

# A unifying framework for robust synchronisation of heterogeneous networks via integral quadratic constraints

Sei Zhen Khong, Enrico Lovisari, and Anders Rantzer

**Abstract**—A general framework for analysing robust synchronisation in large-scale heterogeneous networks is proposed based on the theory of integral quadratic constraints (IQCs). Dynamic agents are represented as linear time-invariant single-input-single-output systems. The agents exchange information according to a sparse dynamical interconnection operator in order to achieve synchronisation, where their outputs are steered to the same, possibly time-varying, signal. The main technical hindrance to applying IQCs in this context lies with the presence of the marginally stable dynamics which define the trajectory to which the agents’ outputs synchronise. It is shown that by working with conditions defined on modified signal spaces of interest and exploiting the graph structure underlying the connections between the dynamic systems, IQC methods can be applied directly to synchronisation analysis without recourse to loop transformations, which may obscure the inherent structural properties of the multi-agent networked systems. Decentralised and scalable conditions for synchronisation are proposed within this setting. The IQC framework is demonstrated to unify and generalise some of the existing results in the literature, including certain Nyquist-type consensus certificates for time-delay systems. Moreover, it allows the role of feedback in robustness against uncertainty to be better manifested within the context of synchronisation.

**Index Terms**—Synchronisation, consensus, heterogeneous multi-agent networks, integral quadratic constraints, distributed analysis

## I. INTRODUCTION

*Synchronisation* in large-scale interconnected networks is a ubiquitous phenomenon that takes place both in natural and engineered contexts, such as biological, energy-exchanging networks [1], clock synchronisation, and power network phase locking [2]. One of the most studied synchronisation problems is *consensus*, in which agents in a large-scale network exchange information in order to collectively reach an agreement on object of interest over time. Starting from the seminal work [3], in which the authors prove consensus for fixed directed topologies, various linear consensus algorithms have been proposed for single and double integrator multi-agent systems [4], [5]. They also find applications in a number of more complex tasks such as formation control, distributed estimation, load balancing, distributed optimisation, distributed demodulation [6], [7], [8], [9]. In the last decade there appeared a number of studies on higher order consensus networks, in which agents are represented as generic linear time-invariant (LTI) single-input-single-output (SISO) systems. These include [10], [11], which are concerned with *homogeneous* networks, and [12], [13], [14], where the *heterogeneous* case is considered.

In this paper, we examine heterogeneous networks of stable LTI SISO dynamic agents for which the objective is to synchronise the system outputs according to a persistent signal trajectory defined by poles on the imaginary axis. This setting encompasses consensus as a special case, where the synchronised trajectory is towards a constant value characterised by a simple pole at the origin. The agents

are interconnected through a possibly dynamical LTI interconnection operator corresponding to an underlying communication graph. We propose a unifying framework within which to analyse synchronisation using a well-known robustness analysis tool known as integral quadratic constraints (IQCs) [15]. Within the literature addressing the problem of synchronisation, works that employ similar techniques are [16], which exploits passivity (itself a particular type of IQC) to achieve synchronisation in constrained sets, and [17], which proposes criteria based on the incremental IQC notion of co-coerciveness for the study of synchronisation in biological networks.

The theory of integral quadratic constraints (IQCs) introduces a computationally attractive approach to encapsulating structural uncertainties of open-loop systems. It presents itself as a useful tool in closed-loop stability/performance analysis. The IQC stability conditions in [15] are applicable only to open-loop *stable* components, i.e. possessing no singularities in the closed right-half complex plane. The marginally stable dynamics corresponding to the imaginary-axis poles are therefore an impediment to the use of IQC analysis on synchronisation problems. One workaround is to employ loop transformations to the systems so as to yield a feedback interconnection whose stability implies synchronisation of the original setup [13], [14] — a related idea is exploited in [18] to study the stability of systems with rate limiters. Specifically, the work [14] considers the problem of synchronisation of heterogeneous linear time-invariant (LTI) systems perturbed by nonlinear uncertainties. The work [13] proposes a scalable consensus certificate for heterogeneous LTI systems interconnected on a possibly time-varying graph. In the case where the network interconnection matrix is normal, a certain factorisation can be exploited to transform the systems to a form to which IQC analysis is applicable to conclude higher-order consensus (multiple poles at the origin) [19].

A main theme of this paper is to establish that the theory of integral quadratic constraints (IQCs) [15] can be applied directly to the study of synchronisation of LTI systems without appealing to loop transformations to accommodate the marginally stable dynamics in the open-loop agents, or relying on specific structures of the interconnection matrices as in [19]. This input-output approach serves as an alternative to the results in the literature, which are chiefly based on the generalised Nyquist criterion. The idea involves modifying the definition of the standard frequency-domain  $L_2$  signal space with an indented integration contour that avoids the poles on the imaginary-axis and apply IQC theory to the open-loop systems, which are input-output stable when defined on the new space. The proof method differs from [15] in that graph-topological results of [20] are used to establish closed-loop well-posedness for LTI systems, thereby simplifying the IQC conditions for synchronisation. It is noted here that the proofs in this paper are based on frequency-domain methods, and hence do not naturally extend to nonlinear and time-varying systems. Nevertheless, it is hoped that the presented results open the door to future generalisations to accommodate nonlinearities.

To demonstrate the generality and utility of the proposed IQC framework, distributed synchronisation certificates for heterogeneous networks that generalise a result in [13] are proposed using ideas from the recent work [21]. These certificates are scalable with respect to

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S. Z. Khong and A. Rantzer are with the Department of Automatic Control, Lund University, SE-221 00 Lund, Sweden. seizhen@control.lth.se; rantzer@control.lth.se

E. Lovisari is with the Univ. Grenoble Alpes, GIPSA-Lab, F-38000 Grenoble, and CNRS, GIPSA-Lab, F-38000 Grenoble, and NeCS team, INRIA Grenoble Rhone-Alpes, 38330 Montbonnot, France enrico.lovisari@gipsa-lab.fr

the size of the network in the sense that the addition or removal of any agent affects only local conditions and a centralised analysis is not required to establish network synchronisation. The possibility to exploit the structure of the system, offered by the IQC analysis carried out in this paper, has already appeared in the literature in the context of stabilisation of network systems. In particular, [22], [23] investigate the notion of David-Wieland shell, a higher dimension generalisation of the concept of numerical range, and propose distributed stability certificates based on graph separation (a notion also employed in IQCs results) of the David-Wieland shells of the agents and of the interconnection operator. Similar concepts are exploited in [24], [25] in which the network is considered as bipartite (on one side, the agents, on the other, the interconnection matrix) and such a structure is employed to derive scalable stability results. Finally, we show that the IQC framework incorporates certain Nyquist contour type consensus criteria for multi-agent networks with different types of communication delays. Synchronisation under delays is a problem of great importance in the field of distributed control, and has already been addressed in specific scenarios for consensus [26] and synchronisation of Euler-Lagrange systems [27].

The paper is organised as follows. Notation is defined in the following section. Some preliminaries on linear analysis and graph theoretic concepts are also provided therein. In Section III the problem of synchronisation is formulated. The main IQC framework for analysing synchronisation is introduced in Section IV. Section V establishes the abovementioned distributed synchronisation certificates. In Section VI, the IQC conditions developed in Section IV are shown to specialise to certain Nyquist type criteria for consensus for networks with time-delay communication commonly employed in the literature. Some illustrative examples on synchronisation analysis via IQCs are given in Section VII. Finally, concluding remarks are provided in Section VIII.

## II. NOTATION AND PRELIMINARIES

### A. Matrices

Let  $\mathbb{R}$  and  $\mathbb{C}$  denote the real and complex numbers respectively.  $j\mathbb{R}$  denotes the imaginary axis,  $\mathbb{C}_+$  (resp.  $\bar{\mathbb{C}}_+$ ) the open (resp. closed) right half complex plane, and  $\|\cdot\|$  the Euclidean norm. Given an  $A \in \mathbb{C}^{m \times n}$  (resp.  $\mathbb{R}^{m \times n}$ ),  $A^* \in \mathbb{C}^{n \times m}$  (resp.  $A^T \in \mathbb{R}^{n \times m}$ ) denotes its complex conjugate transpose (resp. transpose).  $A_{ij}$  denotes the  $(i, j)$  entry of  $A$ . The  $i^{\text{th}}$  row and  $j^{\text{th}}$  column of  $A$  are denoted respectively by  $A_{i\bullet}$  and  $A_{\bullet j}$ . Given a vector  $v \in \mathbb{C}^n$ ,  $\text{diag}(v) \in \mathbb{C}^{n \times n}$  denotes the diagonal matrix whose diagonal entries are  $v_1, \dots, v_n$ . Let  $\otimes$  denote the Kronecker product and  $\oplus$  the direct sum of matrices. Define  $\bigoplus_{i=1}^n A_i := A_1 \oplus A_2 \oplus \dots \oplus A_n$ .  $I_n$  denotes the identity matrix of dimensions  $n \times n$ .

### B. Function spaces

Define the Lebesgue space

$$\mathbf{L}_\infty := \left\{ \phi : j\mathbb{R} \rightarrow \mathbb{C} \mid \|\phi\|_\infty := \sup_{\omega \in \mathbb{R}} |\phi(j\omega)| < \infty \right\}$$

and the Hardy space

$$\mathbf{H}_\infty := \left\{ \phi \in \mathbf{L}_\infty \mid \begin{array}{l} \phi \text{ has analytic continuation into } \mathbb{C}_+ \\ \text{with } \sup_{s \in \mathbb{C}_+} |\phi(s)| = \|\phi\|_\infty < \infty \end{array} \right\}.$$

Let  $\mathbf{C}$  be the class of functions continuous on  $j\mathbb{R} \cup \{\infty\}$ , and  $\mathbf{S} := \mathbf{H}_\infty \cap \mathbf{C}$ . Note that  $\mathbf{C} \subset \mathbf{L}_\infty$ . An  $H \in \mathbf{C}^{n \times n}$  is said to be Hermitian if  $H(j\omega) = H(j\omega)^*$  for all  $\omega \in \mathbb{R} \cup \{\infty\}$  and positive semidefinite if in addition,  $H(j\omega) \geq 0$  and positive definite if  $H(j\omega) \geq \gamma I_n$  for some  $\gamma > 0$ .

Given an  $\epsilon > 0$  and a point  $jq \in j\mathbb{R}$ , define the semi-circle of radius  $\epsilon$  in the right-half plane as

$$\mathcal{S}_\epsilon(jq) := \{s \in \mathbb{C} : |s - jq| = \epsilon, \Re(s) > 0\}$$

and  $\mathcal{S}_0(jq) := \emptyset$ . Given a finite ordered set  $j\mathcal{Q} = \{jq_1, jq_2, \dots, jq_K\} \subset j\mathbb{R}$  with  $q_1 > q_2 > \dots > q_K$ , define a contour parameterised by  $\epsilon \geq 0$  as

$$\begin{aligned} \mathcal{C}_\epsilon(j\mathcal{Q}) := & j[q_1 + \epsilon, \infty) \cup \mathcal{S}_\epsilon(jq_1) \cup j[q_2 + \epsilon, q_1 - \epsilon] \\ & \cup \mathcal{S}_\epsilon(jq_2) \cup j[q_3 + \epsilon, q_2 - \epsilon] \\ & \vdots \\ & \cup \mathcal{S}_\epsilon(jq_K) \cup j(-\infty, q_K - \epsilon]. \end{aligned}$$

that is, a straight line on the imaginary axis indented to the right of every point in  $j\mathcal{Q}$  by a semi-circle of radius  $\epsilon$ . In particular, notice that  $\mathcal{C}_0(j\mathcal{Q}) = j\mathbb{R}$  for any  $j\mathcal{Q} \subset j\mathbb{R}$ . Denote by  $\mathcal{C}_\epsilon^+(j\mathcal{Q})$  the open half plane that lies to the right of  $\mathcal{C}_\epsilon(j\mathcal{Q})$ , i.e.

$$\mathcal{C}_\epsilon^+(j\mathcal{Q}) := \{s = \sigma + j\omega \in \mathbb{C} \mid \bar{\sigma} + j\omega \in \mathcal{C}_\epsilon(j\mathcal{Q}) \implies \sigma > \bar{\sigma}\},$$

and  $\bar{\mathcal{C}}_\epsilon^+(j\mathcal{Q})$  its closure. Let  $\mathbf{C}_\epsilon(j\mathcal{Q})$  be the class of functions continuous on  $\mathcal{C}_\epsilon(j\mathcal{Q}) \cup \{\infty\}$ . Given  $X \in \mathbf{C}_\epsilon(j\mathcal{Q})^{n \times m}$ , define  $\|X\|_{\mathbf{C}_\epsilon(j\mathcal{Q})} := \sup_{s \in \mathcal{C}_\epsilon(j\mathcal{Q})} \bar{\sigma}(X(s))$ , where  $\bar{\sigma}(\cdot)$  denotes the maximum singular value. An  $H \in \mathbf{C}_\epsilon(j\mathcal{Q})^{n \times n}$  is said to be Hermitian if  $H(s) = H(s)^*$  for all  $s \in \mathcal{C}_\epsilon(j\mathcal{Q}) \cup \{\infty\}$ . Let  $\mathbf{S}_\epsilon(j\mathcal{Q})$  be the subclass of  $\mathbf{C}_\epsilon(j\mathcal{Q})$  containing functions that have analytic continuation into  $\mathcal{C}_\epsilon^+(j\mathcal{Q})$ . Note that  $\mathbf{S} \subset \mathbf{S}_\epsilon(j\mathcal{Q})$  for all  $\epsilon \geq 0$ .

