcal{E}_\mathcal{D}^-$ the outflow $f_e^{\text{out}}(\rho^e)$ depends only the density on link e itself and the links downstream to it, and if $e \in \mathcal{E}_\mathcal{D}^-$ then $f_e^{\text{out}}(\rho^e)$ only depends on ρ_e. Similarly, the inflow

$$f_e^{\text{in}}(\rho) := f_{e_{\sigma_e}\to e}(\rho^{e_{\sigma_e}}) + \sum_{j\in\mathcal{E}_e^-} f_{j\to e}(\rho^j)$$

of a link $e \in \mathcal{E}$ depends on the density on all the links in $\mathcal{E}$ incoming to or outgoing from σ_e (including link e itself). Also notice that the flow $f_{e_v\to j}$ from e_v to a link $j \in \mathcal{E}_v^+$

[1] The value of flow at a node is the same as the usual notion of external inflow at that node. In this paper, we use this terminology because of the necessity to interpret nodes with positive external inflow, i.e., origin nodes, as links.

depends on the densities of the links in $\mathcal{E}_v^+$ only, and that by (3) it holds $f_{e_v}^{\text{out}}(\rho^{e_v}) \equiv \lambda_v$, i.e., the outflow from every link e_v is constantly equal to λ_v. Finally, (4) and (6) imply that $f_e^{\text{out}}(\rho^e) = 0$ if $\rho_e = 0$, i.e., there is no outflow from a link e which is empty, or if $\rho_j = B_j, \forall j \in \mathcal{E}_e^+$, i.e., if the densities on all the links outgoing from τ_e are at their buffer capacities.

For $\rho \in \mathcal{R}$, let $F(\rho) \in \mathbb{R}_+^{(\mathcal{E}\cup\mathcal{O}_\lambda)\times(\mathcal{E}\cup\mathcal{E}_\mathcal{D}^+)}$ be defined as

$$F_{ej}(\rho) = \begin{cases} f_{e\to j}(\rho^e) & \text{if } j \in \mathcal{E}_e^+ \,, \\ f_{e\to e_d}(\rho_e) & \text{if } e \in \mathcal{E}_d^-, d \in \mathcal{D}, j = e_d \,, \\ 0 & \text{otherwise.} \end{cases}$$

Imposing mass conservation $\dot{\rho}_e = f_e^{\text{in}} - f_e^{\text{out}}$ on every link $e \in \mathcal{E}$ leads one to consider the dynamical system

$$\dot{\rho} = (F(\rho)'\mathbb{1})_\mathcal{E} - (F(\rho)\mathbb{1})_\mathcal{E} = \Phi(\rho) \,. \tag{7}$$

We shall refer to it as the *dynamical flow network*. Observe that, thanks to the Lipschitzianity assumption on the routing policies, standard analytical results (Picard's Existence Theorem) imply, for every initial density $\rho(0) = \rho^\circ \in \mathcal{R}$, existence and uniqueness of a solution $\{\rho(t) : 0 < t < \kappa(\rho^\circ)\}$ of (7) up to $\kappa(\rho^\circ) := \sup\{t \geq 0 : \rho(t) \in \mathcal{R}, \rho(0) = \rho^\circ\}$, i.e., as long as $\rho(t)$ stays within $\mathcal{R}$. Moreover, (4) implies invariance of the nonnegative orthant, i.e., $\rho(t) \succeq 0$ for all $\rho^\circ \in \mathcal{R}$ and $t \leq \kappa(\rho^\circ)$. Hence, $\kappa(\rho^\circ)$ coincides with the first time the solution hits the buffer capacity on some link.

Remark 1. *In this paper, we study the behavior of dynamical flow networks only for $t \in [0, \kappa(\rho^0))$. Some initial work on the complex behavior of dynamical flow networks, such as cascading failures, for $t > \kappa(\rho^0)$ is reported in our companion papers [16], [17].*

Remark 2. *In our previous work [12], [13], we formulated the dynamical flow network framework for acyclic network topologies and where the links have infinite buffer capacities. We considered routing policies under which the outflow from a link j is independent of the densities on the links downstream from link j. This, combined with the fact that all the links have infinite buffer capacities, implied that there is no backward propagation of congestion effects. In this paper, we extend and modify the framework from [12], [13] to allow for finite buffer capacities and cyclic network topologies, and also allow the routing policies to completely control (subject to capacity constraints) the flow transfer between links, i.e., the routing policies can also control the inflow arriving at nodes. This allows for backward propagation of congestion effects, and hence yields stronger results in comparison to [12], [13].*

We shall be interested in a special class of distributed routing policies, as per the following.

Definition 2. *A distributed routing policy f is* monotone *if, for all $e \in \mathcal{E} \cup \mathcal{O}_\lambda$, $\rho^e \in \mathcal{R}_e^\bullet$, the functions $\{f^e\}$ satisfy*

$$\frac{\partial f_{e\to j}}{\partial \rho_k}(\rho^e) \geq 0, \qquad \forall\, j \in \mathcal{E}_e^+, k \in \mathcal{E}_e \setminus \{j\} \,, \tag{8}$$

$$\frac{\partial}{\partial \rho_k} f_e^{\text{out}}(\rho^e) \leq 0, \qquad \forall\, k \in \mathcal{E}_e^+ \,, \tag{9}$$

for almost every $\rho^e \in \mathcal{R}_e$. A monotone distributed policy is strongly monotone *if, for all $e \in \mathcal{E} \cup \mathcal{O}_\lambda$, and almost every $\rho^e \in \mathcal{R}_e$, the inequalities in* (8) *and* (9) *are strict.*

Example 1. *For every link $e \in \mathcal{E}$, let $\varphi_e : [0, B_e] \to [0, +\infty]$ be Lispchitz continuous, strictly increasing, and such that $\varphi_e(0) = 0$ and $\varphi_e(B_e) = +\infty$. Example of such a φ_e is $\varphi_e(\rho_e) = \beta_e\rho_e/(B_e - \rho_e)$ if $B_e < +\infty$, and $\varphi_e(\rho_e) = \beta_e\rho_e$ if $B_e = +\infty$, for some $\beta_e > 0$, $e \in \mathcal{E}$. Define*

$$f_{e\to j}(\rho^e) = \begin{cases} C_e\,(1-\gamma_e)\,\gamma_j/Z \text{ if } e \in \mathcal{E} \setminus \mathcal{E}_\mathcal{D}^- \,, \\ C_e\,(1-\gamma_e) \text{ if } e \in \mathcal{E}_d^-, d \in \mathcal{D}, j = e_d \,, \\ \lambda_v\gamma_j/Z \text{ if } e = e_v \in \mathcal{O}_\lambda \,, \end{cases}$$

where $\gamma_i := \exp(-\varphi_i(\rho_i))$ and $Z := \sum_{k\in\mathcal{E}_e}\gamma_k$. Then $\{f^e\}_{e\in\mathcal{E}\cup\mathcal{O}_\lambda}$ is a strongly monotone distributed routing policy.

Notice that, under monotone distributed routing policies, (7) defines a *cooperative* dynamical system in the sense of Hirsch [18], [19], i.e,

$$\frac{\partial \Phi_e}{\partial \rho_k}(\rho) \geq 0, \qquad \forall e, k \in \mathcal{E}, e \neq k \,. \tag{10}$$

Then, Kamke's theorem [19, Theorem 1.2], [20] implies that (7) is a monotone system [18], i.e.,

$$\rho(0) \preceq \tilde{\rho}(0) \quad \Rightarrow \quad \rho(t) \preceq \tilde{\rho}(t) \,, \quad \forall t \in [0, \kappa(\tilde{\rho}(0)))\,. \tag{11}$$

Also, observe that monotonicity implies that $\kappa(\rho^\circ) \leq \kappa(\mathbf{0})$ for all $\rho^\circ \in \mathcal{R}$.

We conclude this section by showing how our framework of dynamical flow networks allows analysis of dynamic traffic models, as well as distributed routing in data networks.

C. Dynamic Traffic Models

Our framework allows to analyze a dynamic traffic model closely related to the well known *cell transmission model* [11], [8]. We explain this for simple line networks. For $n \geq 1$, let $\mathcal{V} = \{v_i : 0 \leq i \leq n\}$, $\mathcal{E} := \{1, 2, \ldots, n\}$, and the indices of the nodes and links be such that $\tau_e = \sigma_{e+1} = e$, for $1 \leq e \leq n$. In the cell transmission model terminology, link e represents a *cell* with homogeneous traffic density within it. For $1 \leq e \leq n$, let $B_e, C_e \in (0, +\infty)$, and $\overline{\rho}_e \in (0, B_e)$. Let $\psi_e : [0, B_e] \to [0, C_e]$ be a Lipschitz continuous function such that $\psi_e(0) = \psi_e(B_e) = 0$, $\psi_e(\bar{\rho}_e) = C_e$, and ψ_e is nondecreasing in $[0, \overline{\rho}_e]$ and nonincreasing in $[\overline{\rho}_e, B_e]$. Define

$$\begin{aligned} &f_{e\to e+1}(\rho^e) := \min\{d_e(\rho_e), s_{e+1}(\rho_{e+1})\}\,, \qquad 1 \leq e < n\,, \\ &f_{n\to e_d}(\rho_n) := d_n(\rho_n), \end{aligned} \tag{12}$$

where

$$\begin{array}{llll} d_e(x) = \psi_e(x), & s_e(x) = C_e & \text{if} & x \in [0, \bar{\rho}_e] \\ d_e(x) = C_e, & s_e(x) = \psi_e(x) & \text{if} & x \in [\bar{\rho}_e, B_e]\,. \end{array}$$

In the transportation literature, ψ_e is referred to as the *fundamental diagram*, e.g., see [8], [10], while d_e and s_e are referred to as the supply and the demand functions respectively, of link e. It is easily seen that (12) satisfies (4), (6), as well as (8) and (9). (12) satisfies (5) under special conditions as follows.

