Synchronization of Networks of Heterogeneous Agents with Common Nominal Behavior

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Abstract—**This paper deals with the problem of synchronization in networks of heterogeneous agents with common nominal behavior. The agents are modeled as (possibly nonlinear) perturbed versions of a common SISO nominal linear time–invariant operator, and they are interconnected via a sparse memoryless interconnection operator, coherent with the communication graph underlying the network. The network is said to synchronize if the outputs of the agents tend to align along given directions, the most important case being consensus, or agreement. The paper provides a general result which ensures synchronization of the network with robustness w.r.t uncertainties in the interconnection and in each agent's dynamics. Scalability issues are discussed in the popular scenario where the interconnection operator is a constant normal matrix, yielding the generalization of the very popular linear consensus algorithm. The wide range of applicability of the proposed criterion is shown by providing synchronization conditions in two important examples. Whenever possible, simple graphical criteria are proposed for checking the required conditions.**

I. Introduction

The last decade has seen a huge effort put by the scientific community to the study of large–scale dynamical systems, in which many autonomous agents interact in order to achieve a global goal. Each agent is usually constrained to cooperate and exchange information within a subset of the entire network, its *neighborhood*. Such constraints are usually represented by a *communication graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, in which $\mathcal{V}$ is the set of agents and $\mathcal{E}$, the set of edges, tells which pairs of agents can communicate. Namely, the presence of the edge $(k, j) \in \mathcal{E}$ implies that the agent j can receive information from the agent k. According to the information coming from its neighbors, each agent implements a *local* decision law which must be designed so that the global task is accomplished.

A typical example of global task is the decentralized stabilization of large–scale systems [1], [2], [3], such as the Internet [4]. For such systems, applying classical methods usually results in inefficient algorithms as the large number of agents yields extremely high computational load. For this reason, research is focused on *scalable* results, for which the computational burden grows linearly w.r.t the dimension of the network. In addition, the results should be *robust* w.r.t uncertainties and nonidealities, as the aggregate effect of small perturbations at the level of the agents might be much worse than their sum.

Another typical distributed task is agreement, or *consensus*. In this case, given initial values or measurements of some physical quantity, the agents exchange information in order to agree on a common value. Such a procedure can be used in a number of applications such as rendezvous of robots [5], [6], distributed estimation [7], load balancing [8], [9], sensor calibration for sensor networks [10], [11], distributed optimization [12], distributed demodulation [13], thus explaining why the consensus problem has been paid so much attention in the last years (see [14] and references within).

One of the most popular and studied algorithms to solve the consensus problem is *linear consensus*. Consider a network of N agents. Their outputs evolve according to $\mathbf{x}(t+1) = P\mathbf{x}(t)$, where vector $\mathbf{x}$ is the collection of outputs, and the matrix P is chosen to be be row–stochastic, primitive[1], and consistent with the communication graph in the sense that $P_{kj} > 0$ only if $(j, k) \in \mathcal{E}$. Under these assumptions, it can be shown that the outputs of all agents asymptotically converge to $\xi^T\mathbf{x}(0)$, where ξ is the eigenvector of P^T associated with eigenvalue 1 [15]. Many variations of this algorithm have been studied in detail, such as consensus with channel noise or quantization and consensus with switching topology [16], [17], [18].

In this paper we study a broad generalization of the consensus problem, called the *higher-order consensus problem. In particular, we model the agents as SISO systems which cooperate and exchange information to reach agreement on their outputs. Agents are not required to be stable systems, and are affected by possibly nonlinear perturbations.* Motivating examples come from the formation control problem [6], in which robots can be modeled as second order systems, clock synchronization, in which clocks are modeled as double integrators [10], [19], power network control, in which generators can be seen as oscillators with different natural frequencies [20].