Let the Lebesgue space  $\mathbf{L}_2^n$  denote the class of functions  $f : [0, \infty) \rightarrow \mathbb{R}^n$  with finite energy, i.e. square-integrable, satisfying  $\|f\|_2^2 := \int_0^\infty |f(t)|^2 dt < \infty$ . The Fourier transform of  $f \in \mathbf{L}_2^n$  is denoted  $\hat{f}(j\omega) := \int_0^\infty e^{-j\omega t} f(t) dt$ . Note that  $\|\hat{f}\|_2 = \|f\|_2$  and  $\hat{f}$  has analytic continuation into  $\mathbb{C}_+$  and  $\sup_{\sigma > 0} \|\hat{f}(\sigma + \cdot)\|_2 = \|\hat{f}\|_2 < \infty$ . The set of Fourier transforms of functions in  $\mathbf{L}_2^n$  is denoted  $\mathbf{H}_2^n$ . A linear operator mapping between Banach spaces  $X : \mathcal{X} \rightarrow \mathcal{Y}$  is said to be bounded if the induced norm

$$\|X\|_{\mathcal{X} \rightarrow \mathcal{Y}} := \sup_{f \in \mathcal{X} : \|f\|_{\mathcal{X}} = 1} \|Xf\|_{\mathcal{Y}} < \infty.$$

Note that multiplication by a transfer function in  $\mathbf{S}$  as an operator on  $\mathbf{H}_2$  defines a corresponding causal and bounded LTI operator on  $\mathbf{L}_2$  in the time domain via the Laplace transform isomorphism [28].

For  $\epsilon \geq 0$  and finite subset  $j\mathcal{Q} \subset j\mathbb{R}$ , define  $\mathbf{H}_{2\epsilon}^n(j\mathcal{Q})$  to be the set of functions  $\hat{f} : \bar{\mathcal{C}}_\epsilon(j\mathcal{Q}) \rightarrow \mathbb{C}^n$  that are analytic on  $\mathcal{C}_\epsilon^+(j\mathcal{Q})$  and square-integrable on  $\mathcal{C}_\epsilon(j\mathcal{Q})$ , i.e.  $\|\hat{f}\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}^2 := \int_{\mathcal{C}_\epsilon(j\mathcal{Q})} |\hat{f}(s)|^2 ds < \infty$ . For notational simplicity, the spatial dimension  $n$  and the set of imaginary-axis poles ( $j\mathcal{Q}$ ) are often dropped from  $\mathbf{H}_{2\epsilon}^n(j\mathcal{Q})$ . Note that  $\mathbf{H}_2^n = \mathbf{H}_{2\epsilon}^n(j\mathcal{Q})$  when  $\epsilon = 0$ . Moreover, for all  $\epsilon \geq 0$ , multiplication by a transfer function  $X \in \mathbf{S}_\epsilon(j\mathcal{Q})$  defines a bounded operator on  $\mathbf{H}_{2\epsilon}$  with its induced norm equals to  $\|X\|_{\mathbf{C}_\epsilon(j\mathcal{Q})}$ .  $\mathbf{H}_{2\epsilon}$  is a Hilbert space with inner product  $\langle u, v \rangle_{\mathcal{C}_\epsilon(j\mathcal{Q})} := \int_{\mathcal{C}_\epsilon(j\mathcal{Q})} u(s)^* v(s) ds$ . It can be seen that multiplication by an  $X \in \mathbf{S}$  is bounded on  $\mathbf{H}_{2\epsilon}$  for all  $\epsilon \geq 0$ . On the other hand, given a  $q \in \mathbb{R}$ , multiplication by  $\frac{1}{s-jq}$  is bounded on  $\mathbf{H}_{2\epsilon}(\{jq\})$  for all  $\epsilon > 0$  but not on  $\mathbf{H}_2$ . In the following, we will not notationally distinguish between a transfer function and its associated multiplication operator. For instance, an  $X \in \mathbf{S}$  defines a bounded operator  $X : \mathbf{H}_{2\epsilon} \rightarrow \mathbf{H}_{2\epsilon}$  for all  $\epsilon \geq 0$ .

Given an  $\epsilon \geq 0$  and  $X \in \mathbf{C}_\epsilon(j\mathcal{Q})^{n \times m}$ , define the graph of the operator  $X : \mathbf{H}_{2\epsilon}^n(j\mathcal{Q}) \rightarrow \mathbf{H}_{2\epsilon}^m(j\mathcal{Q})$  to be

$$\mathcal{G}_\epsilon(X) := \begin{bmatrix} I_m \\ X \end{bmatrix} \mathbf{H}_{2\epsilon}^m(j\mathcal{Q}) = \left\{ \begin{bmatrix} u \\ y \end{bmatrix} \in \mathbf{H}_{2\epsilon}^{n+m}(j\mathcal{Q}) : y = Xu \right\}.$$

Similarly, define the (inverse) graph

$$\mathcal{G}'_\epsilon(X) := \begin{bmatrix} X \\ I_m \end{bmatrix} \mathbf{H}_{2\epsilon}^m(j\mathcal{Q}) = \left\{ \begin{bmatrix} u \\ y \end{bmatrix} \in \mathbf{H}_{2\epsilon}^{n+m}(j\mathcal{Q}) : u = Xy \right\}.$$

### C. Graph theory

A graph is denoted by  $\mathcal{G} = (V, E)$ , where  $V = \{v_1, \dots, v_n\}$  is the set of nodes and  $E \subset V \times V$ ,  $E = \{e_1, \dots, e_m\}$  is the set of edges such that  $e_k = \{v_i, v_j\} \in E$  if node  $v_j$  can receive information from node  $v_i$ . For the edge  $\{v_i, v_j\}$ ,  $v_i$  is called the *parent node* and  $v_j$  the *child node*. A graph is *undirected* if  $\{v_i, v_j\} \in E$  then  $\{v_j, v_i\} \in E$ . A path on  $\mathcal{G}$  of length  $N$  is an ordered set of distinct vertices  $\{v_0, v_1, \dots, v_N\}$  such that  $\{v_i, v_{i+1}\} \in E$  for all  $i \in \{0, 1, \dots, N-1\}$ . An undirected graph is said to be *connected* if any two nodes in  $V$  are connected by a path. The adjacency matrix  $A = [A_{ij}] \in \mathbb{R}^{n \times n}$  is defined by  $A_{ij} = 1$  if  $\{v_j, v_i\} \in E$  and  $A_{ij} = 0$  otherwise. Note that  $A$  is symmetric for an undirected graph. A *directed tree* is a directed graph in which every node has exactly one parent except for one node, called the *root*, which has no parent and which has a directed path to every other node. A *subgraph*  $(V_s, E_s)$  of  $(V, E)$  is a graph such that  $V_s \subset V$  and  $E_s \subset E \cap (V_s \times V_s)$ . A (*rooted*) *directed spanning tree*  $(V_s, E_s)$  of the directed graph  $(V, E)$  is a subgraph such that  $(V_s, E_s)$  is a directed tree and  $V_s = V_p$ .

In an undirected graph, let the neighbours of node  $v_i \in V$  be defined as  $N_i := \{v_j \in V : \{v_i, v_j\} \in E\}$  and denote its degree by  $|N_i|$ . The graph Laplacian is defined as  $L := \text{diag}(|N_i|) - A$ .  $L$  has a zero eigenvalue corresponding to the vector of ones  $\mathbf{1}_n \in \mathbb{R}^n$ . The multiplicity of the zero eigenvalue is one if the graph is connected [29]. The Laplacian matrix can be factorised as  $L = DD^T$ , where  $D = [D_{ik}] \in \mathbb{R}^{n \times m}$  is the oriented incidence matrix. It is defined by associating an orientation to every edge of the graph: for each  $e_k = \{v_i, v_j\} = \{v_j, v_i\}$ , one of  $v_i, v_j$  is defined to be the head and the other tail of the edge:

$$D_{ik} := \begin{cases} +1 & \text{if } v_i \text{ is the head of } e_k \\ -1 & \text{if } v_i \text{ is the tail of } e_k \\ 0 & \text{otherwise.} \end{cases}$$

Note that the Laplacian matrix is invariant to the choice of orientation. Define also the unoriented incidence matrix  $\bar{D} \in \mathbb{R}^{n \times m}$  whose entries are the absolute value of those of  $D$ .

### III. SYNCHRONISATION PROBLEM FORMULATION

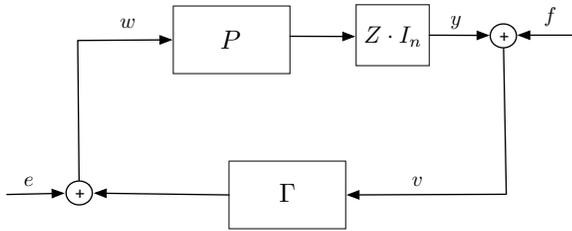


Fig. 1. Feedback setup for synchronization.

Consider the feedback interconnection in Figure 1. There,  $P := \bigoplus_{i=1}^n P_i = \text{diag}(P_i)$  with the SISO dynamical agents  $P_i \in \mathbf{S}$  and  $\Gamma \in \mathbf{S}^{n \times n}$  denotes the interconnection matrix.  $Z$  is a SISO proper rational transfer function that has a finite number of poles on  $j\mathbb{R}$ . Throughout the paper,  $j\mathcal{Q} = \{jq_1, jq_2, \dots, jq_K\}$  is used to denote the set of poles of  $Z$  on the imaginary axis. These poles/modes describe the trajectory of the output signal  $y$  under synchronization. The interactions between the agents is determined by an underlying graph  $\mathcal{G} = (V, E)$  containing a directed spanning tree, where each

node  $v_i \in V$  is associated with a corresponding  $P_i$  and the edges describe the communication/connections between the agents. Figure 1 models the problem of synchronization of a network of heterogeneous agents interconnected through a dynamic matrix. In subsequent parts of the paper,  $\mathcal{G}$  is often taken to be a connected undirected graph when special cases are considered.

The following standing assumption is made throughout the paper.  
*Assumption 3.1:* For every  $jq \in j\mathcal{Q}$ ,

$$\lim_{s \rightarrow jq} \frac{1}{(s - jq)^{m_q - 1}} \Gamma(s) \mathbf{1}_n = 0 \quad \text{and} \quad \lim_{s \rightarrow jq} \frac{1}{(s - jq)^{m_q}} \Gamma(s) = \infty,$$

where  $m_q$  denotes the multiplicity of the pole  $jq$  of  $Z$ . Furthermore, there exists no  $x \notin \text{span}\{\mathbf{1}_n\}$ , such that  $\Gamma(jq)x = 0$ . In other words,  $\det(\Gamma(s))$  has a zero at every  $s = jq \in j\mathcal{Q}$  of multiplicity  $m_q$  corresponding to the null space  $\text{span}\{\mathbf{1}_n\}$ .

For instance, in the case where  $Z$  has non-repeated poles on  $j\mathbb{R}$ ,  $\Gamma$  can be set to  $L$ , the graph Laplacian matrix for a connected undirected graph  $\mathcal{G}$ . Dynamics can be included via the expression  $\Gamma = D \text{diag}(\Gamma_i) D^T$ , where  $D$  denotes the incidence matrix and  $\Gamma_i \in \mathbf{S}$  for  $i = 1, \dots, m$ ; see Figure 2. This models a heterogeneous network configuration of agents interconnected via dynamically weighted matrices. Note that for both cases  $\Gamma$  satisfies Assumption 3.1 by the connectedness of the graph  $\mathcal{G}$ .

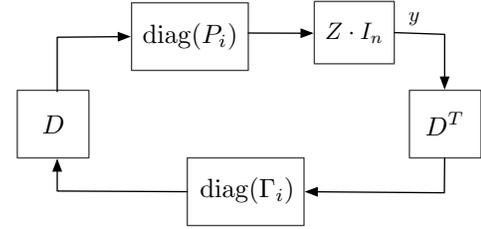


Fig. 2. A synchronization setup with dynamical interconnection matrix.

*Definition 3.2:* The interconnection in Figure 1 is said to reach synchronization if

$$|y_i(t) - y_j(t)| \rightarrow 0 \quad \text{as } t \rightarrow \infty$$

for all  $i, j \in \{1, 2, \dots, n\}$  and  $e, f \in \mathbf{L}_2$ .

In other words,  $y(t)$  converges to the subspace spanned by  $\mathbf{1}_n$ , i.e.  $\text{span}\{\mathbf{1}_n\}$ . This means the output  $y_i$  of each of the agent  $P_i$  synchronises to the same trajectory defined by the imaginary-axis poles of  $Z$ .

*Remark 3.3:* If  $Z(s) = 1$ , one recovers the standard setup of feedback interconnection, whereby synchronization in the definition above corresponds to feedback stability. By defining  $Z(s) := \frac{1}{s}$ , one recovers the standard consensus problem where all  $y_i$ 's are to asymptotically converge to the same constant value. By contrast, if  $Z(s) := \frac{\omega_0}{s^2 + \omega_0^2}$  and synchronization takes place, then each  $y_i$  will converge to a sinusoid of frequency  $\omega_0$  and the same phase/magnitude. Another example is  $Z(s) := \frac{1}{s^2}$ , where the system outputs synchronise to a ramp function.

### IV. INTEGRAL QUADRATIC CONSTRAINT BASED ANALYSIS OF SYNCHRONISATION

This section introduces a unified framework within which to analyse the problem of synchronization using integral quadratic constraints (IQCs) [15]. To this end, some results from robustness of closed-loop interconnections are needed and provided next.