Consider the case when all the cells are identical in the sense that $B_e = B$, $\bar{\rho}_e = \bar{\rho}$ and $C_e = C$ for all $e \in \mathcal{E}$, for some $C > 0$, $B > 0$ and $\bar{\rho} \in (0, B)$. Let the initial condition be such that $\rho^\circ \in \Pi_{e\in\mathcal{E}}[0, \bar{\rho}_e) \subset \mathcal{R}$. (12) implies that $\dot{\rho}_n = f_{n-1\to n}(\rho^{n-1}) - f_{n\to e_d}(\rho_n)$. Since $f_{n-1\to n}(\rho^{n-1}) \leq C$ for all ρ^{n-1}, and $d_n(\bar{\rho}) = C$, this implies that, as $\rho_n \to \bar{\rho}$, $\dot{\rho}_n$ is non-positive. Hence, $\rho_n(t) \in [0, \bar{\rho}]$ for all $t \geq 0$. This implies that $s_n(\rho_n(t)) = C$ for all $t \geq 0$. Therefore, $f_{n-1\to n}(\rho^{n-1}) = d_{n-1}(\rho_{n-1})$. By performing backward induction on the indices of the cells, one can establish that $\Pi_{e\in\mathcal{E}}[0, \bar{\rho}_e)$ is positively invariant, and hence $f_{e\to e+1}(\rho^e) = d_e(\rho_e)$ for all $e \in \{1, \ldots, n-1\}$, and $f_{n\to e_d}(\rho_n) = d_n(\rho_n)$. (12) then satisfies (5) by interpreting $\bar{\rho}$ as B_e in (5). The invariance of $\Pi_{e\in\mathcal{E}}[0, \bar{\rho}_e)$ also implies that (6) is irrelevant when $\rho^\circ \in \Pi_{e\in\mathcal{E}}[0, \bar{\rho}_e)$. Using our results from Section III and IV, one can show that there exists a unique equilibrium ρ^* in $\Pi_{e\in\mathcal{E}}[0, \bar{\rho}_e)$ which is asymptotically stable for all initial conditions $\rho^\circ \in \Pi_{e\in\mathcal{E}}[0, \bar{\rho}_e)$ – this can be extended to global asymptotic stability by $\bar{\rho} \to B$.

D. Distributed Routing in Data Networks

It is possible to extend our framework to include well-known distributed routing algorithms in data networks, e.g., see [3], [4], [5]. We now explain the procedure to fit our setup within the framework of [3]. We define a new network $\tilde{\mathcal{G}} = (\tilde{\mathcal{V}}, \tilde{\mathcal{E}}, \tilde{C})$ based on the given $\mathcal{G} = (\mathcal{V}, \mathcal{E}, C)$ as follows. For every $e \in \mathcal{E}$, assign a node $\tilde{v}(e)$ in $\tilde{\mathcal{V}}$. For every pair of links e and j in $\mathcal{E}$ such that $\tau_e = \sigma_j$, define a link in $\tilde{\mathcal{G}}$ from node $\tilde{v}(e)$ to node $\tilde{v}(j)$. For every $\tilde{e} \in \tilde{\mathcal{E}}$, $\tilde{C}_{\tilde{e}}$ is defined to be equal to C_e for $e \in \mathcal{E}$ such that $\tilde{v}(e) = \sigma_{\tilde{e}}$. Under this definition, the capacities of all links outgoing from the same node are equal in $\tilde{\mathcal{G}}$. Therefore, in order to impose constraints on the simultaneous utilization of the links outgoing from a common node in $\tilde{\mathcal{V}}$, we define the constraint set $\mathcal{S}_{\tilde{v}} := \{x \in \mathbb{R}_+^{\mathcal{E}_{\hat{e}}^+} : \sum_{j\in\mathcal{E}_{\hat{e}}^+} x_j \leq 1\}$, where $\hat{e} \in \tilde{\mathcal{E}}$ is such that $\tau_{\hat{e}} = \tilde{v}$. The distributed routing policy at node $\tilde{v}$ as per Definition 1 then corresponds to choosing an *activation vector* from the constraint set and multiplying it by $\tilde{C}_{\tilde{e}}$ for $\tilde{e}$ such that $\sigma_{\tilde{e}} = \tilde{v}$. The back pressure routing policy proposed in [3] relies on the same local information as the distributed routing policy in the context of $\tilde{\mathcal{G}}$. However, the constraint set for the back pressure routing policy is different than $\mathcal{S}_{\tilde{v}}$, and in general is equal to the set of all possible binary vectors defined over the links $\tilde{\mathcal{E}}$. When restricted to local information around a node, it is equal to the set of all possible binary vectors over the set of outgoing links from the node. While such a general notion of activation vectors allows for modeling richer class of constraints, the constraint set when restricted to $\mathcal{S}_{\tilde{v}}$ is simply the union of the vertices of the corresponding simplex and $\mathbf{0}$. In fact, recalling Example 1 with $B_e = +\infty$ as is the case in [3], in the limit as $\beta \to \infty$, we get that: if $\varphi_e(\rho_e) \geq \varphi_j(\rho_j)$ for all $j \in \mathcal{E}_e^+$ then $f_{e\to j}(\rho^e) \to 0^+$ for all $j \in \mathcal{E}_e^+$; otherwise $f_{e\to j}(\rho^e) \to C_e 1_{\{\rho_e>0\}} G_j(\rho^e)$ where $G_j(\rho^e) = 1$ if $\varphi_j(\rho_j) > \varphi_k(\rho_k)$ for all $k \in \mathcal{E}_e^+ \setminus \{j\}$ and zero otherwise, with the ties resolved arbitrarily. With $\varphi_e(x) = x$ for all $e \in \mathcal{E}$, this gives us the back-pressure policy, and with $\varphi_e(x) = x^\alpha$, $\alpha > 0$, for all $e \in \mathcal{E}$, this gives us the maxweight-α policy [5]. Therefore, in our continuous time and continuous state setup, the monotone distributed routing policy generalizes existing well-known distributed routing policies. Moreover, using our results from Section III, we can establish throughput optimality under finite buffer capacities using distributed routing.

III. Main results

In this section, we present the main contributions of the paper. The first result is Theorem 1, which states a dichotomy. If the inflow is less than the capacity of every cut, then there exists a globally asymptotically stable equilibrium density $\rho^* \in \mathcal{R}$. Otherwise, the network is divided in two parts by a cut $\mathcal{S}$, such that the densities on the links in $\mathcal{E}_{\mathcal{S}}^+$ approach their buffer capacities simultaneously.

Theorem 1. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, C)$ be a network satisfying Assumption 1, and f be a monotone distributed routing policy with value λ. For $\rho^\circ \in \mathcal{R}$, let $\{\rho(t) : 0 \leq t < \kappa(\rho^\circ)\}$ be the solution of the dynamical flow network* (7) *with initial condition $\rho(0) = \rho^\circ$. Then,*

(i) *if $\max_{\mathcal{U}} (\lambda_{\mathcal{U}} - C_{\mathcal{U}}) < 0$, then $\kappa(\rho^\circ) = +\infty$ for every initial density $\rho^\circ \in \mathcal{R}$; moreover, if the distributed routing policy is strongly monotone, then there exists an equilibrium density $\rho^* \in \mathcal{R}$ such that* $\lim_{t\to\infty} \rho(t) = \rho^*$ *for every initial density vector $\rho^\circ \in \mathcal{R}$.*

(ii) *if $\max_{\mathcal{U}} \{\lambda_{\mathcal{U}} - C_{\mathcal{U}}\} > 0$, or if $\max_{\mathcal{U}} \{\lambda_{\mathcal{U}} - C_{\mathcal{U}}\} = 0$ and the routing policy is strongly monotone, then, for every initial density $\rho^\circ \in \mathcal{R}$, there exists a cut $\mathcal{S}$ such that*

$$\lim_{t\to\kappa(\rho^\circ)} \rho_e(t) = B_e, \quad \forall e \in \mathcal{E}_{\mathcal{S}}^+ . \tag{13}$$

Remark 3. *1) Part i) of Theorem 1 strengthens results on stability of dynamical flow networks from our previous work [12], [13] as follows. First, in [12], [13], we considered acyclic network topologies and infinite buffer capacities on links, whereas Theorem 1 is valid for cyclic network topologies, and infinite as well as finite buffer capacities. And, second, the routing policies in [12], [13] do not give strong guarantees for existence or stability of equilibria, whereas (strongly) monotone routing policies guarantee existence and (global) asymptotic stability of equilibria when* $\max_{\mathcal{U}} (\lambda_{\mathcal{U}} - C_{\mathcal{U}}) < 0$. *At an equilibrium ρ^*, the throughput of the dynamical network is* $\sum_{d\in\mathcal{D}} f_{e_d}^{\text{out}}(\rho^*) = \sum_{v\in\mathcal{V}\setminus\mathcal{D}} \lambda_v$. *Therefore, in conjunction with the max-flow min-cut theorem, part i) of Theorem 1 implies that monotone distributed routing policies are throughput optimal. It is important to emphasize that this throughput optimality is achieved under a 'distributed' routing policy. The stronger results in this paper are possible due to backward propagation of congestion effects facilitated by a routing policy architecture under which the outflow from a link j depends on the density of links downstream from j.*

2) The throughput optimality result can also be interpreted from the point of view of resilience. For simplicity, consider a network having only one node $o \in \mathcal{V}$ such that $\lambda_o > 0$. If one defines, as in [12], [13], the margin of resilience as the minimum sum of link-wise flow capacity losses under which the throughput of the network is asymptotically strictly less than λ_o, then Theorem 1 implies that, under the monotone *distributed routing policy, the margin of resilience is equal to the network residual capacity* $\min_{cut\,\mathcal{U}} C_{\mathcal{U}} - \lambda_o$. *A simple application of the mass balance equation to the min-cut also implies that the network residual capacity is indeed the maximum possible margin of resilience under any (not necessarily distributed) routing policy. The margin of resilience under the routing policies in [12], [13] is equal to minimum node residual capacity, which is less than or equal to the network residual capacity. This illustrates that, by allowing backward propagation of congestion effects, monotone routing policies give the maximum possible margin of resilience, even by using only local information.*

A. Overload behavior with finite buffer capacities

The following proposition gives a more detailed characterization of what happens when the capacity constraints are violated in the case of finite buffer capacities.