#### A. Feedback robustness

*Definition 4.1:* Given  $\epsilon \geq 0$ ,  $\Delta : \mathbf{H}_{2\epsilon}^n(j\mathcal{Q}) \rightarrow \mathbf{H}_{2\epsilon}^m(j\mathcal{Q})$  and  $G : \mathbf{H}_{2\epsilon}^m(j\mathcal{Q}) \rightarrow \mathbf{H}_{2\epsilon}^n(j\mathcal{Q})$ , the feedback interconnection of  $\Delta$  and

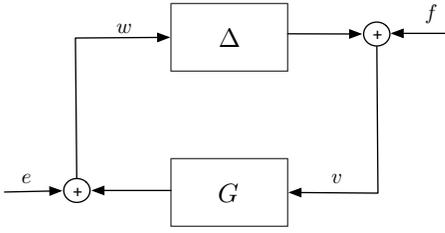


Fig. 3. Standard feedback configuration.

$G$  in Figure 3, denoted  $[\Delta, G]$ :

$$\begin{cases} v = \Delta w + f \\ w = Gv + e \end{cases} \quad (1)$$

is said to be  $\mathbf{H}_{2\epsilon}$ -stable if the map  $(v, w) \mapsto (f, e)$  has a bounded inverse on  $\mathbf{H}_{2\epsilon}^{2n}$ .

Given an  $\mathbf{H}_{2\epsilon}$ -stable  $[\Delta, G]$ , define the generalised robustness margin with the ambient space taken to be  $\mathbf{H}_{2\epsilon}(j\mathcal{Q})$  by

$$b_{\Delta, G} := \inf_{v \in \mathcal{G}_\epsilon(\Delta), w \in \mathcal{G}'_\epsilon(G)} \frac{\|v + w\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}}{\|v\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}}. \quad (2)$$

Furthermore, given two systems  $\Delta_1 : \mathbf{H}_{2\epsilon}^n(j\mathcal{Q}) \rightarrow \mathbf{H}_{2\epsilon}^m(j\mathcal{Q})$  and  $\Delta_2 : \mathbf{H}_{2\epsilon}^n(j\mathcal{Q}) \rightarrow \mathbf{H}_{2\epsilon}^m(j\mathcal{Q})$ , define the generalised gap metric as follows:

$$\begin{aligned} \delta(\Delta_1, \Delta_2) &:= \|\Pi_{\mathcal{G}_\epsilon(\Delta_1)} - \Pi_{\mathcal{G}_\epsilon(\Delta_2)}\|_{\mathcal{C}_\epsilon(j\mathcal{Q})} \\ &= \max \left\{ \bar{\delta}(\Delta_1, \Delta_2), \bar{\delta}(\Delta_2, \Delta_1) \right\}, \end{aligned} \quad (3)$$

where the directed gap

$$\begin{aligned} \bar{\delta}(\Delta_k, \Delta_l) &:= \gamma \left( \Pi_{(\mathcal{G}_\epsilon(\Delta_l))^\perp} \Pi_{\mathcal{G}_\epsilon(\Delta_k)} \right) \\ &= \sup_{x_k \in \mathcal{G}_\epsilon(\Delta_k)} \inf_{x_l \in \mathcal{G}_\epsilon(\Delta_l)} \frac{\|x_k - x_l\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}}{\|x_k\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}}. \end{aligned} \quad (4)$$

See [20] for the original definitions of the robustness margin and gap metric with respect to the ambient space  $\mathbf{H}_2$ .

*Proposition 4.2:* Suppose  $[\Delta_1, G]$  is  $\mathbf{H}_{2\epsilon}$ -stable with  $b_{\Delta_1, G} > \delta(\Delta_1, \Delta_2)$ , then  $[\Delta_2, G]$  is  $\mathbf{H}_{2\epsilon}$ -stable.

*Proof:* The claim can be established following the arguments in [20, Thm. 3] or [30, Prop. III.1], where the result is proven with respect to the ambient space  $\mathbf{H}_2$ . ■

The following lemma is used to establish the main IQC result in the next subsection.

*Lemma 4.3:* Given  $\Delta \in \mathbf{S}_\epsilon(j\mathcal{Q})^{n \times m}$ , the mapping

$$\lambda \in [0, 1] \mapsto \Delta_\lambda := \lambda \Delta \in \mathbf{S}_\epsilon(j\mathcal{Q})^{n \times m} \quad (5)$$

is continuous with respect to gap metric  $\delta(\cdot, \cdot)$ .

*Proof:* Observe that for any  $\lambda_0, \lambda_1 \in [0, 1]$ , (4) gives

$$\begin{aligned} &\bar{\delta}(\Delta_{\lambda_0}, \Delta_{\lambda_1}) \\ &= \sup_{\substack{x_1 \in \mathcal{G}_\epsilon(\lambda_1 \Delta) \\ \|x_1\|_{\mathcal{C}_\epsilon(j\mathcal{Q})} \neq 0}} \inf_{\substack{x_0 \in \mathcal{G}_\epsilon(\lambda_0 \Delta) \\ \|x_0\|_{\mathcal{C}_\epsilon(j\mathcal{Q})} \neq 0}} \frac{\|x_1 - x_0\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}}{\|x_1\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}} \\ &= \sup_{\substack{\begin{bmatrix} u_1 \\ y_1 \end{bmatrix} \in \mathcal{G}_\epsilon(\Delta) \\ \left\| \begin{bmatrix} u_1 \\ y_1 \end{bmatrix} \right\|_{\mathcal{C}_\epsilon(j\mathcal{Q})} \neq 0}} \inf_{\substack{\begin{bmatrix} u_0 \\ y_0 \end{bmatrix} \in \mathcal{G}_\epsilon(\Delta) \\ \left\| \begin{bmatrix} u_0 \\ y_0 \end{bmatrix} \right\|_{\mathcal{C}_\epsilon(j\mathcal{Q})} \neq 0}} \frac{\left\| \begin{bmatrix} \lambda_1 y_1 \\ \lambda_1 y_1 \end{bmatrix} - \begin{bmatrix} \lambda_0 y_0 \\ \lambda_0 y_0 \end{bmatrix} \right\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}}{\left\| \begin{bmatrix} u_1 \\ y_1 \end{bmatrix} \right\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}} \\ &= \sup_{\substack{\begin{bmatrix} u_1 \\ y_1 \end{bmatrix} \in \mathcal{G}_\epsilon(\Delta) \\ \left\| \begin{bmatrix} u_1 \\ y_1 \end{bmatrix} \right\|_{\mathcal{C}_\epsilon(j\mathcal{Q})} \neq 0}} \inf_{\substack{\begin{bmatrix} u_0 \\ y_0 \end{bmatrix} \in \mathcal{G}_\epsilon(\Delta) \\ \left\| \begin{bmatrix} u_0 \\ y_0 \end{bmatrix} \right\|_{\mathcal{C}_\epsilon(j\mathcal{Q})} \neq 0}} \left( \frac{\|\lambda_1 y_1 - \lambda_0 y_0\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}^2 + \|u_1 - u_0\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}^2}{\|\lambda_1 y_1\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}^2 + \|u_1\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}^2} \right)^{\frac{1}{2}} \\ &\leq \sup_{\substack{\begin{bmatrix} u_1 \\ y_1 \end{bmatrix} \in \mathcal{G}_\epsilon(\Delta), \left\| \begin{bmatrix} u_1 \\ y_1 \end{bmatrix} \right\|_{\mathcal{C}_\epsilon(j\mathcal{Q})} \neq 0}} \left( \frac{\|\lambda_1 y_1 - \lambda_0 y_1\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}^2}{\|\lambda_1 y_1\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}^2 + \|u_1\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}^2} \right)^{\frac{1}{2}} \\ &\leq \sup_{\substack{\begin{bmatrix} u_1 \\ y_1 \end{bmatrix} \in \mathcal{G}_\epsilon(\Delta), \|u_1\|_{\mathcal{C}_\epsilon(j\mathcal{Q})} \neq 0}} |\lambda_1 - \lambda_0| \frac{\|y_1\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}}{\|u_1\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}} \\ &= |\lambda_1 - \lambda_0| \|\Delta\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}, \end{aligned}$$

where the first inequality follows from setting  $\begin{bmatrix} y_0 \\ u_0 \end{bmatrix} = \begin{bmatrix} y_1 \\ u_1 \end{bmatrix}$ . Since this holds for any  $\lambda_0, \lambda_1 \in [0, 1]$ , it follows from (3) that

$$\delta(\Delta_{\lambda_0}, \Delta_{\lambda_1}) \leq |\lambda_1 - \lambda_0| \|\Delta\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}$$

for any  $\tau \in \mathbb{R}$ ,  $\lambda_0, \lambda_1 \in [0, 1]$ . Continuity of the mapping (5) with respect to  $\delta(\cdot, \cdot)$  thus follows from the inequality above. ■

### B. IQC conditions for synchronisation

Recall that  $j\mathcal{Q} = \{jq_1, jq_2, \dots, jq_K\}$  is the finite set of poles on  $j\mathbb{R}$  of  $Z$  and the shorthand notation  $ZP$  is used to denote  $(Z \cdot I_n)P$  in the synchronisation setup of Figure 1. First, an IQC result on the generalised  $\mathbf{H}_{2\epsilon}$  feedback stability is established below.

*Theorem 4.4:* Given  $\epsilon > 0$ ,  $\Delta \in \mathbf{S}_\epsilon(j\mathcal{Q})^{n \times m}$  and  $G \in \mathbf{S}_\epsilon(j\mathcal{Q})^{m \times n}$ , the feedback interconnection of  $\Delta : \mathbf{H}_{2\epsilon}^n(j\mathcal{Q}) \rightarrow \mathbf{H}_{2\epsilon}^m(j\mathcal{Q})$  and  $G : \mathbf{H}_{2\epsilon}^m(j\mathcal{Q}) \rightarrow \mathbf{H}_{2\epsilon}^n(j\mathcal{Q})$  in Figure 3 is  $\mathbf{H}_{2\epsilon}$ -stable if there exists a Hermitian  $\Pi \in \mathbf{C}_\epsilon(j\mathcal{Q})^{(n+m) \times (n+m)}$  such that the following complementary IQC conditions hold:

- (i)  $\langle v, \Pi v \rangle_{\mathcal{C}_\epsilon(j\mathcal{Q})} \geq 0$  for all  $v \in \mathcal{G}_\epsilon(\Delta)$ ;
- (ii) there exists a  $\gamma > 0$  for which  $\langle w, \Pi w \rangle_{\mathcal{C}_\epsilon(j\mathcal{Q})} \leq -\gamma \|w\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}^2$  for all  $w \in \mathcal{G}'_\epsilon(\tau G)$  and  $\tau \in [0, 1]$ .

*Proof:* Mimicking an argument in the proof of [31, Lem. 5.1], let  $\Psi := 2\Pi + \gamma I$ , the IQC conditions thus become

$$\langle v, \Psi v \rangle_{\mathcal{C}_\epsilon(j\mathcal{Q})} \geq \gamma \|v\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}^2 \quad \forall v \in \mathcal{G}_\epsilon(\Delta)$$

and

$$\langle w, \Psi w \rangle_{\mathcal{C}_\epsilon(j\mathcal{Q})} \leq -\gamma \|w\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}^2 \quad \forall w \in \mathcal{G}'_\epsilon(\tau G), \tau \in [0, 1].$$

It follows that for any  $v \in \mathcal{G}_\epsilon(\Delta)$ ,  $w \in \mathcal{G}'_\epsilon(\tau G)$  and  $\tau \in [0, 1]$ ,

$$\begin{aligned} &\gamma (\|v\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}^2 + \|w\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}^2) \\ &\leq \langle v, \Psi v \rangle_{\mathcal{C}_\epsilon(j\mathcal{Q})} - \langle w, \Psi w \rangle_{\mathcal{C}_\epsilon(j\mathcal{Q})} \\ &= \langle v + w, \Psi(v + w) \rangle_{\mathcal{C}_\epsilon(j\mathcal{Q})} - 2 \langle w, \Psi(v + w) \rangle_{\mathcal{C}_\epsilon(j\mathcal{Q})} \\ &\leq \|\Psi\|_{\mathcal{C}_\epsilon(j\mathcal{Q})} \|v + w\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}^2 \\ &\quad + 2 \|\Psi\|_{\mathcal{C}_\epsilon(j\mathcal{Q})} \|w\|_{\mathcal{C}_\epsilon(j\mathcal{Q})} \|v + w\|_{\mathcal{C}_\epsilon(j\mathcal{Q})} \\ &\leq \|\Psi\|_{\mathcal{C}_\epsilon(j\mathcal{Q})} \|v + w\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}^2 + \frac{2 \|\Psi\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}^2 \|v + w\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}^2}{\gamma} \\ &\quad + \frac{\gamma}{2} \|w\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}^2, \end{aligned}$$

where the last inequality holds since  $2xy \leq \frac{x^2}{\beta} + \beta y^2$  for any  $x, y, \beta \in \mathbb{R}$ . This implies

$$\begin{aligned} \left(1 + \frac{2}{\gamma} \|\Psi\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}\right) \|\Psi\|_{\mathcal{C}_\epsilon(j\mathcal{Q})} \|v + w\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}^2 & \quad (6) \\ & \geq \gamma \|v\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}^2 + \frac{\gamma}{2} \|w\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}^2 \\ & \geq \frac{\gamma}{2} \|w\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}^2 \end{aligned}$$

$$\implies \|v + w\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}^2 \geq \frac{\gamma}{2} \|w\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}^2, \quad (7)$$

for any positive  $\eta \leq \frac{\gamma}{\sqrt{2\|\Psi\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}(\gamma+2\|\Psi\|_{\mathcal{C}_\epsilon(j\mathcal{Q})})}}$ .