Proposition 1. *Let* $\mathcal{G} = (\mathcal{V}, \mathcal{E}, C)$ *be a network satisfying Assumption 1, and* f *be a monotone distributed routing policy with value* λ *and finite buffer capacities* $B_e \in (0, +\infty)$, $e \in \mathcal{E}$. *Assume that* $\max_{\mathcal{U}} (\lambda_{\mathcal{U}} - C_{\mathcal{U}}) > 0$. *Then, for every* $\rho^\circ \in \mathcal{R}$,

$$\kappa(\rho^\circ) \leq \min_{\mathcal{U}:\, \lambda_{\mathcal{U}} > C_{\mathcal{U}}} \frac{\sum_{e \in \mathcal{E}^+_{\mathcal{U}}} (B_e - \rho^\circ_e)}{\lambda_{\mathcal{U}} - C_{\mathcal{U}}}, \tag{14}$$

and there exists a cut $\mathcal{S}$, *possibly depending on* ρ°, *such that* $\lambda_{\mathcal{S}} > C_{\mathcal{S}}$ *and*

$$\begin{aligned} &\rho_e(t) < B_e\,,\ \forall e \in \mathcal{E}\,,\ 0 \leq t < \kappa(\rho^\circ)\,,\\ &\lim_{t \to \kappa(\rho^\circ)} \rho_e(t) = B_e, \quad \forall e \in \mathcal{E}^+_{\mathcal{S}}\,, \end{aligned} \tag{15}$$

where $\{\rho(t) : 0 \leq t < \kappa(\rho^\circ)\}$ *is the solution of the dynamical flow network (7) with initial condition* $\rho(0) = \rho^\circ$.

Proposition 1 states that, if the buffer capacities are finite and some cut constraints are violated, then, for every initial density ρ°, all the links in $\mathcal{E}^+_{\mathcal{S}}$, where $\mathcal{S}$ is a cut such that $\lambda_{\mathcal{S}} > C_{\mathcal{S}}$, will reach their buffer capacities simultaneously at time $\kappa(\rho^\circ)$. It is important to stress that, when there are multiple cuts violating the capacity constraint, then the cut $\mathcal{S}$ in the proposition may depend on the initial condition ρ°. Observe that dependence on the initial density ρ° is also evident in (14). While it may be tempting to identify the cut $\mathcal{U}$ minimizing the right hand side of (14) with the cut $\mathcal{S}$ of (15), it is worth stressing that (14) is merely an upper bound on $\kappa(\rho^\circ)$. In fact, in contrast to the right-hand side of (14), the cut $\mathcal{S}$ of (15) may depend on finer details of the routing policy, rather than just its value and buffer capacities.

Remark 4. *It is interesting to compare Proposition 1 with the framework of our previous work [16], where we consider links with finite buffer capacities, but the the routing policies are such that the outflow from a link* j *is independent of densities on links downstream from* j. *As a consequence, in the framework of [16], even if* $C_{\mathcal{U}} > \lambda_{\mathcal{U}}$ *for every cut* $\mathcal{U}$, *there might be a link* e *on which the density hits the buffer capacity which, in turn, could trigger a backward cascade. Part i) of Theorem 1 implies that this cannot happen under the routing policies presented in this paper if* $C_{\mathcal{U}} > \lambda_{\mathcal{U}}$ *for every cut* $\mathcal{U}$. *Moreover, as part ii) of Theorem 1 and Proposition 1 imply, the whole cut* $\mathcal{S}$ *fills simultaneously when* $\max_{\mathcal{U}} \{\lambda_{\mathcal{U}} - C_{\mathcal{U}}\} > 0$, *and hence the network collapse is abrupt, and does not involve any* cascading *phenomena.*

B. Overload behavior with infinite buffer capacities

The following result, similar to Proposition 1, characterizes the way congestion occurs in case of infinite buffer capacities.

Proposition 2. *Let* $\mathcal{G} = (\mathcal{V}, \mathcal{E}, C)$ *be a network satisfying Assumption 1, and* f *be a strongly monotone distributed routing policy with value* λ *and buffer capacities* $B_e = +\infty$, *for* $e \in \mathcal{E}$. *Assume that* $\max_{\mathcal{U}} (\lambda_{\mathcal{U}} - C_{\mathcal{U}}) \geq 0$. *Let*

$$\mathcal{U}^* := \bigcup\nolimits_{\mathcal{U} \in \mathcal{M}} \mathcal{U}\,, \qquad \mathcal{M} := \underset{\mathcal{U}}{\operatorname{argmax}}\, (\lambda_{\mathcal{U}} - C_{\mathcal{U}})\,. \tag{16}$$

Then, for every $\rho^\circ \in \mathcal{R}$, *the solution* $\rho(t)$ *of the dynamical flow network* (7) *with initial condition* $\rho(0) = \rho^\circ \in \mathcal{R}$ *is such that* $\kappa(\rho^\circ) = +\infty$ *and*

$$\begin{aligned} &\lim_{t \to +\infty} \rho_e(t) = +\infty\,, \qquad \forall e \in \mathcal{E}^+_{\mathcal{U}^*}\,,\\ &\lim_{t \to +\infty} \frac{1}{t} \sum_{e \in \mathcal{E}^+_{\mathcal{U}^*}} \rho_e(t) = \lambda_{\mathcal{U}^*} - C_{\mathcal{U}^*}\,. \end{aligned} \tag{17}$$

Moreover, there exist $\rho^*_e \in [0, +\infty)$, $e \in \mathcal{E} \setminus (\mathcal{E}^+_{\mathcal{U}^*} \cup \partial^-_{\mathcal{U}^*})$, *such that*

$$\lim_{t \to +\infty} \rho_e(t) = \rho^*_e\,, \qquad \forall e \in \mathcal{E} \setminus (\mathcal{E}^+_{\mathcal{U}^*} \cup \partial^-_{\mathcal{U}^*})\,, \tag{18}$$

for every initial density $\rho^\circ \in \mathcal{R}$.

Proposition 2 implies that, when the buffer capacities on all the links are infinite, then there exists a cut $\mathcal{U}^*$, independent of initial condition ρ° such that, asymptotically, all the links in $\mathcal{E}^+_{\mathcal{U}^*}$ get congested. This is to be contrasted with the finite buffer capacity case, which has a similar result, however, the cut there depends on the initial condition ρ°. Proposition 2 also implies that the total density in $\mathcal{E}^+_{\mathcal{U}^*}$ grows linearly in time, and that the densities on the links which do not get congested approach a unique limit point. A comparison is due with [5], which studies an acyclic queuing network with set of queues $\mathcal{Q}$ employing max-weight algorithm. It is shown that if $q(t) \in \mathbb{R}^{\mathcal{Q}}_+$ is the vector of queue lengths, then $q(t)/t \to \hat{q}$ where $\hat{q} \in \mathbb{R}^{\mathcal{Q}}_+$ is the solution to an optimization problem related to the parameters of the max-weight algorithm.

IV. Proofs

To prove our results, we first provide a novel l_1 contraction principle for monotone dynamical systems under conservation laws. We then characterize the behavior of dynamical flow networks under the assumption that the vector of densities converges to a limit point. This will then help us to prove the main results.

A. An l_1-contraction principle for monotone conservation laws

We state and prove an l_1-contraction principle for a class of monotone dynamical systems under conservation laws, which includes system (7) under monotone distributed routing policy. As such, it will be instrumental in proving existence and stability of equilibria for dynamical flow networks.

Lemma 1. *For a non-empty closed hyper–rectangle $\Omega \subseteq \mathbb{R}^n$, let $g : \Omega \to \mathbb{R}^n$ be Lipschitz and such that*

$$\frac{\partial}{\partial x_j} g_i(x) \geq 0 \,, \qquad \forall\, i \neq j \in \{1, \ldots, n\} \tag{19}$$

$$\sum_{1 \leq i \leq n} \frac{\partial}{\partial x_j} g_i(x) \leq 0 \,, \qquad \forall\, j \in \{1, \ldots, n\} \tag{20}$$

for almost every $x \in \Omega$. Then

$$\sum_{1 \leq i \leq n} \mathrm{sgn}\,(x_i - y_i)\,(g_i(x) - g_i(y)) \leq 0, \qquad \forall x, y \in \Omega \,. \tag{21}$$

Moreover, if

(i) *there exists some $j \in \{1, \ldots, n\}$ such that the inequality (20) is strict for almost all $x \in \Omega$,*

then inequality (21) is strict for all $x, y \in \Omega$ such that $x_j \neq y_j$.