Now observe that  $\tau \in [0, 1] \mapsto \tau G$  is continuous in the graph topology induced by the gap metric by Lemma 4.3. Since the feedback interconnection  $[\Delta, \tau G]$  is  $\mathbf{H}_{2\epsilon}$ -stable for  $\tau = 0$ , inequality (7) implies that the corresponding robust stability margin  $b_{\Delta, 0} \geq \eta > 0$ ; see (2). By continuity in the graph topology, there exists an  $\zeta > 0$  such that  $\delta(hG, (h + \tau)G) < \eta$  for all  $\tau \in [0, \zeta]$  and  $h \in [0, 1 - \zeta]$ . Application of Proposition 4.2 then leads to the feedback interconnection of  $\Delta$  and  $\tau G$  being  $\mathbf{H}_{2\epsilon}$ -stable for  $\tau \in [0, \zeta]$ . By (7), it follows again that  $b_{\Delta, \zeta G} \geq \eta > 0$ . Repetitively applying the aforementioned arguments yields stability of the feedback interconnection  $[\Delta, \tau G]$  for  $\tau \in [\zeta, 2\zeta], [2\zeta, 3\zeta], \dots$  in succession, and eventually for  $\tau = 1$ , as required. ■

The main IQC-based result on synchronisation is stated next.

*Theorem 4.5:* The feedback configuration in Figure 1, where  $Z$  is a proper rational scalar transfer function analytic with imaginary-axis poles in  $j\mathcal{Q}$ ,  $P := \bigoplus_{i=1}^n P_i : P_i \in \mathbf{S}; P_i(j\omega) \neq 0 \forall j\omega \in j\mathcal{Q}$ , and  $\Gamma \in \mathbf{S}^{n \times n}$  satisfies Assumption 3.1 reaches synchronisation if there exists a Hermitian  $\Pi \in \mathbf{C}^{2n \times 2n}$  such that for all  $\omega \in \mathbb{R} \setminus \mathcal{Q} = (q_1, \infty) \cup (q_2, q_1) \cup \dots \cup (q_K, q_{K-1}) \cup (-\infty, q_K)$ ,

- (i)  $\begin{bmatrix} I_n \\ Z(j\omega)P(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} I_n \\ Z(j\omega)P(j\omega) \end{bmatrix} \geq 0$ ;
- (ii)  $\begin{bmatrix} \tau\Gamma(j\omega) \\ I_n \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \tau\Gamma(j\omega) \\ I_n \end{bmatrix} \leq -\gamma \forall \tau \in [0, 1]$ , where  $\gamma$  is some strictly positive constant.

*Proof:* Let  $\Psi := 2\Pi + \gamma I$ . The quadratic inequalities above can thus be restated as

$$\begin{aligned} \begin{bmatrix} I_n \\ Z(j\omega)P(j\omega) \end{bmatrix}^* \Psi(j\omega) \begin{bmatrix} I_n \\ Z(j\omega)P(j\omega) \end{bmatrix} & \geq \gamma \\ \begin{bmatrix} \tau\Gamma(j\omega) \\ I_n \end{bmatrix}^* \Psi(j\omega) \begin{bmatrix} \tau\Gamma(j\omega) \\ I_n \end{bmatrix} & \leq -\gamma, \end{aligned}$$

for all  $\omega \in \mathbb{R} \setminus \mathcal{Q}$  and  $\tau \in [0, 1]$ . Given  $\epsilon > 0$ , define  $\bar{\Psi}_\epsilon \in \mathbf{C}_\epsilon(j\mathcal{Q})$  by  $\bar{\Psi}_\epsilon(s) := \Psi(j\omega)$  for  $s = \sigma + j\omega \in \mathcal{C}_\epsilon(j\mathcal{Q})$ . Since  $Z, P$ , and  $\Gamma$  are analytic on  $\mathbb{C}_+$ , it follows that there exists a sufficiently small  $\epsilon^* > 0$  such that

$$\begin{aligned} \begin{bmatrix} I_n \\ Z(s)P(s) \end{bmatrix}^* \bar{\Psi}_\epsilon(s) \begin{bmatrix} I_n \\ Z(s)P(s) \end{bmatrix} & \geq \frac{\gamma}{2}; \\ \begin{bmatrix} \tau\Gamma(s) \\ I_n \end{bmatrix}^* \bar{\Psi}_\epsilon(s) \begin{bmatrix} \tau\Gamma(s) \\ I_n \end{bmatrix} & \leq -\frac{\gamma}{2} \end{aligned}$$

for all  $s \in \mathcal{C}_\epsilon(j\mathcal{Q})$ ,  $\tau \in [0, 1]$ , and  $0 < \epsilon \leq \epsilon^*$ . These imply that  $\langle v, \bar{\Psi}_\epsilon v \rangle_{\mathcal{C}_\epsilon(j\mathcal{Q})} \geq 0$  for all  $v \in \mathcal{G}_\epsilon(ZP)$  and  $\langle w, \bar{\Psi}_\epsilon w \rangle_{\mathcal{C}_\epsilon(j\mathcal{Q})} \leq -\frac{\gamma}{2} \|w\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}^2$  for all  $w \in \mathcal{G}'_\epsilon(\tau\Gamma)$  and  $\tau \in [0, 1]$ . By Theorem 4.4, it follows that the feedback configuration  $[ZP, \Gamma]$  is  $\mathbf{H}_{2\epsilon}$ -stable for all  $0 < \epsilon \leq \epsilon^*$ . In turn, this implies that

$$Z(s)P(s) (I - \Gamma(s)Z(s)P(s))^{-1} = P(s) \left( \frac{1}{Z(s)} I - \Gamma(s)P(s) \right)^{-1} \quad (8)$$

has no poles on  $\bar{\mathbb{C}}_+ \setminus j\mathcal{Q}$ , i.e.  $\det(\frac{1}{Z(s)} I - \Gamma(s)P(s))$  has no zeros on  $\bar{\mathbb{C}}_+ \setminus j\mathcal{Q}$ . Moreover, by Assumption 3.1,  $\det(\frac{1}{Z(s)} I - \Gamma(s)P(s))$  has

a zero at every  $s \in j\mathcal{Q}$  corresponding to the null space  $\mathcal{N}$  satisfying  $P(s)\mathcal{N} \subset \text{span}\{1_n\}$ , and the multiplicity of the zero is the same as that of the pole  $s$  of  $Z$ . This implies that for any  $u \in \mathbf{L}_2$ ,  $\hat{x}(s) := P(s) \left( \frac{1}{Z(s)} I - \Gamma(s)P(s) \right)^{-1} \hat{u}(s)$  has poles in  $j\mathcal{Q}$  if and only if the output of  $P(s)$  is in  $\text{span}\{1_n\}$ , i.e. they arise as an element of  $Z(s)\text{span}\{1_n\}$  (up to a difference by possibly some zeros).  $\hat{x}(s)$  may also have open-left-half-plane poles which contribute to the transients in  $x(t)$ , but any sustained behaviour as  $t \rightarrow \infty$  must correspond to an object in  $Z(s)\text{span}\{1_n\}$ .

Now note that for any  $e, f \in \mathbf{L}_2$ , it can be derived from (1) with  $\Delta = ZP$  and  $G = \Gamma$  (cf. Figure 1) that

$$\hat{y} = ZP(I - \Gamma ZP)^{-1}(\hat{e} + \Gamma\hat{f}) = P \left( \frac{1}{Z} I - \Gamma P \right)^{-1} (\hat{e} + \Gamma\hat{f}).$$

By the analysis above,  $\hat{y}$  has no poles on  $\bar{\mathbb{C}}_+ \setminus j\mathcal{Q}$ . Also, either (i)  $\hat{y}$  has poles in the open left half plane only, in which case  $y \in \mathbf{L}_2$  and  $y(t)$  converges to 0 (the trivial synchronised equilibrium), or (ii)  $\hat{y}$  has marginally stable poles at every point in  $j\mathcal{Q}$  with the same multiplicity of the corresponding pole of  $Z$  and they arise as an element of  $Z(s)\text{span}\{1_n\}$ . Note that in this case the open-left-half-plane poles of  $\hat{y}$  give rise to modes in  $y(t)$  that exponentially decay to 0 as  $t \rightarrow \infty$  while the marginally stable modes in  $Z(s)\text{span}\{1_n\}$  lead to sustained asymptotic behaviour. In other words, the feedback reaches synchronisation where the asymptotic behaviour is defined by the imaginary-axis poles of  $Z$ , as required. ■

*Remark 4.6:* It can be seen from the proof of Theorem 4.4 that the conditions of Theorem 4.5 may also be written as

- (i)  $\begin{bmatrix} I_n \\ Z(j\omega)P(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} I_n \\ Z(j\omega)P(j\omega) \end{bmatrix} \geq \gamma > 0$ ;
- (ii)  $\begin{bmatrix} \tau\Gamma(j\omega) \\ I_n \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \tau\Gamma(j\omega) \\ I_n \end{bmatrix} \leq 0 \forall \tau \in [0, 1]$ ,

for all  $\omega \in \mathbb{R} \setminus \mathcal{Q}$ ; or

- (i)  $\begin{bmatrix} I_n \\ \tau Z(j\omega)P(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} I_n \\ \tau Z(j\omega)P(j\omega) \end{bmatrix} \geq \gamma \forall \tau \in [0, 1]$ ;
- (ii)  $\begin{bmatrix} \Gamma(j\omega) \\ I_n \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \Gamma(j\omega) \\ I_n \end{bmatrix} \leq 0$ ,

for all  $\omega \in \mathbb{R} \setminus \mathcal{Q}$ .

*Remark 4.7:* The frequency-dependent quadratic inequalities in Theorem 4.5 are often verified numerically when distributed-parameter transfer functions are involved. In the case where the transfer functions are proper and rational, the inequalities with a fixed  $\tau \in [0, 1]$  can be equivalently transformed into linear matrix inequalities (LMIs) via the generalised Kalman-Yakubovich-Popov (KYP) lemma [32]. The feasibility of LMIs can then be verified efficiently via semidefinite programming.

## V. DISTRIBUTED SYNCHRONISATION CERTIFICATES

In this section we derive synchronisation certificates that can be verified in a distributed manner by employing the IQC result established earlier. These certificates also scale well with the addition or removal of dynamic agents. Some Nyquist-type results on multi-agent consensus are subsequently shown to be special cases of the certificates.

### A. IQC stability conditions

Consider the feedback interconnection in Figure 2, where  $P := \bigoplus_{i=1}^n P_i : P_i \in \mathbf{S}$ ,  $Z$  is a scalar proper rational transfer function analytic with a finite number of imaginary-axis poles  $j\mathcal{Q}$ ,  $\Gamma := \bigoplus_{i=1}^m \Gamma_i : \Gamma_i \in \mathbf{S}$ , and  $D$  denotes the incidence matrix of a connected undirected graph  $\mathcal{G}$ . Suppose  $\Gamma$  satisfies Assumption 3.1 and  $P_i(j\omega) \neq 0 \forall j\omega \in j\mathcal{Q}, i = 1, \dots, n$ .

Given a  $B \in \mathbb{C}^{m \times n}$  such that  $B_{ij} \neq 0$  whenever  $D_{ij}^T \neq 0$  and  $|B_{\bullet j}| = 1$  for  $i = 1, 2, \dots, n$ ,  $j = 1, 2, \dots, m$ , i.e. columns of  $B$  are normalised, let

$$C_{ij} := \begin{cases} 0 & \text{if } B_{ij} = 0 \\ B_{ij}^{-1} & \text{otherwise.} \end{cases} \quad (9)$$

*Theorem 5.1:* Suppose there exist  $B \in \mathbb{C}^{m \times n}$  as above,  $H := \bigoplus_{i=1}^n H_i$ ,  $J := \bigoplus_{i=1}^n J_i$  with  $H_i, J_i \in \mathbb{C}$  and  $K \in \mathbb{C}^{n \times n}$  such that  $H_i + H_i^*$  is positive definite,  $J_i, K$  are positive semidefinite for  $i = 1, \dots, n$ , and

- (i)  $[D\Gamma(j\omega)^* D^T](\tau J(j\omega) - K(j\omega))[D\Gamma(j\omega) D^T] \leq 0$  for all  $\tau \in [0, 1]$  and  $\omega \in \mathbb{R} \setminus \mathcal{Q}$ ;
- (ii) for all  $i = 1, \dots, m$  and  $\omega \in \mathbb{R} \setminus \mathcal{Q}$ ,

$$\begin{bmatrix} I_n \\ I_n \end{bmatrix}^* \Pi_i(j\omega) \begin{bmatrix} I_n \\ I_n \end{bmatrix} \geq \gamma > 0, \quad (10)$$

where

$$\Pi_i = \begin{bmatrix} H + H^* + J & \Phi_i \\ \Phi_i^* & \Omega_i \end{bmatrix},$$

with

$$\begin{aligned} \Phi_i &:= -H(\text{diag}(C_{i\bullet}^*) D_{\bullet i}) \Gamma_i(D_{i\bullet}^T \text{diag}(C_{i\bullet})) ZP, \\ \Omega_i &:= -(ZP)^* \text{diag}(C_{i\bullet}^*) D_{\bullet i} \Gamma_i^* D_{i\bullet}^T K D_{\bullet i} \Gamma_i(D_{i\bullet}^T \text{diag}(C_{i\bullet})) ZP, \end{aligned} \quad (11)$$

and  $C$  is as defined in (9). Then the feedback connection in Figure 2 reaches synchronisation.

*Proof:* It is established below that the hypothesis implies

$$\begin{bmatrix} \tau D\Gamma(j\omega) D^T \\ I_n \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \tau D\Gamma(j\omega) D^T \\ I_n \end{bmatrix} \leq 0 \quad (12)$$

and

$$\begin{bmatrix} I_n \\ Z(j\omega) P(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} I_n \\ Z(j\omega) P(j\omega) \end{bmatrix} \geq \gamma > 0 \quad (13)$$

for all  $\tau \in [0, 1]$  and  $\omega \in \mathbb{R} \setminus \mathcal{Q}$ , where

$$\Pi := \begin{bmatrix} H + H^* + J & -HD\Gamma D^T \\ -D\Gamma^* D^T H^* & -D\Gamma^* D^T K D\Gamma D^T \end{bmatrix}. \quad (14)$$

Synchronisation then follows from Theorem 4.5 and Remark 4.6.