If

(ii) *for every proper subset $\mathcal{K} \subseteq \{1, \ldots, n\}$, there exist $i \in \mathcal{K}$, and $j \in \{1, \ldots, n\} \setminus \mathcal{K}$ such that inequality (19) is strict for almost all $x \in \Omega$,*

then inequality (21) is strict for all $x \neq y$ such that $x \not\prec y$ and $y \not\prec x$.

Finally, if (i) and (ii) hold true, then inequality (21) *is strict for all $x, y \in \Omega$ such that $x \neq y$.*

Proof. First note that, according to Rademacher's theorem, e.g., see [21], Lipschitz continuity implies differentiability almost everywhere. For $\mathcal{A} \subseteq \{1, \ldots, n\}$, put $\mathcal{A}^c := \{1, \ldots, n\} \setminus \mathcal{A}$, and $g_{\mathcal{A}}(z) := \sum_{a \in \mathcal{A}} g_a(z)$. Fix some $x, y \in \Omega$, and put $\mathcal{I} = \{i : x_i > y_i\}$, $\mathcal{J} = \{i : x_i < y_i\}$. Let $\xi \in \Omega$ be such that $\xi_i = x_i$ for $i \in \mathcal{I}$ and $\xi_i = y_i$ for $i \in \mathcal{I}^c$. Consider the segments $\gamma_{\mathcal{I}}$ from y to ξ and $\gamma_{\mathcal{J}}$ from x to ξ. For $\mathcal{A} \subseteq \{1, \ldots, n\}$, and $\mathcal{B} \in \{\mathcal{I}, \mathcal{J}\}$, define the path integral

$$\Gamma^{\mathcal{A}}_{\mathcal{B}} := \int_{\gamma_{\mathcal{B}}} \nabla g_{\mathcal{A}}(z) \cdot \mathrm{d}z \,.$$

Then, (20) implies that

$$g_{\mathcal{I}}(x) - g_{\mathcal{I}}(y) = \Gamma^{\mathcal{I}}_{\mathcal{I}} - \Gamma^{\mathcal{I}}_{\mathcal{J}} \leq -\Gamma^{\mathcal{I}^c}_{\mathcal{I}} - \Gamma^{\mathcal{I}}_{\mathcal{J}}, \tag{22}$$

$$g_{\mathcal{J}}(x) - g_{\mathcal{J}}(y) = \Gamma^{\mathcal{J}}_{\mathcal{I}} - \Gamma^{\mathcal{J}}_{\mathcal{J}} \geq \Gamma^{\mathcal{J}}_{\mathcal{I}} + \Gamma^{\mathcal{J}^c}_{\mathcal{J}} \,. \tag{23}$$

Combining the above gives

$$\begin{aligned} \sum_i s_i\,(g_i(x) - g_i(y)) &= g_{\mathcal{I}}(x) - g_{\mathcal{I}}(y) - g_{\mathcal{J}}(x) + g_{\mathcal{J}}(y) \\ &\leq -\Gamma^{\mathcal{I}^c}_{\mathcal{I}} - \Gamma^{\mathcal{I}}_{\mathcal{J}} - \Gamma^{\mathcal{J}}_{\mathcal{I}} - \Gamma^{\mathcal{J}^c}_{\mathcal{J}} \,, \end{aligned}$$

with $s_i := \mathrm{sgn}\,(x_i - y_i)$. Observe that, by (19), $\mathcal{A} \cap \mathcal{B} = \emptyset$ implies $\Gamma^{\mathcal{A}}_{\mathcal{B}} \geq 0$, so that (21) follows immediately.

Notice that, if there exists some $j \in \{1, \ldots, n\}$ such that inequality (20) is strict for almost every $x \in \Omega$, and $x_j > y_j$ ($x_j < y_j$), then (22) (respectively, (23)) is a strict inequality, hence so is (21), thus proving the second claim.

Now, assume that $x \neq y$, $x \not\prec y$ and $y \not\prec x$. Then, it follows from the definition of the sets $\mathcal{I}$ and $\mathcal{J}$ that the sets $\mathcal{I}^c$ and $\mathcal{J}^c$ are non-empty. We also have that $\mathcal{I}^c \cap \mathcal{J}^c = \{i \in \{1, \ldots, n\} \mid x_i = y_i\}$. Since $x \neq y$, this implies that $\mathcal{I}^c \cap \mathcal{J}^c \neq \{1, \ldots, n\}$. Therefore, at least one of $\mathcal{I}^c$ and $\mathcal{J}^c$ is a proper subset of $\{1, \ldots, n\}$. If say $\mathcal{I}^c$ is a proper subset, then the condition in (ii) in the statement of the lemma implies that (19) is strict for some $i \in \mathcal{I}$ and $j \in \mathcal{I}^c$. Therefore, $\Gamma(\mathcal{I}^c, \mathcal{I}) > 0$, and the third claim follows.

Finally, the last claim is implied by the previous two: if $x \prec y$ or $y \prec x$, then trivially $x_j \neq y_j$ for all $j \in \{1, \ldots, n\}$ and the strict inequality in (21) follows from the claim associated with condition (i); if $x \not\prec y$ and $y \not\prec x$, then the strict inequality in (21) follows from the claim associated with condition (ii). □

Lemma 1 implies the following l_1-contraction principle for dynamic networks with monotone distributed routing policies.

Lemma 2. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, C)$ be a network satisfying Assumption 1, f be a monotone distributed routing policy, and $\hat{\rho}^\circ, \tilde{\rho}^\circ \in \mathcal{R}$. Let $\hat{\rho}(t)$ and $\tilde{\rho}(t)$ be the solutions to the system* (7) *with initial conditions $\hat{\rho}(0) = \hat{\rho}^\circ$, and $\tilde{\rho}(0) = \tilde{\rho}^\circ$, respectively. Define $\varphi(t) := ||\hat{\rho}(t) - \tilde{\rho}(t)||_1$ for $0 \leq t < \min\{\kappa(\hat{\rho}^\circ), \kappa(\tilde{\rho}^\circ)\}$. Then $\dot{\varphi}(t) \leq 0$. Moreover, if the routing policy is strongly monotone, then $\dot{\varphi}(t) = 0$ if and only if $\hat{\rho}(t) = \tilde{\rho}(t)$.*

Proof. It is easily verified that the properties of monotone distributed routing policies (8) and (9) imply (19) and (20) for the function $\Phi(\cdot)$. Therefore, the first claim in Lemma 1 gives

$$\dot{\varphi}(t) = \sum_e \mathrm{sgn}\,(\hat{\rho}_e(t) - \tilde{\rho}_e(t))\,(\Phi_e(\hat{\rho}(t)) - \Phi_e(\tilde{\rho}(t))) \leq 0$$

if the distributed routing policy is monotone.

We now show that conditions (i) and (ii) in Lemma 1 follow from the strong monotonicity property of the distributed routing policies. To that effect, for any $j \in \mathcal{E}^-_{\mathcal{D}}$, we have that

$$\begin{aligned} \frac{\partial}{\partial \rho_j} \sum_{i \in \mathcal{E}} \Phi_i(\rho) &= \frac{\partial}{\partial \rho_j} \left(\sum_{v \in \mathcal{V} \setminus \mathcal{D}} \lambda_v - \sum_{i \in \mathcal{E}^-_{\mathcal{D}}} f^{\mathrm{out}}_i(\rho^i) \right) \\ &= -\frac{\partial}{\partial \rho_j} f^{\mathrm{out}}_j(\rho^j) < 0, \end{aligned}$$

where the strict inequality follows from the strict version of (8) characterizing strongly monotone routing policies. This establishes condition (i) in Lemma 1. In order to connect condition (ii) in Lemma 1, consider any proper subset $\mathcal{K} \subsetneq \mathcal{E}$. It is easily seen that there exist $i \in \mathcal{K}$ and $j \in \mathcal{K}^c$ such that: either (a) $\tau_j = \sigma_i$ or $\sigma_j = \sigma_i$; or (b) $j \in \mathcal{E}^+_i$. In case (a),

$$\frac{\partial \Phi_i}{\partial \rho_j}(\rho) = \frac{\partial f^{\mathrm{in}}_i}{\partial \rho_j}(\rho) = \sum_{e \in \mathcal{E}^-_i} \frac{\partial f_{e \to i}}{\partial \rho_j}(\rho) + \frac{\partial f_{e_{\sigma_i} \to i}}{\partial \rho_j}(\rho^{e_{\sigma_i}}) > 0,$$

where the strict inequality follows from the strict version of (8) that holds true for a strongly monotone routing policy. In case (b), we have that $\frac{\partial}{\partial \rho_j} \Phi_i(\rho) = -\frac{\partial}{\partial \rho_j} f^{\mathrm{out}}_i(\rho^i) > 0$, where the strict inequality follows from the strict version of

Equation (9) that holds true for a strongly monotone routing policy. The second claim in Lemma 2 now follows from the last claim in Lemma 1. □

B. Properties of limit density vectors

For an initial density $\rho^\circ \in \mathcal{R}$, let us consider the following subsets of $\mathcal{E}$:

$$\begin{aligned} &\mathcal{B} := \{\lim \rho_e(t) = B_e\}\,, && \mathcal{W} := \{\limsup \rho_e(t) < B_e\}\,, \\ &\mathcal{Z}_\mathrm{o} := \{\lim f_e^{\mathrm{out}}(\rho^e(t)) = 0\}\,, && \mathcal{Z}_\mathrm{i} := \{\lim f_e^{\mathrm{in}}(\rho^e(t)) = 0\}\,, \\ &\mathcal{C} := \{\lim f_e^{\mathrm{out}}(\rho^e(t)) = C_e\}\,, && \mathcal{Z} := \mathcal{Z}_\mathrm{i} \cup \mathcal{Z}_\mathrm{o}, \end{aligned} \tag{24}$$

were the limits are meant as $t \uparrow \kappa(\rho^\circ)$ and the curly brackets are meant as defining the sets of those links e such that the enclosed condition is satisfied.

Observe that the definitions in (24) do not assume existence of a limit density. However, if a limit $\rho^* = \lim_{t\uparrow\kappa(\rho^\circ)} \rho(t)$ exists, then clearly $\mathcal{E} = \mathcal{B} \cup \mathcal{W}$. On the other hand, in general, existence of the limit density ρ^* does not necessarily imply existence of the limit outflow $\lim_{t\uparrow\kappa(\rho^\circ)} f_e^{\mathrm{out}}(\rho^e(t))$ or the limit inflow $\lim_{t\uparrow\kappa(\rho^\circ)} f_e^{\mathrm{in}}(\rho^e(t))$ for every $e \in \mathcal{E}$. Finally, observe that $\mathcal{C} \cap \mathcal{Z}_\mathrm{o} = \emptyset$, and that $\mathcal{B} \cap \mathcal{C} \cap \mathcal{Z}_\mathrm{i} = \emptyset$, since $\lim_{t\uparrow\kappa(\rho^\circ)} \dot\rho_e(t) = -C_e < 0$ for all $e \in \mathcal{C} \cap \mathcal{Z}_\mathrm{i}$ which is incompatible with $e \in \mathcal{B}$.

The following result characterizes the behavior of $\rho(t)$ starting from some initial condition $\rho(0) = \rho^\circ \in \mathcal{R}$, as t approaches $\kappa(\rho^\circ)$.

Lemma 3. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, C)$ be a network satisfying Assumption 1, and f be a monotone distributed routing policy. Let $\rho^\circ \in \mathcal{R}$ be such that the solution $\rho(t)$ of the dynamical flow network* (7) *with initial condition $\rho(0) = \rho^\circ$ admits a limit $\rho^* = \lim_{t\uparrow\kappa(\rho^\circ)} \rho(t)$. Let $\mathcal{B}, \mathcal{W}, \mathcal{C}, \mathcal{Z} \subseteq \mathcal{E}$ be defined as in* (24). *Then,*

1) *if $e \in \mathcal{B}$, then $e \in \mathcal{C}$, or $e \notin \mathcal{E}_\mathcal{D}^-$ and $\mathcal{E}_e^+ \subseteq \mathcal{B}$;*
2) *if $e \in \mathcal{B}$, then $e \in \mathcal{Z}_i$, or $\mathcal{E}_{\sigma_e}^+ \subseteq \mathcal{B}$;*
3) *if $e \in \mathcal{W} \setminus \mathcal{E}_\mathcal{D}^-$ and $\mathcal{E}_e^+ \subseteq \mathcal{B}$, then $e \in \mathcal{Z}_o$.*

Proof. 1) First consider the case $e \in \mathcal{E}_\mathcal{D}^-$. Then, (5) implies that, if $e \in \mathcal{B}$, then $e \in \mathcal{C}$. On the other hand, assume that $e \notin \mathcal{E}_\mathcal{D}^-$. Then, if $e \in \mathcal{B}$ and $\mathcal{E}_e^+ \nsubseteq \mathcal{B}$, necessarily $\rho^*_{\mathcal{E}_e} \in \mathcal{R}_e^\bullet$, so that property (5) implies that $e \in \mathcal{C}$.
2) Let e be such that $\mathcal{E}_{\sigma_e}^+ \nsubseteq \mathcal{B}$. Then, property (6) implies that $\lim_{t\uparrow\kappa(\rho^\circ)} f_e^{\mathrm{in}}(\rho(t)) = 0$.
3) If $e \in \mathcal{W} \setminus \mathcal{E}_\mathcal{D}^-$ and $\mathcal{E}_e^+ \subseteq \mathcal{B}$, then property (6) implies that $\lim_{t\uparrow\kappa(\rho^\circ)} f_e^{\mathrm{out}}(\rho(t)) = 0$. □

We now prove the following fundamental result that either $\mathcal{B} = \emptyset$, or there exists a cut on the origin side of which the densities hit the buffer capacities.

Lemma 4. *Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, C)$ be a network satisfying Assumption 1, and f be a monotone distributed routing policy with value λ. Let $\rho^\circ \in \mathcal{R}$ be such that the solution $\rho(t)$ of the dynamical flow network* (7) *with initial condition $\rho(0) = \rho^\circ$ admits a limit $\rho^* = \lim_{t\uparrow\kappa(\rho^\circ)} \rho(t)$. Let $\mathcal{B}, \mathcal{W}, \mathcal{C}, \mathcal{Z} \subseteq \mathcal{E}$ be defined as in* (24). *Then, either $\mathcal{E} = \mathcal{W}$, or there exists a cut $\mathcal{S}$ with $C_\mathcal{S} \leq \lambda_\mathcal{S}$ such that $\mathcal{E}_\mathcal{S}^+ \subseteq \mathcal{B}$, $\partial_\mathcal{S}^+ \subseteq \mathcal{C}$, $\partial_\mathcal{S}^- \subseteq \mathcal{Z}$, and $\mathcal{E} \setminus (\mathcal{E}_\mathcal{S}^+ \cup \partial_\mathcal{S}^-) \subseteq \mathcal{W}$.*

Proof. Existence of the limit density ρ^* implies that $\mathcal{E} = \mathcal{B} \cup \mathcal{W}$. Assume that $\mathcal{E} \neq \mathcal{W}$, and hence $\mathcal{B} \neq \emptyset$. Let $\mathcal{S} := \{v \in \mathcal{V} \setminus \mathcal{D} : \mathcal{E}_v^+ \subseteq \mathcal{B}\}$. To start with, we prove that $\mathcal{S} \neq \emptyset$. To see this, consider a link $e \in \mathcal{B}$. If also $e \in \mathcal{E}_\mathcal{D}^-$, then statement 1 of Lemma 3 implies that $e \in \mathcal{C}$, and hence $e \notin \mathcal{Z}_i$. This combined with statement 2 of Lemma 3 implies that $\mathcal{E}_{\sigma_e}^+ \subseteq \mathcal{B}$, and hence $\sigma_e \in \mathcal{S} \neq \emptyset$. On the other hand, if $e \in \mathcal{B} \setminus \mathcal{E}_\mathcal{D}^-$, then statement 1 of Lemma 3 implies that $\mathcal{E}_e^+ \subseteq \mathcal{B}$ or $e \in \mathcal{C}$. In the former case, $\tau_e \in \mathcal{S} \neq \emptyset$. In the latter case, $e \in \mathcal{C} \cap \mathcal{B}$ implies again $e \notin \mathcal{Z}_i$, so that, statement 2 of Lemma 3 yields $\mathcal{E}_{\sigma_e}^+ \subseteq \mathcal{B}$, hence $\sigma_e \in \mathcal{S} \neq \emptyset$. Hence, $\mathcal{S} \neq \emptyset$ and, since $\mathcal{S} \cap \mathcal{D} = \emptyset$ by construction, $\mathcal{S}$ is a cut. Also, by construction, $\mathcal{E}_\mathcal{S}^+ \subseteq \mathcal{B}$.

Now, it is easily seen that $\partial_\mathcal{S}^+ \subseteq \mathcal{C}$. In fact, if $e \in \partial_\mathcal{S}^+$, then $\mathcal{E}_e^+ \nsubseteq \mathcal{B}$ for otherwise one would have $\tau_e \in \mathcal{S}$ so that $e \notin \partial_\mathcal{S}^+$. Therefore, $e \in \partial_\mathcal{S}^+$ implies $\rho^*_{\mathcal{E}_e} \in \mathcal{R}_e^\bullet$. Then, property (5) implies that $e \in \mathcal{C}$.

On the other hand, for every $e \in \partial_\mathcal{S}^-$, one has $\mathcal{E}_{\sigma_e}^+ \nsubseteq \mathcal{B}$ (since $\sigma_e \notin \mathcal{S}$) and $\mathcal{E}_e^+ \subseteq \mathcal{B}$ (since $\tau_e \in \mathcal{S}$). Therefore, statement 2 of Lemma 3 implies that $\partial_\mathcal{S}^- \cap \mathcal{B} \subseteq \mathcal{Z}_\mathrm{i}$, while statement 3 of Lemma 3 implies that $\partial_\mathcal{S}^- \cap \mathcal{W} \subseteq \mathcal{Z}_\mathrm{o}$.

To show that $\mathcal{E} \setminus (\mathcal{E}_\mathcal{S}^+ \cup \partial_\mathcal{S}^-) \subseteq \mathcal{W}$, it is sufficient to prove that, for every $e \in \mathcal{B}$ with $\sigma_e \notin \mathcal{S}$, necessarily $\tau_e \in \mathcal{S}$, so that $e \in \partial_\mathcal{S}^-$. Indeed, it follows from statement 2 of Lemma 3 that $e \in \mathcal{B}$ and $\sigma_e \notin \mathcal{S}$ (i.e., $\mathcal{E}_{\sigma_e}^+ \nsubseteq \mathcal{B}$) imply that $e \in \mathcal{Z}_\mathrm{i}$, so that $e \notin \mathcal{C}$ and statement 1 of Lemma 3 implies that $\tau_e \in \mathcal{S}$.