Now note that

$$\begin{aligned} & \begin{bmatrix} \tau D\Gamma(j\omega) D^T \\ I_{pm} \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} \tau D\Gamma(j\omega) D^T \\ I_{pm} \end{bmatrix} \\ &= \tau(\tau - 1)(D\Gamma(j\omega)^* D^T H(j\omega)^* D\Gamma(j\omega) D^T) \\ & \quad + \tau(\tau - 1)(D\Gamma(j\omega)^* D^T H(j\omega) D\Gamma(j\omega) D^T) \\ & \quad + [D\Gamma(j\omega)^* D^T](\tau^2 J(j\omega) - K(j\omega))[D\Gamma(j\omega) D^T] \\ & \leq 0 \quad \forall \omega \in \mathbb{R} \setminus \mathcal{Q} \text{ and } \tau \in [0, 1], \end{aligned} \quad (15)$$

where hypothesis (i) has been used for the last inequality. This establishes (12). The rest of the proof shows that hypothesis (ii) leads to (13).

Notice that

$$\begin{aligned} \Phi &:= -HD\Gamma D^T ZP = -H \sum_{i=1}^p D_{\bullet i} \Gamma_i D_{i\bullet}^T ZP \\ &= \sum_{i=1}^p \text{diag}(B_{i\bullet}^*) \Phi_i \text{diag}(B_{i\bullet}). \end{aligned} \quad (16)$$

Likewise,

$$\Omega := -P^* Z^* D\Gamma^* D^T K D\Gamma D^T ZP = \sum_{i=1}^p \text{diag}(B_{i\bullet}^*) \Omega_i \text{diag}(B_{i\bullet}). \quad (17)$$

Since  $|B_{\bullet j}| = 1$  for  $j = 1, 2, \dots, p$ , it follows that

$$\sum_{i=1}^p \text{diag}(B_{i\bullet}^*) \text{diag}(B_{i\bullet}) = I_n.$$

As such,

$$\begin{aligned} & \begin{bmatrix} I_n \\ ZP \end{bmatrix}^* \Pi \begin{bmatrix} I_n \\ ZP \end{bmatrix} \\ &= \begin{bmatrix} I_n \\ I_n \end{bmatrix}^* \begin{bmatrix} H + H^* + J & \Phi \\ \Phi^* & \Omega \end{bmatrix} \begin{bmatrix} I_n \\ I_n \end{bmatrix} \\ &= \sum_{i=1}^p \begin{bmatrix} I_n \\ I_n \end{bmatrix}^* \begin{bmatrix} \text{diag}(B_{i\bullet}^*) & 0 \\ 0 & \text{diag}(B_{i\bullet}^*) \end{bmatrix} \begin{bmatrix} H + H^* + J & \Phi_i \\ \Phi_i^* & -\Omega_i \end{bmatrix} \\ & \quad \begin{bmatrix} \text{diag}(B_{i\bullet}) & 0 \\ 0 & \text{diag}(B_{i\bullet}) \end{bmatrix} \begin{bmatrix} I_n \\ I_n \end{bmatrix}, \end{aligned}$$

where (16) and (17) have been used in the last equality. It thus follows that hypothesis (ii) implies (13). This completes the proof.  $\blacksquare$

The prowess of Theorem 5.1 lies in the decentralised nature and scalability of condition (ii). In particular, the stability test involves only the frequency responses of the weighting transfer function  $\Gamma_i$  on the  $i^{\text{th}}$  edge and the associated two vertices or agents in  $P$ , as related via the  $i^{\text{th}}$  column of incidence matrix  $D$ ; see (11). If an additional agent joins the network, only the corresponding certificates for the new edges need to be verified to conclude synchronisation.

## B. Specialisation to Nyquist-based results

Define for  $i = 1, \dots, m$ , the weighted loop gain or return ratio

$$L_i := -(\text{diag}(C_{i\bullet}^*) D_{\bullet i}) \Gamma_i(D_{i\bullet}^T \text{diag}(C_{i\bullet})) ZP. \quad (18)$$

Also let  $\ell_\alpha$  be a straight line passing through the  $(-1, 0j)$  point on the complex plane in the direction  $\alpha \in \mathbb{C}$

$$\ell_\alpha := \{r\alpha - 1 : r \in \mathbb{R}\}. \quad (19)$$

Given a matrix  $M \in \mathbb{C}^{n \times n}$  and a positive definite Hermitian  $\hat{H} \in \mathbb{C}^{n \times n}$ , define the generalised inner product variation of the numerical range or field of values of  $M$  by  $\hat{H}$  as

$$\mathbf{N}_{\hat{H}}(M) := \{x^* \hat{H} M x : x \in \mathbb{C}^n, x^* \hat{H} x = 1\}, \quad (20)$$

which is a closed subset of  $\mathbb{C}^n$ .

*Proposition 5.2:* Suppose there exist an  $\alpha \in \mathbb{C}$  and an  $\hat{H} := \bigoplus_{i=1}^n \hat{H}_i$  with positive definite  $\hat{H}_i \in \mathbb{C}$  such that for every  $i \in \{1, \dots, m\}$  and  $\omega \in \mathbb{R} \setminus \mathcal{Q}$ ,

$$\mathbf{N}_{\hat{H}(j\omega)}(L_i(j\omega)) \cap \ell_\alpha = \emptyset. \quad (21)$$

Then the network given by the feedback interconnection of  $ZP$  and  $D\Gamma D^T$  in Figure 2 reaches synchronisation.

*Proof:* Suppose  $\alpha$  has non-zero real and imaginary parts, i.e.  $\Re(\alpha) \neq 0$  and  $\Im(\alpha) \neq 0$ , then for a fixed  $\omega \in \mathbb{R} \setminus \mathcal{Q}$ , (21) implies

$$\frac{\Re(\beta(z+1))}{|\beta||z+1|} \geq \bar{\gamma} \quad (22)$$

for all  $z \in \mathbf{N}_{\hat{H}(j\omega)}(L_i(j\omega))$  and some  $\bar{\gamma} > 0$ , where  $\beta = \alpha^*$  if  $\mathbf{N}_{\hat{H}(j\omega)}(L_i(j\omega))$  lies to the right of  $\ell_\alpha$  and  $\beta = -\alpha^*$  otherwise. To see this, note that (21) implies that the angle between  $\beta$  and  $z+1$  is strictly between  $-\frac{\pi}{2}$  and  $\frac{\pi}{2}$ , which is equivalent to (22). The case for zero real or imaginary part of  $\alpha$  can be treated similarly. In particular,  $\beta$  can be chosen to be  $\alpha e^{j\frac{\pi}{2}}$  or  $\alpha e^{-j\frac{\pi}{2}}$  depending on the orientation of  $\mathbf{N}_{\hat{H}(j\omega)}(L_i(j\omega))$  with respect to  $\ell_\alpha$ .

Now observe that (21) and (22) hold only if  $\Re(\beta(z+1)) \geq \hat{\gamma}$  for all  $z \in \mathbf{N}_{\hat{H}(j\omega)}(L_i(j\omega))$  and some  $\hat{\gamma} > 0$ . That is, for all  $x \neq 0$ ,

$$\begin{aligned} & \Re \left( \beta \left( \frac{x^* \hat{H}(j\omega) L_i(j\omega) x}{x^* \hat{H}(j\omega) x} + 1 \right) \right) \geq \hat{\gamma} \\ \implies & \Re(\beta(x^* \hat{H}(j\omega) L_i(j\omega) x + x^* \hat{H}(j\omega) x)) \geq \hat{\gamma} x^* \hat{H}(j\omega) x \\ \implies & \beta(x^* \hat{H}(j\omega) L_i(j\omega) x + x^* \hat{H}(j\omega) x) \\ & + (\beta(x^* \hat{H}(j\omega) L_i(j\omega) x + x^* \hat{H}(j\omega) x))^* \geq \hat{\gamma} x^* \hat{H}(j\omega) x. \end{aligned}$$

This in turn implies that

$$\beta \hat{H}(j\omega) L_i(j\omega) + \beta^* L_i(j\omega)^* \hat{H}(j\omega)^* + \beta \hat{H}(j\omega) + \beta^* \hat{H}(j\omega)^* \geq \gamma,$$

where  $\gamma := \hat{\gamma} \inf_{\omega \in \mathbb{R}} \underline{\sigma}(\hat{H}) > 0$  and  $\underline{\sigma}(\cdot)$  denotes the smallest eigenvalue of a Hermitian matrix. Letting  $H := \beta \hat{H}$  and noting by (11) and (18) that  $\Phi_i = H L_i$  yields

$$\begin{bmatrix} I_n \\ I_n \end{bmatrix}^* \begin{bmatrix} H(j\omega) + H(j\omega)^* & \Phi_i(j\omega) \\ \Phi_i(j\omega)^* & 0 \end{bmatrix} \begin{bmatrix} I_n \\ I_n \end{bmatrix} \geq \gamma, \quad (23)$$

which is (10) with  $J = 0$  and  $K = 0$ . As such, (21) implies condition (ii) of Theorem 5.1. Furthermore, note that for this choice of  $J$  and  $K$ , it is immediate that condition (i) of the same theorem holds trivially. The claim on synchronisation thus follows from Theorem 5.1.  $\blacksquare$

Proposition 5.2 can be seen as a generalisation of [21, Thm. 1]. To be specific, the latter derives distributed conditions of the form (21) for feedback stability while the former extends these to the problem of synchronisation. In [21, Section III.E], the field of values  $\mathbf{N}_{\hat{H}(j\omega)}(L_i(j\omega))$  in (21) is shown to be an ellipse on the complex plane, since only SISO agents are considered. Correspondingly, [21, Lem. 4] establishes that the optimal  $\hat{H}$  which gives rise to the least conservative test (21) (or the ‘smallest’ ellipse  $\mathbf{N}_{\hat{H}(j\omega)}(L_i(j\omega))$ ) takes the form:

$$\hat{H}(j\omega) := \bigoplus_{i=1}^n ((Z P_i(j\omega))^* Z P_i(j\omega) + \eta)^{\frac{1}{2}}, \quad (24)$$

where  $\eta$  is a small positive constant to ensure the positive definiteness of  $\hat{H}$ . In fact, by choosing the optimal  $\hat{H}$  and setting  $B_{ij} := D_{ij}^T / |D_{\bullet j}^T|$  in (9), one obtains the following corollary of Proposition 5.2.

*Corollary 5.3:* Suppose there exist an  $\alpha \in \mathbb{C}$  such that the line  $\ell_\alpha$  does not intersect the ellipse with foci at the origin and at  $f_i(j\omega)$ , and major axis length  $a_i(j\omega)$  for all  $i = 1, \dots, m, \omega \in \mathbb{R} \setminus \mathcal{Q}$ , where

$$\begin{aligned} f_i(j\omega) &:= -\Gamma_i(j\omega) \sum_{j: D_{ij}^T \neq 0} n_j Z(j\omega) P_j(j\omega) \\ a_i(j\omega) &:= |\Gamma_i(j\omega)| \sum_{j: D_{ij}^T \neq 0} |n_j Z(j\omega) P_j(j\omega)| \end{aligned}$$

and  $n_j := |D_{\bullet j}^T|^2$  denotes the number of neighbours of  $P_j$ , then the feedback interconnection in Figure 2 reaches synchronisation.

*Proof:* The result follows from the fact that for the choice of  $\hat{H}$  in (24) and  $C$  in (9) defined by  $B_{ij} := D_{ij}^T / |D_{\bullet j}^T|$ , the numerical range  $\mathbf{N}_{\hat{H}(j\omega)}$  in Proposition 5.2 equals to the ellipse described in the statement of the corollary. The details can be found in [21, Section III.E].  $\blacksquare$

*Remark 5.4:* In the case where  $Z(s) = \frac{1}{s}$ , the distributed consensus certificates derived in Corollary 5.3 are tighter than those in [13, Thm. 1], by virtue of the fact that the ellipses in the former are subsets of the  $S$ -hulls employed in the latter; see [21, Appendix B]. This parallels the claim therein that [21, Prop. 1] is less conservative than the results in [33], where feedback stability is concerned.

## VI. CONSENSUS FOR SYSTEMS WITH COMMUNICATION DELAYS

This section demonstrates that some of the standard Nyquist stability criteria for systems with feedback delays imply the IQC conditions developed earlier. Together with the distributed synchronisation certificates in the previous section, this serves to affirm the claim that the main IQC framework in Section IV unifies various related results in the literature.

Consider Figure 1 with  $Z(s) := \frac{1}{s}$  and  $n$  transfer functions

$$y_i(s) = \frac{1}{s} P_i(s) u_i(s),$$

where  $u_i$  and  $y_i$  are the input and output of agent  $i$ , respectively, and  $P_i \in \mathbf{S}$  for  $i = 1, \dots, n$ . Define  $P := \text{diag}(P_i)$ . Let the topology of the network be described by an connected undirected graph  $\mathcal{G} = (V, E)$  with a weighted adjacency matrix  $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ , where  $a_{ij} > 0$  if agent  $i$  and  $j$  are connected and  $a_{ij} = 0$  otherwise. The valency of agent  $i$  is denoted  $b_i := \sum_{j=1}^n a_{ji}$  and define the valency matrix by  $B := \text{diag}(b_i) \in \mathbb{R}^{n \times n}$ . Consider the following three types of time-delay feedback control laws based on [26]:

1) feedback control *without self-delay*

$$u_i(t) = - \sum_{j=1}^n \frac{a_{ji}}{b_i} (y_i(t) - y_j(t - \tau_{ji}));$$

2) feedback control *with identical self-delay*

$$u_i(t) = - \sum_{j=1}^n \frac{a_{ji}}{b_i} (y_i(t - \tau_{ji}) - y_j(t - \tau_{ji}))$$

3) and feedback control *with different self-delay*

$$u_i(t) = - \sum_{j=1}^n \frac{a_{ji}}{b_i} (y_i(t - T_{ji}) - y_j(t - \tau_{ji})),$$

where  $\tau_{ji} \geq 0$  and  $T_{ji} \geq 0$  for  $i, j = 1, \dots, n$  denote the constant delay terms. The feedback without self-delay models transmission delays for data sent from one agent to another over a communication network. Feedback with identical self-delay is commonly used for situations where there are computation or reaction delays in the agent’s own state behaviour. The situation modelled by a feedback with different self-delay arises when the delay associated with the agent’s own behaviour differs from that of its neighbours.