Finally, it follows from $\mathcal{E}_\mathcal{S}^+ \subseteq \mathcal{B}$ and $\mathcal{E} \setminus (\mathcal{E}_\mathcal{S}^+ \cup \partial_\mathcal{S}^-) \subseteq \mathcal{W}$ that $\mathcal{B} = \mathcal{E}_\mathcal{S}^+ \cup \partial_\mathcal{S}^- \cap \mathcal{B}$. Then, using $\partial_\mathcal{S}^+ \subseteq \mathcal{C}$, $\partial_\mathcal{S}^- \cap \mathcal{B} \subseteq \mathcal{Z}_\mathrm{i}$, and $\partial_\mathcal{S}^- \cap \mathcal{W} \subseteq \mathcal{Z}_\mathrm{o}$, one gets that

$$\sum_{e\in\mathcal{B}} \dot\rho_e(t) = \lambda_\mathcal{S} + \sum_{e\in\partial_\mathcal{S}^-\cap\mathcal{W}} f_e^{\mathrm{out}}(t) + \sum_{e\in\partial_\mathcal{S}^-\cap\mathcal{B}} f_e^{\mathrm{in}}(t) - \sum_{e\in\partial_\mathcal{S}^+} f_e^{\mathrm{out}}(t) \overset{t\uparrow\kappa(\rho^\circ)}{\longrightarrow} \lambda_\mathcal{S} - C_\mathcal{S}\,,$$

Since $\rho_e(t) < B_e$ for $t \in [0, \kappa(\rho^\circ))$ and $\lim_{t\uparrow\kappa(\rho^\circ)} \rho_e(t) = B_e$ for all $e \in \mathcal{B}$, the above implies that $\lambda_\mathcal{S} - C_\mathcal{S} \geq 0$. □

C. Proof of Theorem 1

The results in the previous subsection assume existence of a limit density, which, in principle, is not guaranteed for every initial condition $\rho(0) = \rho^\circ \in \mathcal{R}$. However, for monotone distributed routing policies, existence of a limit density is ensured for the initial condition $\rho(0) = \mathbf{0}$. Indeed, for every $\rho^\circ \in \mathcal{R}$ and $0 \leq t < \kappa(\rho^\circ)$, let $\phi^t(\rho^\circ) = \rho(t)$ be the solution of (7) with initial condition $\rho(0) = \rho^\circ$. Then, for monotone distributed routing policies, (11) implies that $\phi^{t+s}(\mathbf{0}) = \phi^t(\phi^s(\mathbf{0})) \succeq \phi^t(\mathbf{0})$, for $0 \leq t < \kappa(\rho^\circ)$ and $0 \leq s < \kappa(\rho^\circ) - t$, i.e., $\phi^t(\mathbf{0})$ is component-wise non-decreasing and hence convergent to some limit, to be denoted, with slight abuse of notation, by $\rho^* := \lim_{t\to\kappa(\mathbf{0})} \phi^t(\mathbf{0})$.

Let $\mathcal{B}$, $\mathcal{W}$, $\mathcal{C}$, $\mathcal{Z}_\mathrm{i}$, and $\mathcal{Z}_\mathrm{o}$ be defined as in (24) for $\rho^\circ = \mathbf{0}$. First, consider the case $\max_\mathcal{U} (\lambda_\mathcal{U} - C_\mathcal{U}) < 0$. Then, Lemma 4 implies that $\mathcal{E} = \mathcal{W}$, as otherwise there would exist a cut $\mathcal{S}$ such that $C_\mathcal{S} \leq \lambda_\mathcal{S}$. Then, ρ^* is an equilibrium, i.e., $\Phi(\rho^*) = \mathbf{0}$. For an arbitrary initial condition $\rho^\circ \in \mathcal{R}$, it cannot be that $\kappa(\rho^\circ) < \infty$, as then the limit $\lim_{t\uparrow\kappa(\rho^\circ)} \phi^t(\rho^\circ) \notin \mathcal{R}$ would exist, and Lemma 4 would imply that $\lambda_\mathcal{S} \geq C_\mathcal{S}$ for some cut $\mathcal{S}$. Therefore, $\kappa(\rho^\circ) = \infty$, for all $\rho^\circ \in \mathcal{R}$. By Lemma 2, we also have $||\phi^t(\rho^\circ)-\rho^*||_1 \leq ||\rho^\circ-\rho^*||_1$, for all $t\geq 0$, so that in particular $\phi^t(\rho^\circ)$ remains bounded. If the distributed routing policy is strongly monotone, then Lemma 2 allows one to use LaSalle's theorem showing that $\lim_{t\to\infty}\phi^t(\rho^\circ)=\rho^*$ for any initial condition $\rho^\circ\in\mathcal{R}$.

Conversely, if $\rho^*\in\mathcal{R}$, then, for every cut $\mathcal{U}$, mass balance on $\mathcal{E}^+_{\mathcal{U}}$ implies that $0=\Phi(\rho^*)=\frac{\mathrm{d}}{\mathrm{d}t}\sum_{e\in\mathcal{E}^+_{\mathcal{U}}}\phi^t_e(\rho^*)=\lambda_{\mathcal{U}}-\sum_{e\in\partial^+_{\mathcal{U}}}f^{\mathrm{out}}_e(\rho^*)+\sum_{e\in\partial^-_{\mathcal{U}}}f^{\mathrm{out}}_e(\rho^*)\geq\lambda_{\mathcal{U}}-C_{\mathcal{U}}$. This proves that, if $\lambda_{\mathcal{U}}>C_{\mathcal{U}}$ for some cut $\mathcal{U}$, then necessarily $\rho^*\notin\mathcal{R}$. The same conclusion carries over if $\max_{\mathcal{U}}\{\lambda_{\mathcal{U}}-C_{\mathcal{U}}\}=0$ and the routing policy is strongly monotone, for in that case $\sum_{e\in\partial^+_{\mathcal{U}}}f^{\mathrm{out}}_e(\rho^*)<C_{\mathcal{U}}$ if $\rho^*\in\mathcal{R}$. Therefore, $\mathcal{W}\neq\mathcal{E}$, so that Lemma 4 implies (13) for $\rho^\circ=\mathbf{0}$. For arbitrary initial density $\rho^\circ\in\mathcal{R}$, consider the following two cases: $\kappa(\rho^\circ)<+\infty$ and $\kappa(\rho^\circ)=+\infty$. In the former, $\lim_{t\uparrow\kappa(\rho^\circ)}\rho(t)$ exists, hence (13) is implied by Lemma 4. In the latter, $\kappa(\mathbf{0})\geq\kappa(\rho^\circ)=\infty$, hence (13) for $\rho^\circ=\mathbf{0}$ also implies (13) for arbitrary $\rho^\circ\in\mathcal{R}$.

D. Proof of Proposition 1

Observe that, for every cut $\mathcal{U}$,

$$\sum_{e\in\mathcal{E}^+_{\mathcal{U}}}\dot\rho_e=\lambda_{\mathcal{U}}+\sum_{e\in\partial^-_{\mathcal{U}}}f^{\mathrm{out}}_e(\rho^e(t))-\sum_{e\in\partial^+_{\mathcal{U}}}f^{\mathrm{out}}_e(\rho^e(t))\geq\lambda_{\mathcal{U}}-C_{\mathcal{U}}\,,$$

so that $\sum_{e\in\mathcal{E}^+_{\mathcal{U}}}\rho_e\geq\sum_{e\in\mathcal{E}^+_{\mathcal{U}}}\rho^\circ_e+t(\lambda_{\mathcal{U}}-C_{\mathcal{U}})$, from which (14) follows. On the other hand, (15) is an immediate consequence of claim ii) of Theorem 1 and the definition of $\kappa(\rho^\circ)$.

E. Proof of Proposition 2

Let $\mathcal{U}^*$ be defined as in (16), and $\mathcal{S}$ be a cut whose existence is guaranteed by Lemma 4 for $\rho^\circ=\mathbf{0}$. The key step in proving Proposition 2 will be to show that $\mathcal{U}^*=\mathcal{S}$. That will be done in Lemma 6, after the following technical result.

Lemma 5. *For a network $\mathcal{G}=(\mathcal{V},\mathcal{E},C)$ satisfying Assumption 1, let $\mathcal{U}^*$ and $\mathcal{M}$ be as in* (16)*. Then, $\mathcal{U}^*\in\mathcal{M}$.*

Proof. We will prove that $\mathcal{U}_1\cup\mathcal{U}_2\in\mathcal{M}$ for $\mathcal{U}_1,\mathcal{U}_2\in\mathcal{M}$. For $\mathcal{A},\mathcal{H}\subseteq\mathcal{V}$, let $C^{\mathcal{A}}_{\mathcal{H}}:=\sum_{e:\sigma_e\in\mathcal{A},\tau_e\in\mathcal{H}}C_e$. It is easy to seen that

$$\lambda_{\mathcal{A}\cup\mathcal{H}}-C_{\mathcal{A}\cup\mathcal{H}}=\lambda_{\mathcal{A}}+\lambda_{\mathcal{H}\setminus\mathcal{A}}-C_{\mathcal{A}}+C^{\mathcal{A}}_{\mathcal{H}\setminus\mathcal{A}}-C^{\mathcal{H}\setminus\mathcal{A}}_{\mathcal{V}\setminus(\mathcal{A}\cup\mathcal{H})}\,. \quad (25)$$

For $\mathcal{U}_1,\mathcal{U}_2\in\mathcal{M}$, put $\mathcal{I}:=\mathcal{U}_1\cap\mathcal{U}_2$, $\mathcal{J}:=\mathcal{U}_1\cup\mathcal{U}_2$, $\mathcal{K}:=\mathcal{U}_2\setminus\mathcal{U}_1$. Observe that $\lambda_{\mathcal{J}}-C_{\mathcal{J}}\leq\lambda_{\mathcal{U}_1}-C_{\mathcal{U}_1}$ since $\mathcal{U}_1\in\mathcal{M}$. Assume by contradiction that $\lambda_{\mathcal{J}}-C_{\mathcal{J}}<\lambda_{\mathcal{U}_1}-C_{\mathcal{U}_1}=\lambda_{\mathcal{U}_2}-C_{\mathcal{U}_2}$. Then, (25) with $\mathcal{A}=\mathcal{U}_1$ and $\mathcal{H}=\mathcal{U}_2$ gives

$$\lambda_{\mathcal{U}_1}+\lambda_{\mathcal{K}}-C_{\mathcal{U}_1}+C^{\mathcal{U}_1}_{\mathcal{K}}-C^{\mathcal{K}}_{\mathcal{V}\setminus\mathcal{J}}<\lambda_{\mathcal{U}_1}-C_{\mathcal{U}_1}$$

which yields

$$\lambda_{\mathcal{K}}+C^{\mathcal{U}_1}_{\mathcal{K}}-C^{\mathcal{K}}_{\mathcal{V}\setminus\mathcal{J}}<0\,. \quad (26)$$

Similarly, applying (25) with $\mathcal{A}=\mathcal{K}$ and $\mathcal{H}=\mathcal{I}$, noting that $\mathcal{K}\cap\mathcal{I}=\emptyset$, and using $C_{\mathcal{K}}=C^{\mathcal{K}}_{\mathcal{V}\setminus\mathcal{J}}+C^{\mathcal{K}}_{\mathcal{U}_1}$ yields

$$\lambda_{\mathcal{U}_2}-C_{\mathcal{U}_2}=\lambda_{\mathcal{K}}+\lambda_{\mathcal{I}}-C^{\mathcal{K}}_{\mathcal{V}\setminus\mathcal{J}}-C^{\mathcal{K}}_{\mathcal{U}_1}+C^{\mathcal{K}}_{\mathcal{I}}-C^{\mathcal{I}}_{\mathcal{V}\setminus\mathcal{U}_2}. \quad (27)$$

Combining (27) and (26), some algebraic steps lead to

$$\begin{aligned}\lambda_{\mathcal{U}_2}-C_{\mathcal{U}_2}&<\lambda_{\mathcal{I}}-C^{\mathcal{U}_1}_{\mathcal{K}}-C^{\mathcal{K}}_{\mathcal{U}_1}+C^{\mathcal{K}}_{\mathcal{I}}-C^{\mathcal{I}}_{\mathcal{V}\setminus\mathcal{U}_2}\\&=\lambda_{\mathcal{I}}-C_{\mathcal{I}}-C^{\mathcal{U}_1\setminus\mathcal{U}_2}_{\mathcal{K}}-C^{\mathcal{K}}_{\mathcal{U}_1\setminus\mathcal{U}_2}\quad<\lambda_{\mathcal{I}}-C_{\mathcal{I}}.\end{aligned}$$

Hence, $\lambda_{\mathcal{I}}-C_{\mathcal{I}}>\lambda_{\mathcal{U}_2}-C_{\mathcal{U}_2}$, which contradicts $\mathcal{U}_2\in\mathcal{M}$. This proves that $\lambda_{\mathcal{J}}-C_{\mathcal{J}}=\lambda_{\mathcal{U}_2}-C_{\mathcal{U}_2}=\lambda_{\mathcal{U}_1}-C_{\mathcal{U}_1}$. □

Lemma 6. *Let $\mathcal{G}=(\mathcal{V},\mathcal{E},C)$ be a network satisfying Assumption 1, and f be a strongly monotone distributed routing policy with value λ such that $\max_{\mathcal{U}}\{\lambda_{\mathcal{U}}-C_{\mathcal{U}}\}\geq 0$. Let $\mathcal{U}^*$ be defined as in* (16) *and $\mathcal{B},\mathcal{W},\mathcal{C},\mathcal{Z}_o\subseteq\mathcal{E}$ be defined as in* (24) *for $\rho^\circ=\mathbf{0}$. If $\kappa(\mathbf{0})=+\infty$, then $\mathcal{E}^+_{\mathcal{U}^*}\subseteq\mathcal{B}$, $\partial^+_{\mathcal{U}^*}\subseteq\mathcal{C}$, $\partial^-_{\mathcal{U}^*}\subseteq\mathcal{Z}_o$, and $\mathcal{E}\setminus(\mathcal{E}^+_{\mathcal{U}^*}\cup\partial^-_{\mathcal{U}^*})\subseteq\mathcal{W}$.*

Proof. Let $\rho(t)$ be the solution of (7) with initial condition $\rho(0)=\mathbf{0}$. Let $\mathcal{S}:=\{v\in\mathcal{V}\setminus\mathcal{D}:\,\mathcal{E}^+_v\subseteq\mathcal{B}\}$. Observe that, as argued in Sect. IV-C, $\dot\rho_e=f^{\mathrm{in}}_e(\rho)-f^{\mathrm{out}}_e(\rho^e)\geq 0$ for all e, so that in particular $\mathcal{Z}_i\subseteq\mathcal{Z}_o$. On the other hand, Barbalat's lemma implies that $\dot\rho_e\to 0$ for $e\in\mathcal{W}$, so that $\mathcal{W}\cap\mathcal{Z}_{\mathrm{o}}\subseteq\mathcal{Z}_i$. Then, it follows from Lemma 4 that $\partial^+_{\mathcal{S}}\subseteq\mathcal{C}$, $\partial^-_{\mathcal{S}}\subseteq\mathcal{Z}_{\mathrm{o}}\cap\mathcal{Z}_{\mathrm{i}}$, and $\mathcal{E}\setminus(\mathcal{E}^+_{\mathcal{S}}\cup\partial^-_{\mathcal{S}})\subseteq\mathcal{W}$.

We start by proving that $\mathcal{S}\subseteq\mathcal{U}^*$. Define $\mathcal{H}:=\mathcal{S}\setminus\mathcal{U}^*$, $\mathcal{I}:=\partial^-_{\mathcal{H}}\cap\mathcal{E}^+_{\mathcal{S}}$, and $\mathcal{J}:=\partial^+_{\mathcal{H}}\cap\partial^+_{\mathcal{S}}$. Then,

$$0\leq\sum_{e\in\mathcal{E}^+_{\mathcal{H}}}\dot\rho_e(t)\leq\lambda_{\mathcal{H}}+\sum_{e\in\partial^-_{\mathcal{S}}}f^{\mathrm{out}}_e(t)+\sum_{i\in\mathcal{I}}f^{\mathrm{out}}_i(t)-\sum_{j\in\mathcal{J}}f^{\mathrm{out}}_j(t)\,.$$

Passing to the limit of large t, $\partial^-_{\mathcal{S}}\subseteq\mathcal{Z}_{\mathrm{o}}$ and $\partial^+_{\mathcal{S}}\subseteq\mathcal{C}$ imply $0\leq\lambda_{\mathcal{H}}+\sum_i C_i-\sum_j C_j$. Let $\hat{\mathcal{U}}:=\mathcal{S}\cup\mathcal{U}^*\supseteq\mathcal{U}^*$ and notice that $\mathcal{K}:=\partial^+_{\mathcal{H}}\setminus\partial^-_{\mathcal{U}^*}\subseteq\mathcal{J}$ and $\mathcal{I}\subseteq\partial^-_{\mathcal{H}}\cap\partial^+_{\mathcal{U}^*}=:\mathcal{L}$. Then,

$$C_{\hat{\mathcal{U}}}=C_{\mathcal{U}^*}+c_{\mathcal{K}}-c_{\mathcal{L}}\leq C_{\mathcal{U}^*}+c_{\mathcal{J}}-c_{\mathcal{I}}\leq C_{\mathcal{U}^*}+\lambda_{\mathcal{H}}\,,$$

where $c_{\mathcal{X}}:=\sum_{x\in\mathcal{X}}C_c$ for $\mathcal{X}=\mathcal{I},\mathcal{J},\mathcal{K},\mathcal{L}$. This implies that

$$\lambda_{\hat{\mathcal{U}}}-C_{\hat{\mathcal{U}}}=\lambda_{\mathcal{U}^*}+\lambda_{\mathcal{H}}-C_{\hat{\mathcal{U}}}\geq\lambda_{\mathcal{U}^*}-C_{\mathcal{U}^*}\,,$$

so that $\hat{\mathcal{U}}\in\mathcal{M}$, and then $\hat{\mathcal{U}}=\mathcal{U}^*$. Therefore, $\mathcal{S}\subseteq\mathcal{U}^*$.