To represent the aforementioned feedback laws in the Laplace domain, define the delay-dependent adjacency matrix  $A_\tau(s) := [a_{ji} e^{-\tau_{ji}s}]$  and valency matrices

$$\begin{aligned} B_\tau(s) &:= \text{diag} \left( \sum_{j=1}^n a_{ji} e^{-\tau_{ji}s} \right) \quad \text{and} \\ B_T(s) &:= \text{diag} \left( \sum_{j=1}^n a_{ji} e^{-T_{ji}s} \right). \end{aligned}$$

By defining  $u(s) := (u_1(s), \dots, u_n(s))^T$  and  $y(s) := (y_1(s), \dots, y_n(s))^T$ , it is straightforward to verify that the feedback controller without self-delay, feedback controller with identical self-delay, and feedback controller with different self-delay can be realised respectively by the following transfer functions  $\Gamma_1$ ,  $\Gamma_2$ , and  $\Gamma_3$  in  $\mathbf{S}^{n \times n}$ :

$$\begin{aligned} u(s) &= -\Gamma_1 y(s) = -(I - B^{-1} A_\tau(s)) y(s); \\ u(s) &= -\Gamma_2 y(s) = -B^{-1} L_\tau(s) y(s); \\ u(s) &= -\Gamma_3 y(s) = -B^{-1} (D_T(s) - A_\tau(s)) y(s), \end{aligned} \quad (25)$$

where  $L_\tau(s) := B_\tau(s) - A_\tau(s)$  denotes the delay-dependent graph Laplacian matrix. Note that  $\Gamma_r$  satisfies Assumption 3.1 for all  $r = 1, 2, 3$  with  $j \in \mathcal{Q}$  being a singleton containing only the origin of the

complex plane. The consensus problems involving different types of communication delays are thus modelled by Figure 1 as feedback interconnections  $[\frac{1}{s}P, \Gamma_r]$  for  $r = 1, 2, 3$ .

Observe from (8) and the arguments in the proof for the synchronisation Theorem 4.5 that consensus is achieved if the characteristic equation of the closed-loop system

$$\det(sI - P(s)\Gamma_r(s))$$

has no zeros in  $\bar{\mathbb{C}}_+ \setminus \{0\}$  and has a simple zero at 0; see also [26, Section 2.2.]. The latter follows from the connectedness of the underlying graph  $G$  as in the aforementioned proof. On the other hand, the former is equivalent to  $\det(I + G_r(s))$  having no zeros in  $\mathbb{C}_+$ , where the return ratio

$$G_r(s) := -\frac{1}{s}P(s)\Gamma_r(s).$$

This can in turn be guaranteed by the generalised Nyquist criterion [34]:

$$\sigma(G_r(j\omega)) \cap \ell_\alpha = \emptyset \quad \forall \omega \in (0, \infty), \quad (26)$$

where  $\sigma(\cdot)$  denotes the spectrum of a matrix,  $\alpha \in \mathbb{C}$  and  $\ell_\alpha$  is a straight line passing through  $(-1, 0j)$  as defined in (19).

It is well-known that for an  $M \in \mathbb{C}^{n \times n}$ ,  $\sigma(M) \subset \mathbf{N}_I(M)$  [29], where  $\mathbf{N}_I$  denotes the field of values defined in (20) with respect to the identity  $I \in \mathbb{R}^{n \times n}$ . Thus, a sufficient condition for (26) is

$$\mathbf{N}_I(G_r(j\omega)) \cap \ell_\alpha = \emptyset \quad \forall \omega \in (0, \infty), \quad (27)$$

which also implies  $\mathbf{N}_I(G_r(j\omega)) \cap \ell_\alpha = \emptyset, \forall \omega \in (-\infty, 0)$  inasmuch as  $G_r(-j\omega) = G_r(j\omega)^*$ . Such a condition is commonly employed in the literature for consensus, see for example [26], [13]. Notice the similarity of this condition to (21). By the same arguments employed in the proof for Proposition 5.2 to show that (21) implies (23), it follows that (27) implies there exists a  $\beta \in \mathbb{C}$  such that

$$\begin{bmatrix} I_n \\ I_n \end{bmatrix}^* \begin{bmatrix} \beta I_n + \beta^* I_n & \beta G_r(j\omega) \\ \beta^* G_r(j\omega)^* & 0 \end{bmatrix} \begin{bmatrix} I_n \\ I_n \end{bmatrix} \geq \gamma,$$

or equivalent,

$$\begin{bmatrix} I_n \\ \frac{1}{j\omega}P(j\omega)I_n \end{bmatrix}^* \begin{bmatrix} \beta I_n + \beta^* I_n & -\beta \Gamma_r(j\omega) \\ -\beta^* \Gamma_r(j\omega)^* & 0 \end{bmatrix} \begin{bmatrix} I_n \\ \frac{1}{j\omega}P(j\omega)I_n \end{bmatrix} \geq \gamma, \quad (28)$$

for all  $\omega \in \mathbb{R} \setminus \{0\}$  and some  $\gamma > 0$ . Note also that

$$\begin{bmatrix} \tau \Gamma_r(j\omega) \\ I_n \end{bmatrix}^* \begin{bmatrix} \beta I_n + \beta^* I_n & -\beta \Gamma_r(j\omega) \\ -\beta^* \Gamma_r(j\omega)^* & 0 \end{bmatrix} \begin{bmatrix} \tau \Gamma_r(j\omega) \\ I_n \end{bmatrix} \leq 0$$

for all  $\tau \in [0, 1]$  and  $\omega \in \mathbb{R} \setminus \{0\}$ . As such, the conditions in Remark 4.6 holds for

$$\Pi := \begin{bmatrix} \beta I_n + \beta^* I_n & -\beta \Gamma_r \\ -\beta^* \Gamma_r^* & 0 \end{bmatrix}. \quad (29)$$

and consensus of the feedback interconnection  $[\frac{1}{s}P, \Gamma_r]$  thus follows from Theorem 4.5 for  $r = 1, 2, 3$ , which correspond to different types of delays given in (25).

In effect the consensus condition (21) merely implies the IQC expression (28) involving a specific type of multiplier  $\Pi$  stated in (29). Theorem 4.5 is more general in the sense that generic multipliers  $\Pi$  can be employed to conclude synchronisation. This can be achieved by exploiting known structures of the open-loop systems as in the classical IQC analysis theory [15], as demonstrated in the succeeding section where agents with uncertain feedback delays are considered.

## VII. ILLUSTRATIVE EXAMPLES

### A. Convergence of a standard consensus algorithm

This subsection analyses the convergence of a standard consensus algorithms in [5, Chapter 2] via IQCs. The weighted adjacency matrix  $A_n = [a_{ij}] \in \mathbb{R}^{n \times n}$  of a directed graph  $\mathcal{G} = (V, E)$  is defined such that  $a_{ii} = 0$ ,  $a_{ij} > 0$  is a positive weight if  $(v_j, v_i) \in E$  and  $a_{ij} = 0$  otherwise. The directed Laplacian matrix  $L_n = [l_{ij}] \in \mathbb{R}^{n \times n}$  is defined as

$$l_{ii} := \sum_{j=1, j \neq i}^n a_{ij}, \quad l_{ij} := -a_{ij}, \quad i \neq j.$$

Suppose that  $\mathcal{G}$  contains a directed spanning tree, then  $L_n$  has a simple zero eigenvalue corresponding to the eigenvector  $1_n$  and all other eigenvalues of  $L_n$  have positive real parts [5, Lem. 2.4]. The following continuous-time consensus algorithm is commonly employed [5, Chapter 2]:

$$\dot{x}(t) = -L_n x(t). \quad (30)$$

It is known that the algorithm achieves consensus if, and only if,  $\mathcal{G}$  has a directed spanning tree [5, Thm. 2.8]. We establish below that sufficiency of this result can be recovered from Theorem 4.5 using a particular multiplier.

Note that (30) can be modelled by the feedback interconnection in Figure 1 with  $P = I_n$ ,  $Z(s) = \frac{1}{s}$ , and  $\Gamma = -L_n$ . That  $\Gamma$  satisfies Assumption 3.1 follows from  $\mathcal{G}$  containing a directed spanning tree. Let  $v \in \mathbb{R}^n$  be such that  $L_n^T v = 0$  and  $v^T 1_n = 1$ . Define

$$\Pi := \begin{bmatrix} (I_n + 1_n v^T)^T \\ L_n^T \end{bmatrix} \begin{bmatrix} I_n + 1_n v^T & L_n \end{bmatrix}.$$

It is straightforward to see that

$$\begin{bmatrix} \Gamma \\ I_n \end{bmatrix}^T \Pi \begin{bmatrix} \Gamma \\ I_n \end{bmatrix} \leq 0.$$

Furthermore, for all  $\tau \in [0, 1]$  and  $\omega \in \mathbb{R}$  such that  $\omega \neq 0$ , observe that

$$\begin{aligned} & \begin{bmatrix} I_n \\ \tau \frac{1}{j\omega} I_n \end{bmatrix}^* \Pi \begin{bmatrix} I_n \\ \tau \frac{1}{j\omega} I_n \end{bmatrix} \\ &= \left( I_n + 1_n v^T + \frac{\tau}{j\omega} L_n \right)^* \left( I_n + 1_n v^T + \frac{\tau}{j\omega} L_n \right) \geq 0. \end{aligned}$$

Given any  $\alpha \in \mathbb{C}$ , note that the eigenvalues of  $1_n v^T + \alpha L_n$  consist of 1 and  $\alpha \lambda$ , where  $\lambda$  is any nonzero eigenvalue of  $L_n$ , which has a positive real part [5, Lem. 2.4]. It follows that  $I_n + 1_n v^T + \frac{\tau}{j\omega} L_n$  has no eigenvalues that are arbitrarily close to 0 for all  $\tau \in [0, 1]$ ,  $\omega \in \mathbb{R} \setminus \{0\}$ . Therefore, there exists a  $\gamma > 0$  such that

$$\left( I_n + 1_n v^T + \frac{\tau}{j\omega} L_n \right)^* \left( I_n + 1_n v^T + \frac{\tau}{j\omega} L_n \right) \geq \gamma$$

for all  $\tau \in [0, 1]$ ,  $\omega \in \mathbb{R} \setminus \{0\}$ . Consensus thus follows from Theorem 4.5 and Remark 4.6.

### B. Output-synchronisation of agents with uncertain communication delays

In this subsection, we consider three examples of synchronisation in heterogeneous networks with uncertain but bounded communication delays. Synchronisation is established using Theorem 4.5. Consider  $n = 5$  agents deployed on a line (connected in series by  $m = 4$  undirected edges) whose dynamics are described by  $h_i(s) = Z(s)P_i(s)$ , for  $i = 1, \dots, 5$ , where  $Z(s)$  represents a nominal model and  $P_i(s)$  a stable multiplicative perturbation.

The interconnection operator takes the form

$$\Gamma(s) = K(s)DE(s)D^T, \quad (31)$$

where  $D$  denotes the incidence matrix,  $K(s)$  represents a pre-processing of the input to each agent, and the diagonal transfer matrix  $E(s) := \text{diag}(e^{-\tau_i s})$ . Each delay  $\tau_i$  is selected randomly uniformly from the interval  $[0, \bar{\tau}]$  for  $i = 1, \dots, 4$ . Figure 4 depicts the feedback interconnection of the network described above.

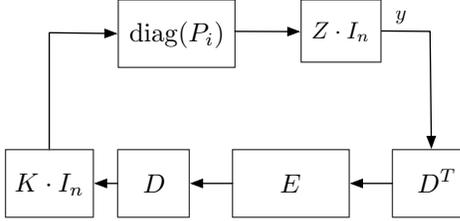


Fig. 4. Feedback interconnection involving uncertain communication delays.

The following multiplier is employed in the IQC analysis:

$$\begin{aligned} \Pi(j\omega) &:= \pi(j\omega) \otimes I_n \\ \pi(j\omega) &:= \gamma_1(j\omega) \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ &+ \gamma_2(j\omega) \begin{bmatrix} -\omega_*^2 \Phi_1(\omega_*) - \Phi_2(\omega_*) & j\omega_* \Phi_1(\omega_*) + \Phi_2(\omega_*) \\ -j\omega_* \Phi_1(\omega_*) + \Phi_2(\omega_*) & \omega_*^2 \Phi_1(\omega_*) - \Phi_2(\omega_*) \end{bmatrix} \end{aligned} \quad (32)$$

where  $\omega_* := \frac{1}{2}\omega\bar{\tau}$ ,

$$\begin{aligned} \Phi_1(j\omega) &:= \begin{cases} \frac{\sin(\omega)}{\omega}, & |\omega| < \pi \\ 0, & |\omega| \geq \pi \end{cases}, \\ \Phi_2(j\omega) &:= \begin{cases} \cos(\omega), & |\omega| < \pi \\ 0, & |\omega| \geq \pi \end{cases}, \end{aligned}$$

$\gamma_1(\cdot)$  is any bounded measurable function on  $j\mathbb{R}$ , and  $\gamma_2(\cdot)$  any non-negative bounded measurable function on  $j\mathbb{R}$ . First of all, note that (here, to avoid confusion,  $\theta$  is the homotopy parameter, previously denoted  $\tau$  in Theorem 4.5)

$$\begin{bmatrix} I_n \\ \theta E(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} I_n \\ \theta E(j\omega) \end{bmatrix} \geq 0 \quad \text{for all } \omega \in \mathbb{R} \text{ and } \theta \in [0, 1]. \quad (33)$$

Indeed, as shown in [15, Section IV.H] and [35, Lem. 15], the SISO operator  $\Delta(v(t)) := v(t-\tau)$  with  $\tau \in [0, \bar{\tau}]$  satisfies the IQC defined by  $\pi$  in (32). The extension to the diagonal  $E(s)$  by means of the Kronecker product is straightforward. Synchronisation would then be established if

$$\begin{bmatrix} D^T Z(j\omega) P(j\omega) K(j\omega) D \\ I \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} D^T Z(j\omega) P(j\omega) K(j\omega) D \\ I \end{bmatrix} \leq -c < 0 \quad (34)$$

for some  $\gamma_1(\cdot)$  and  $\gamma_2(\cdot)$  and all  $\omega \in \mathbb{R} \setminus \mathcal{Q}$ , with  $j\mathcal{Q}$  being the set poles of  $Z$  on  $j\mathbb{R}$ . This follows from the fact that conditions (33) and (34) are equivalent to those in Theorem 4.5.

In the following three examples, condition (34) is numerically verified. The maximum (real) eigenvalue of the matrix on the left hand side of (34) is computed, and its negativity checked across frequency.

*Consensus with delayed communication:* Let  $Z(s) := \frac{1}{s}$ , which corresponds to a perturbed consensus network in Figure 4 in which a high-frequency zero-pole combination is adopted for each perturba-

tion  $P_i(s) := \frac{1+s/a_i}{1+s/K_i}$ , where  $a_i = 0.9K_i$ , and  $K_i$  is taken uniformly randomly from the interval  $[10, 15]$ , for  $i = 1, \dots, 5$ . Let the upper bound of the delay be  $\bar{\tau} := 0.3$  and  $K(s) := 1$ . By defining

$$\gamma_1(j\omega) := \begin{cases} 0.1, & |\omega| < 1 \\ 1, & |\omega| \geq 1 \end{cases} \quad \text{and} \quad \gamma_2(j\omega) := \frac{0.1}{1+\omega^2},$$

inequality (34) is satisfied for  $c = 4 \times 10^{-4}$ . Figure 5 illustrates the results, where the clocks' initial start times are taken uniformly randomly from  $[0, 10]$ . On the left of the figure it can be seen that the outputs of the agents converge to a common constant value. The right half of the figure shows the maximum eigenvalue of the left hand side of (34) on a dense frequency grid in  $\mathbb{R} \setminus \{0\}$ .

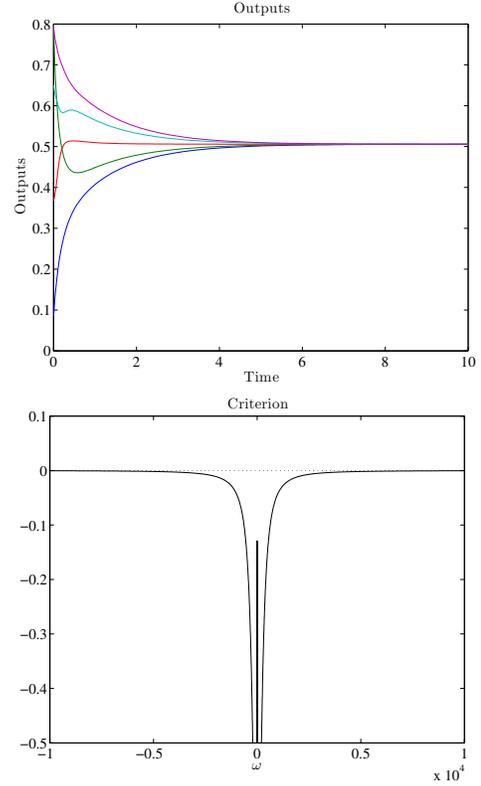


Fig. 5. Numerical simulation for perturbed consensus. At the top, an output simulation plot. At the bottom, numerical evaluation of maximum eigenvalue of the left hand side of (34) for frequencies on a dense grid in  $\mathbb{R} \setminus \{0\}$ .

*Oscillators with delayed communication:* Let  $Z(s) := \frac{1}{s^2 + \omega_0^2}$  with  $\omega_0 := \frac{\pi}{2}$ . In this setting, Figure 4 models a network of perturbed oscillators, which finds applications in power generators [2], for instance. Let the upper bound of the delay be  $\bar{\tau} := 0.1$  and  $K(s) := \frac{s+0.1}{1+s/100}$ . Correspondingly, inequality (34) can be verified to hold with

$$\gamma_1(j\omega) := \begin{cases} 0.099, & |\omega| < 1 \\ 1, & |\omega| \geq 1 \end{cases} \quad \text{and} \quad \gamma_2(j\omega) := \frac{0.1}{1+\omega^2}.$$

Simulation results are given in Figure 6, where the outputs of the agents are shown to agree on a common sinusoid of frequency  $\omega_0$  over time.

*Double integrators with delayed communication:* Let  $Z(s) := \frac{1}{s^2}$ , which yields a network of perturbed double integrators in Figure 4. Such a setup may be applied to the problem of clock synchronisation in the following way [14]: a single clock with uncertain skew  $\delta_i$  with

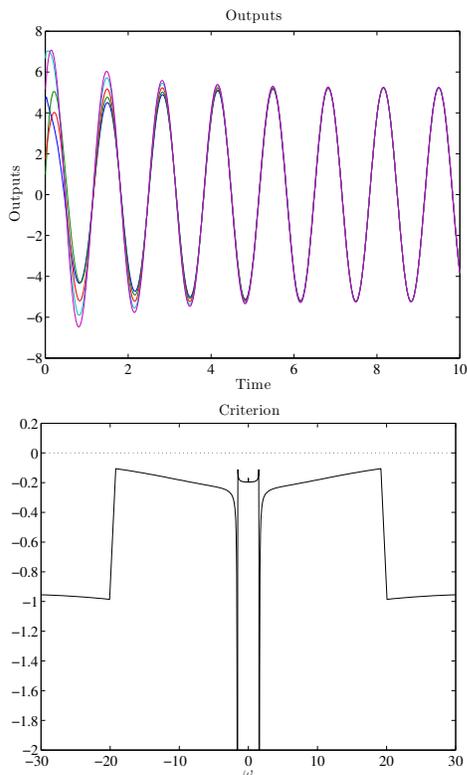


Fig. 6. Numerical simulation for perturbed oscillators with natural frequency  $\omega_0 = \frac{\pi}{2}$ . At the top, an output simulation plot. At the bottom, numerical evaluation of maximum eigenvalue of the left hand side of (34) for frequencies on a dense grid in  $\mathbb{R} \setminus \{\pm\omega_0\}$ .

respect to the absolute time  $t$  provides the following estimate of  $t$ :

$$t_i(t) = t_i(t_0) + \delta_i s_i(t)(t - t_0),$$

where  $s_i(t)$  is a skew-correcting tunable parameter and  $t_0$  is the initial absolute instant at which the clock starts reading the time. If the clock is isolated,  $s_i(t)$  is initialized and kept equal to any constant value. If available, a natural choice is the reciprocal of the nominal value of the skews, denoted  $\delta_0$ , so that the estimate  $t_i(t)$  evolves as a ramp with slope  $\frac{\delta_i}{\delta_0}$  with respect to the absolute time, i.e.,  $t_i(t) = t_i(t_0) + \frac{\delta_i}{\delta_0}(t - t_0)$ . In practice, the estimates of different clocks tend to drift away due to the non-equal skews, so compensation is needed if the goal is to obtain a common estimate of the absolute time. Here it is assumed that neighbouring clocks exchange their time estimates at each instant, compute the controls via the interconnection matrix  $\Gamma(s)$ , and update both the estimate  $t_i(t)$  and the skew correction  $s_i(t)$ .

Such a control scheme has already appeared in the literature [36], but has the undesirable feature that clocks estimates can be subjected to large variations. In order to smooth the influence of the control, the actual time estimate evolves according to

$$\dot{t}_i^r(t) = \kappa(t_i(t) - t_i^r(t)),$$

where  $\kappa > 0$  is a tunable parameter. In state-space form, the augmented system evolves according to

$$\begin{cases} \dot{x}_i(t) = \begin{bmatrix} 0 & \delta_i & 0 \\ 0 & 0 & 0 \\ \kappa & 0 & -\kappa \end{bmatrix} x_i(t) + \begin{bmatrix} b_1 \\ b_2 \\ 0 \end{bmatrix} u_i(t) \\ y_i(t) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x_i(t) = Cx_i(t), \end{cases}$$

where  $x_i(t) = [t_i(t) \quad s_i(t) \quad t_i^r(t)]^T \in \mathbb{R}^3$ ,  $y_i(t) = t_i^r(t)$  is the output of the  $i$ -th clock,  $u_i(t)$  is the control to the  $i$ -th clock, and  $b_1$ ,  $b_2$ ,  $\kappa$  are parameters to be designed. The resulting transfer function of the  $i$ -th clock is

$$P_i(s)Z(s) = \frac{b_1 s + b_2 \delta_i}{s^2(1 + s/\kappa)},$$

where  $Z(s) = \frac{1}{s^2}$  and  $P_i(s) = \frac{b_1 \delta_i s + b_2}{1 + s/\kappa}$ . Notice that the multiplicity of the pole at the origin is 2. While  $\kappa$  may be known to all clocks, each clock is characterised by an uncertain zero location.

Consider now the feedback control scheme in Figure 4, where the control input to each clock is updated based on information regarding the outputs of itself and the neighbouring clocks, subject to uncertain delays. In the following example, each skew is taken uniformly randomly  $\delta_i \sim \mathcal{U}[\delta_0 - \varepsilon, \delta_0 + \varepsilon]$ , where  $\delta_0 = 1$  is the nominal skew and  $\varepsilon = 0.5$ . Let  $b_1 = b_2 = 1$ ,  $\kappa = 50$ . Also, let the upper bound on the delay be  $\bar{\tau} := 0.2$ , and consider the pre-processing filter

$$K(s) := \frac{s}{1 + s/5}.$$

Notice that by the definition in (31),  $\Gamma(0)1_n = 0$  and also  $\lim_{s \rightarrow 0} \frac{1}{s} \Gamma(s)1_n = 0$ , whereby Assumption 3.1 is satisfied. Using

$$\gamma_1(j\omega) := \begin{cases} 0.01, & |\omega| < 1 \\ 1, & |\omega| \geq 1 \end{cases} \quad \text{and} \quad \gamma_2(j\omega) := \frac{0.1}{1 + \omega^2},$$

inequality (34) can be verified to hold. The results are illustrated in Figure 7, where the clock/agent outputs exhibit smooth behaviour and can be seen to agree on a common ramp function over time.

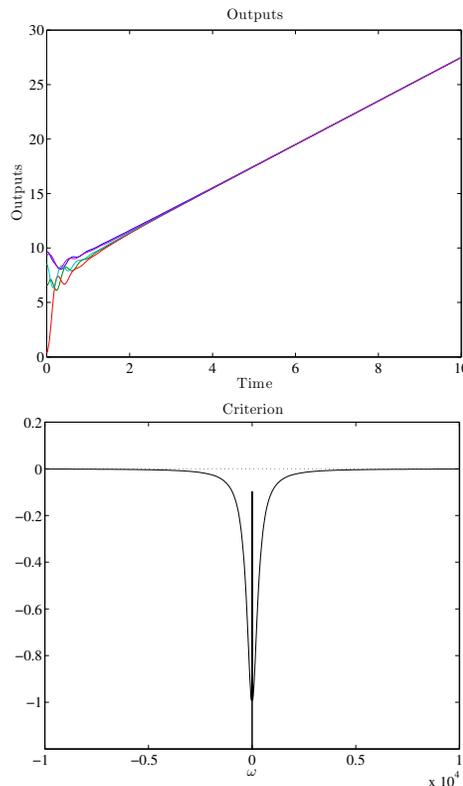


Fig. 7. Numerical simulation for perturbed double integrators. At the top, an output simulation plot. At the bottom, numerical evaluation of maximum eigenvalue of the left hand side of (34) for frequencies on a dense grid in  $\mathbb{R} \setminus \{0\}$ .

### C. Distributed certification of synchronisation for time-delay systems

Consider  $n$  number of systems  $P_1, \dots, P_n$ , whose respective outputs are described by

$$y_i(t) = \left(\frac{2}{\pi}\right)^2 \sin\left(\frac{\pi}{2}t\right) * \frac{1}{n_i} \sum_{j:v_j \in N_i} \mu[(y_i(t-1) - y_j(t-1))] \\ i = 1, \dots, n, \quad (35)$$

where the symbol  $*$  denotes the convolution operation,  $n_i = |N_i|$  denotes the number of neighbours of the node  $v_i$ , and  $\mu > 0$  is a parameter. Here it is assumed that the agents communicate between themselves according to a given connected undirected graph structure  $\mathcal{G} = (V, E)$  with the oriented (resp. unoriented) incidence matrix  $D \in \mathbb{R}^{n \times m}$  (resp.  $\bar{D} \in \mathbb{R}^{n \times m}$ ). The network (35) can be expressed in terms of the feedback interconnection depicted in Figure 2, where  $P = \text{diag}(P_i)$ ,  $P_i := \frac{1}{n_i}$ ,  $Z(s) := \frac{1}{s^2 + (\frac{\pi}{2})^2}$ , and  $\Gamma_j(s) := \mu e^{-s}$  for  $i = 1, \dots, n$  and  $j = 1, \dots, m$ .