We now prove that $\mathcal{U}^*\subseteq\mathcal{S}$. Assume by contradiction that $\mathcal{A}:=\mathcal{U}^*\setminus\mathcal{S}\neq\emptyset$. Let

$$\Upsilon:=\lambda_{\mathcal{A}}+\sum_{e\in\partial^-_{\mathcal{A}}\cap\partial^+_{\mathcal{S}}}C_e+\liminf_t\sum_{k\in\partial^-_{\mathcal{A}}\setminus\partial^+_{\mathcal{S}}}f^{\mathrm{out}}_k(t)-\sum_{j\in\partial^+_{\mathcal{A}}\setminus\partial^-_{\mathcal{S}}}f^{\mathrm{out}}_j(t).$$

Then, the inclusions $\partial^-_{\mathcal{S}}\subseteq\mathcal{Z}_{\mathrm{o}}\cap\mathcal{Z}_i$ and $\partial^+_{\mathcal{S}}\subseteq\mathcal{C}$ imply

$$\begin{aligned}\liminf_t\sum_{e\in\mathcal{E}^+_{\mathcal{A}}\setminus\partial^-_{\mathcal{S}}}\dot\rho_e(t)&=\liminf_t\sum_{e\in\mathcal{E}^+_{\mathcal{A}}\setminus\partial^-_{\mathcal{S}}}\left(f^{\mathrm{in}}_e(t)-f^{\mathrm{out}}_e(t)\right)\\&=\liminf_t\sum_{e\in\mathcal{E}^+_{\mathcal{A}}}\left(f^{\mathrm{in}}_e(t)-f^{\mathrm{out}}_e(t)\right)\;=\Upsilon.\end{aligned}$$

Observe that strict monotonicity implies that

$$\limsup_t f^{\mathrm{out}}_j(t)<C_j\,,\qquad\liminf_t f^{\mathrm{out}}_k(t)>0\,, \quad (28)$$

for all $j\in\mathcal{E}^+_{\mathcal{A}}\setminus\partial^-_{\mathcal{S}}$ and $k\in\partial^-_{\mathcal{A}}$. If $\partial^-_{\mathcal{A}}\setminus\partial^+_{\mathcal{S}}=\partial^+_{\mathcal{A}}\setminus\partial^-_{\mathcal{S}}=\emptyset$, then Assumption 1 implies that $\lambda_{\mathcal{A}}>0$ or $\partial^-_{\mathcal{A}}\cap\partial^+_{\mathcal{S}}\neq\emptyset$, therefore $\Upsilon=\lambda_{\mathcal{A}}+\sum_{e\in\partial^-_{\mathcal{A}}\cap\partial^+_{\mathcal{S}}}C_e>0$. On the other hand if $\partial^-_{\mathcal{A}}\setminus\partial^+_{\mathcal{S}}\neq\emptyset$ or $\partial^+_{\mathcal{A}}\setminus\partial^-_{\mathcal{S}}\neq\emptyset$, then (28) and $\mathcal{S}\subseteq\mathcal{U}^*$ imply

$$\Upsilon>\lambda_{\mathcal{A}}+\sum_{e\in\partial^-_{\mathcal{A}}\cap\partial^+_{\mathcal{S}}}C_e-\sum_{e\in\partial^+_{\mathcal{A}}\setminus\partial^-_{\mathcal{S}}}C_e=\lambda_{\mathcal{U}^*}-\lambda_{\mathcal{S}}-C_{\mathcal{U}^*}+C_{\mathcal{S}}\geq 0\,,$$

the last inequality holding since $\mathcal{U}^*\in\mathcal{M}$ by Lemma 5. In both cases, $\liminf_t\sum_{e\in\mathcal{E}^+_{\mathcal{A}}\setminus\partial^-_{\mathcal{S}}}\dot\rho_e(t)=\Upsilon>0$, which contradicts $\mathcal{E}^+_{\mathcal{A}}\setminus\partial^-_{\mathcal{S}}\subseteq\mathcal{W}$. Then, necessarily $\mathcal{A}=\emptyset$, so that $\mathcal{U}^*\subseteq\mathcal{S}$. □

We now prove Proposition 2. For all $\rho^\circ \in \mathcal{R}$, $i \in \mathcal{E}$, and $t \geq 0$, one has $\rho_i(t) \leq \sum_e \rho_e(t) \leq \sum_e \rho_e^\circ + t\sum_{v\in\mathcal{V}\setminus\mathcal{D}} \lambda_v$, so that $\kappa(\rho^\circ) = \infty$. For $\rho^\circ = \mathbf{0}$, Lemma 4 and Lemma 6 imply the the first part of (17) and (18), while the second part of (17) follows from L'Hopital's rule:

$$\lim_{t\to\infty} \frac{1}{t} \sum_{e\in\mathcal{E}^+_{\mathcal{U}^*}} \rho_e(t) = \lim_{t\to\infty} \sum_{e\in\mathcal{E}^+_{\mathcal{U}^*}} \dot\rho_e(t) = \lambda_{\mathcal{U}^*} - C_{\mathcal{U}^*}\,.$$

In particular, (18) and Barbalat's lemma imply,

$$0 = \lim_t \Phi_e(\rho(t)) = \Phi_e(B_{\mathcal{E}^+_{\mathcal{S}}\cup\partial^-_{\mathcal{S}}}, \rho^*_{\hat{\mathcal{E}}})\,, \qquad \forall e \in \hat{\mathcal{E}}\,, \quad (29)$$

where $\hat{\mathcal{E}} := \mathcal{E} \setminus (\mathcal{E}^+_{\mathcal{S}} \cup \partial^-_{\mathcal{S}})$.

For arbitrary $\rho^\circ \in \mathcal{R}$, the extension of (17) follows from Lemma 2, hence we are left with proving (18). Towards this goal, first note that $\rho^\circ \succeq \mathbf{0}$ implies, by monotonicity, that

$$\liminf_t \rho_e(t) \geq \rho_e^*\,, \qquad e \in \hat{\mathcal{E}}\,. \quad (30)$$

Consider a new network $\hat{\mathcal{G}} = (\hat{\mathcal{V}}, \hat{\mathcal{E}}, \hat{C})$ with $\hat{\mathcal{V}} := \mathcal{V} \setminus \mathcal{S}$ and $\hat{C}_e = C_e$ for $e \in \hat{\mathcal{E}}$. Let $\{\hat{f}_e\}_{e\in\hat{\mathcal{E}}}$ be a distributed routing function for $\hat{\mathcal{G}}$ with buffer capacities $\hat{B}_e = B_e$ for $e \in \hat{\mathcal{E}}$, value $\hat\lambda_{\hat v} := \lambda_{\hat v} + \sum_{e\in\mathcal{E}^-_{\hat v}\cap\partial^+_{\mathcal{S}}} C_e$ for $\hat v \in \hat{\mathcal{V}}$, and such that $\hat{f}_{e\to j}(\hat\rho^e) = f_{e\to j}(\rho^e)$ where $\rho^e \in \mathcal{R}^\bullet_e$ is such that $\rho_j = \hat\rho_j$ for all $j \in \mathcal{E}_e \cap \hat{\mathcal{E}}$, and $\rho_j = B_j$ for all $j \in \mathcal{E}_e \cap \partial^-_{\mathcal{S}}$. It is not difficult to see that $\mathcal{S} = \mathcal{U}^*$ implies $\hat\lambda_{\hat{\mathcal{U}}} < C_{\hat{\mathcal{U}}}$ for every cut $\hat{\mathcal{U}}$ in $\hat{\mathcal{G}}$, where $\hat\lambda_{\hat{\mathcal{U}}} = \sum_{\hat v\in\hat{\mathcal{U}}} \hat\lambda_{\hat v}$. Then, applying part i) of Theorem 1 to the dynamical flow network associated to $\hat{\mathcal{G}}$ and $\{\hat{f}_e\}_{e\in\hat{\mathcal{E}}}$ shows existence of a globally attractive equilibrium, $\hat\rho^* = \lim_t \hat\rho(t)$. Observe that $\dot{\hat\rho}_e(t) = \Phi_e(B_{\mathcal{E}^+_{\mathcal{S}}\cup\partial^-_{\mathcal{S}}}, \hat\rho_{\hat{\mathcal{E}}}(t))$ for $e \in \hat{\mathcal{E}}$, so that (29) implies that $\hat\rho^* = \rho^*_{\hat{\mathcal{E}}}$. Moreover, notice that the new network is a monotone controlled system [22], once we interpret the densities on $\mathcal{E}^+_{\mathcal{S}} \cup \partial^-_{\mathcal{S}}$ as inputs. Since $\rho_e(t) < B_e$ for all $e \in \mathcal{E}^+_{\mathcal{S}} \cup \partial^-_{\mathcal{S}}$ and $t \geq 0$, one gets that

$$\limsup_{t\to\infty} \rho_e(t) \leq \lim_{t\to\infty} \hat\rho_e(t) = \rho_e^*\,, \qquad \forall e \in \hat{\mathcal{E}}\,. \quad (31)$$

Combining (30) and (31) gives (18) for arbitrary $\rho^\circ \in \mathcal{R}$.

V. Conclusion

This paper studies dynamical flow networks with distributed monotone routing policies. Throughput optimality is shown by making use of a novel l_1 contraction argument for monotone dynamical systems. Applications to analysis of existing dynamic traffic models and of well-known distributed routing policies for data networks are also discussed. We also characterized the overload behavior of the network when the external inflow at the origin nodes violates capacity constraint of some cut in the network.

There are several directions of research that we plan to pursue in the future. We plan to derive appropriate conditions under which monotone routing policies can optimize secondary objectives, such as steady-state delay, without comprising throughput optimality. We also plan to formally interpret monotone routing policies as combinations of physical properties and control policies in various application domains, and utilize the characteristic properties of monotone routing policies for synthesis of appropriate control policies. Finally, we plan to extend our formalism to the multi-commodity case, possibly under partial state feedback to model urban traffic networks where the observations typically are aggregates of flows of all commodities.

Acknowledgements

The authors would like to thank Prof.s Bo Bernhardsson, Munther A. Dahleh, Emilio Frazzoli, Sanjoy Mitter, and Anders Rantzer for several valuable comments on this work.

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