*Proposition 7.1:* The feedback interconnection modelling (35) reaches synchronisation for  $\mu \leq 1.23$  and arbitrary network topology, i.e. arbitrary connected undirected graph  $\mathcal{G}$ .

*Proof:* Define  $B = [B_{ij}] \in \mathbb{R}^{n \times m}$  by  $B_{ij} := D_{ij}^T / |D_{\bullet j}^T|$ ,  $C \in \mathbb{R}^{n \times m}$  as in (9),  $H := I_n$ ,  $J := 0$  and  $K := 0$ . Notice that condition (i) of Theorem 5.1 holds trivially for this choice of  $J$  and  $K$ . Furthermore, it can be verified in a distributed manner that condition (ii) also holds for  $\mu \leq 1.23$ . In particular, by expanding the left-hand side of (10), it yields that for all  $i = 1, \dots, m$

$$\begin{aligned} \Lambda(j\omega) &:= H(j\omega) + H(j\omega)^* - H(j\omega)(\text{diag}(C_{i\bullet}^*)D_{\bullet i})\Gamma_i(j\omega) \\ &\quad (D_{i\bullet}^T \text{diag}(C_{i\bullet}))Z(j\omega)P(j\omega) \\ &\quad - P(j\omega)^*Z(j\omega)^*(\text{diag}(C_{i\bullet}^*)D_{\bullet i})\Gamma_i(j\omega)^* \\ &\quad (D_{i\bullet}^T \text{diag}(C_{i\bullet}))H(j\omega)^* \\ &= 2I_n - \left( \mu \frac{e^{-j\omega}}{(\frac{\pi}{2})^2 - \omega^2} - \mu \frac{e^{j\omega}}{(\frac{\pi}{2})^2 - \omega^2} \right) \bar{D}_{\bullet i} \bar{D}_{i\bullet}^T \\ &= 2I_n - \left( \mu \frac{2 \cos(\omega)}{(\frac{\pi}{2})^2 - \omega^2} \right) \bar{D}_{\bullet i} \bar{D}_{i\bullet}^T. \end{aligned}$$

Moreover, it holds that for  $\mu \leq 1.23$  and  $\omega \in \mathbb{R} \setminus \{\frac{\pi}{2}, -\frac{\pi}{2}\}$ , there exists  $\gamma > 0$  such that  $\Lambda(j\omega) \geq \gamma$ . The claim thus follows by Theorem 5.1.  $\blacksquare$

The bound on  $\mu$  obtained by applying the distributed synchronisation certificate of Theorem 5.1 is tight to within two decimal places for the simple time-delay network (35). As an illustration, Figure 8 shows the outputs of  $n = 3$  agents connected by  $m = 2$  communication links for  $\mu = 1.20$ . There the input to the first agent is perturbed by a pulse of magnitude 20 lasting for 1s at the beginning of the simulation. It can be observed that the outputs eventually converge to a synchronised trajectory defined by a sinusoidal signal of period  $\frac{\pi}{2}$ . On the other hand, when  $\mu$  is set to 1.24, Figure 9 shows that the outputs of the agents diverge and never synchronise.

## VIII. CONCLUSIONS

The paper demonstrates the application of integral quadratic constraint based analysis to the study of synchronisation problems for heterogeneous multi-agent networks without the use of loop transformations. To this end, certain gap-metric type results have been utilised to establish IQC conditions guaranteeing a generalised notion of closed-loop stability. We introduced an IQC framework for analysing multi-agent synchronisation. This is subsequently shown to encompass and extend several Nyquist-type results in the literature. We also proposed scalable synchronisation certificates that can be verified in a distributed fashion. A number of simulation examples

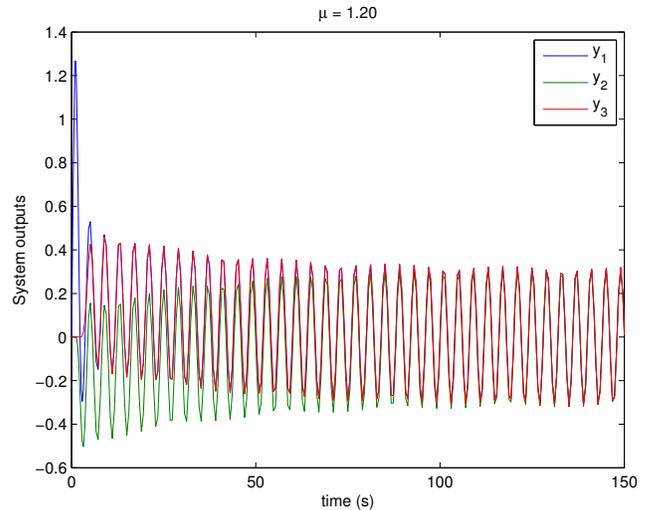


Fig. 8. System outputs for  $\mu = 1.20$ . Synchronisation to a sinusoid takes place.

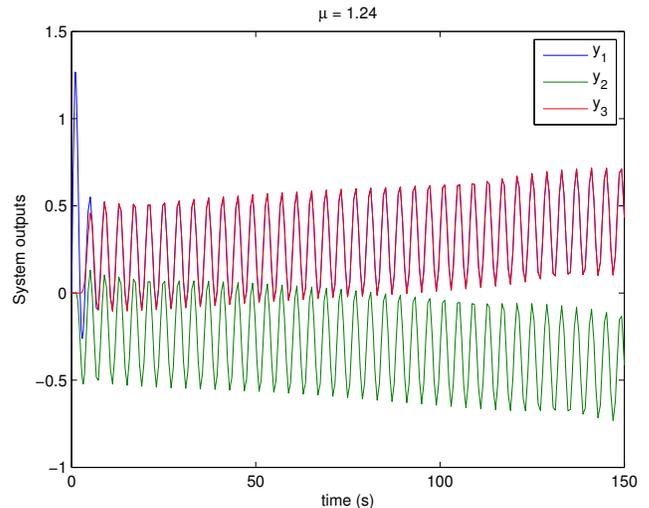


Fig. 9. System outputs for  $\mu = 1.24$ . Divergence occurs.

are provided to illustrate the theoretical results. Future research may involve extending the framework to cooperative formation control, characterising the speed of synchronisation and accommodating open-loop unstable agents and nonlinear systems in the IQC framework.

## REFERENCES

- [1] S. Strogatz, *Sync: How Order Emerges From Chaos In the Universe, Nature, and Daily Life*. Theia, 2003.
- [2] F. Dörfler and F. Bullo, "Synchronization and transient stability in power networks and non-uniform Kuramoto oscillators," *SIAM Journal of Control and Optimization*, vol. 50, no. 3, pp. 1616–1642, 2010.
- [3] B. P. Bertsekas and J. N. Tsitsiklis, *Parallel and Distributed Computation: Numerical Methods*. Prentice Hall, 1989.
- [4] R. Olfati-Saber, A. Fax, and R. M. Murray, "Consensus and cooperation in networked multi-agent systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 215–233, 2007.
- [5] W. Ren and R. W. Beard, *Distributed Consensus in Multi-vehicle Cooperative Control*. Springer, 2008.
- [6] A. Jadbabaie, J. Lin, and A. Morse, "Coordination of groups of mobile autonomous agents using nearest neighbor rules," *IEEE Trans. Autom. Contr.*, vol. 48, no. 6, pp. 988–1001, June 2003.

- [7] G. Cybenko, "Dynamic load balancing for distributed memory multi-processors," *J. Par. Dist. Comp.*, vol. 7, no. 2, pp. 279–301, 1989.
- [8] B. Johansson, M. Rabi, and M. Johansson, "A simple peer-to-peer algorithm for distributed optimization in sensor networks," in *Proc. of 46th IEEE Conf. on Decision and Control*, Dec. 2007, pp. 4705–4710.
- [9] Z. Hao, A. Cano, and G. Giannakis, "Distributed demodulation using consensus averaging in wireless sensor networks," in *Proc. of 42nd Asilomar Conf. on Signals, Systems and Computers*, 2008.
- [10] J. Fax and R. Murray, "Information flow and cooperative control of vehicle formations," *IEEE Trans. Autom. Contr.*, vol. 49, no. 9, pp. 1465–1476, 2004.
- [11] L. Scardovi and R. Sepulchre, "Synchronization in networks of identical linear systems," *Automatica*, vol. 45, no. 11, pp. 2557–2562, 2009.
- [12] H. Trentelman, K. Takaba, and N. Monshizadeh, "Robust synchronization of uncertain linear multi-agent systems," *IEEE Trans. Autom. Contr.*, vol. 58, no. 6, pp. 1511–1523, June 2013.
- [13] I. Lestas and G. Vinnicombe, "Heterogeneity and scalability in group agreement protocols: Beyond small gain and passivity approaches," *Automatica*, vol. 46, pp. 1141–1151, 2010.
- [14] E. Lovisari and U. T. Jönsson, "A framework for robust synchronization in heterogeneous multi-agent networks," in *IEEE Conference on Decision and Control*, 2011, pp. 7268–7274.
- [15] A. Megretski and A. Rantzer, "System analysis via integral quadratic constraints," *IEEE Trans. Autom. Contr.*, vol. 42, no. 6, pp. 819–830, 1997.
- [16] M. Arcaç, "Passivity as a design tool for group coordination," *IEEE Trans. Autom. Contr.*, vol. 52, no. 8, pp. 1380–1390, 2007.
- [17] L. Scardovi, M. Arcaç, and E. Sontag, "Synchronization of interconnected systems with applications to biochemical networks: An input-output approach," *IEEE Trans. Autom. Contr.*, vol. 55, no. 6, pp. 1367–1379, 2010.
- [18] A. Rantzer and A. Megretski, "Analysis of rate limiters using integral quadratic constraints," in *IFAC Nonlinear Control Systems Design Symposium*, 1998.
- [19] U. T. Jönsson and C.-Y. Kao, "Consensus of heterogeneous LTI agents," *IEEE Trans. Autom. Contr.*, vol. 57, no. 8, pp. 2133–2139, 2012.
- [20] C. Foias, T. T. Georgiou, and M. C. Smith, "Robust stability of feedback systems: A geometric approach using the gap metric," *SIAM J. Control Optim.*, vol. 31, pp. 1518–1537, 1993.
- [21] R. Pates and G. Vinnicombe, "Stability certificates for networks of heterogeneous linear systems," in *IEEE Conference on Decision and Control*, 2012, pp. 6915 – 6920.
- [22] I. Lestas, "Large scale heterogeneous networks, the Davis-Wielandt shell and graph separation," *SIAM J. Cont. Opt.*, vol. 50, no. 4, pp. 1753–1774, 2012.
- [23] —, "On network stability, graph separation, interconnection structure and convex shells," in *Proc. IEEE Conference Dec. Cont., CDC'11*, June 2011, pp. 4257–4263.
- [24] U. Jönsson, C.-Y. Kao, and H. Fujioka, "Low dimensional stability criteria for large-scale interconnected systems," in *Proc. European Control Conference, ECC'07*, July 2007.
- [25] U. Jönsson, "Primal and dual criteria for robust stability and their application to systems interconnected over a bipartite graph," in *Proc. American Control Conference, ACC'10*, June 2010, pp. 5458–5464.
- [26] U. Münz, A. Papachristodoulou, and F. Allgöwer, "Delay robustness in consensus problems," *Automatica*, vol. 46, no. 8, pp. 1252 – 1265, 2010.
- [27] E. Nuno, R. Ortega, L. Basanez, and D. Hill, "Synchronization of networks of nonidentical Euler-Lagrange systems with uncertain parameters and communication delays," *IEEE Trans. Autom. Contr.*, vol. 56, no. 4, pp. 935–941, 2011.
- [28] G. Weiss, "Representation of shift-invariant operators on  $L^2$  by  $H^\infty$  transfer functions: An elementary proof, a generalization to  $L^p$ , and a counterexample for  $L^\infty$ ," *Math Control Signals Systems*, vol. 4, no. 2, pp. 193–203, 1991.
- [29] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*. Cambridge: Cambridge University Press, 1991.
- [30] M. Cantoni and G. Vinnicombe, "Linear feedback systems and the graph topology," *IEEE Trans. Autom. Contr.*, vol. 47, pp. 710–719, 2002.
- [31] M. Cantoni, U. T. Jönsson, and S. Z. Khong, "Robust stability analysis for feedback interconnections of time-varying linear systems," *SIAM J. Control Optim.*, vol. 51, no. 1, pp. 353–379, 2013.
- [32] T. Iwasaki and S. Hara, "Generalized KYP lemma: unified frequency domain inequalities with design applications," *IEEE Trans. Autom. Contr.*, vol. 50, no. 1, pp. 41 – 59, 2005.
- [33] I. Lestas and G. Vinnicombe, "Scalable decentralized robust stability certificates for networks of interconnected heterogeneous dynamical systems," *IEEE Trans. Autom. Contr.*, vol. 51, no. 10, pp. 1613–1625, 2006.
- [34] C. Desoer and Y.-T. Wang, "On the generalised Nyquist stability criterion," *IEEE Trans. Autom. Contr.*, vol. 25, no. 2, pp. 187–196, 1980.
- [35] M. Cantoni, U. T. Jönsson, and C.-Y. Kao, "Robustness analysis for feedback interconnections of unstable distributed systems via integral quadratic constraints," *IEEE Trans. Autom. Contr.*, vol. 57, pp. 302–317, 2012.
- [36] R. Carli and S. Zampieri, "Network clock synchronization based on the second order linear consensus algorithm," *IEEE Trans. Autom. Contr.*, vol. 59, no. 2, pp. 409–422, 2014.