This subject has been studied for formation control in [6], where the *homogeneous* network (i.e., all agents have the same dynamics) is considered. In [6], a linear feedback $\mathbf{u} = -L\mathbf{y}$ is proposed, where $\mathbf{y}$ and $\mathbf{u}$ are the output and the input of the network, respectively, and L is the Laplacian associated with the corresponding communication graph. It is shown that the output synchronization is achieved if all the nonzero

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[1] P is primitive if exists an m such that all entries of P^m are strictly positive.

eigenvalues of $-L$ stabilize the dynamics of the individual agent in feedback. Beyond [6], several papers contributed to explore this field of research. In [21], [22], the authors consider a homogeneous network in which the nominal system has a state–space realization and show how to design the input protocol to achieve synchronization of the entire state vectors, not only of the outputs. The paper [23] addresses the consensus problem in case of communication delays, and it establishes graphical conditions for consensus which relate upper bounds on the delay and the spectral structure of the communication graph. In [24], [25], instead, the perturbed consensus problem is studied for an *heterogeneous* network, in which the agents are modeled as perturbed integrators of the type $N_k(s) = h_k(s)/s$, where $h_k(s)$, $k \in \mathcal{V}$, are stable transfer functions. The papers provide scalable and powerful graphical criteria based on a generalized Nyquist criterion which ensures that consensus is achieved if the $h_k(s)$'s satisfy certain conditions. The same perturbed consensus problem, but with static interconnection operator, is studied in [2], where the input–output Integral Quadratic Constraint (IQC) theory is applied. Various other robust stability techniques have also been fruitfully employed, such as passivity [26], numerical range [27], and co-coerciveness [28], [29]. Since stability conditions from robust stability criteria might be computationally difficult to check, researchers have often tried to exploit the structure of the network in order to reduce the computational burden [30], [31], [32]. This is the path followed in the present paper also. Finally, the notion of co-coerciveness, which can be seen as a particular case of IQC, is used to prove synchronization of uncertain complex systems in [28], [29].

As an extension of our conference paper [33], this paper aims to provide a unified framework to study synchronization problems in large–scale *heterogeneous* networks in which agents are *heterogeneous* in the sense that they are perturbed versions of a nominal system. In the proposed framework, the dynamics of the each agent are governed by $h_1 + h_2 \circ \Delta \circ h_3$, where h_1, h_2, h_3 are LTI systems shared by all agents and Δ denotes a bounded perturbation which varies among the agents and could be nonlinear. The interaction among the agents is modelled by a generic operator which can be nonlinear and which lies on a fixed topology. Such a network model covers and extends many scenarios which have been studied separately in the literature. We are interested in output synchronization, in which outputs tend to align along certain directions. This recovers the usual notion of synchronization if such direction is the vector with all the entries equal to 1. We provide a general sufficient criterion for output-synchronization based on the IQC theory, [34] a general framework based on graph-separation which extends and recovers well-established stability criteria such as the small-gain theorem and the passivity theory. The main result in this paper covers the cases where the nominal dynamics of the agents are stable or unstable; the perturbation of the dynamics, as well as the interconnection among the agents, are linear or nonlinear. The criterion is applied to some important and popular scenarios. In particular, if the interconnection operator is a constant normal or reversible matrix, simple and scalable criteria involving the spectral properties of such matrices are obtained. To illustrate the possibilities of IQC we provide a numerical example in which a Popov-like IQC proves synchronization of the system whereas small-gain theorem, passivity theorem and circle criterion all fail to do it. Finally, we also consider a scenario in which the interconnection is a memoryless slope-restricted nonlinearity, which is instrumental to study a case of perturbed consensus.

The paper is organized as follows. The remaining part of this section sets up basic notation. The model we proposed for the heterogeneous networks under consideration is explained in Section II, while the main result of this paper is presented in Section III. Section IV is concerned with the application of the the main result to the case where the interconnection among the agents is governed by normal matrices or reversible matrices. The criterion that follows has a characteristic of scalability, and is applied to study clock synchronization and synchronization of leader-following networks. In Section V, systems with a particular type of nonlinear interconnection are considered and the result is applied to a perturbed consensus scenario. Finally some concluding remarks are drawn in Section VI.

Notation

The notation adopted here is fairly standard. For a set $\mathcal{N}$, $|\mathcal{N}|$ denotes its cardinality. We use symbols $\mathbb{R}$, $\mathbb{R}^n$, and $\mathbb{R}^{n\times m}$ to denote respectively the sets of real numbers, n-dimensional real vectors, and $n \times m$ real matrices. Symbols I_n and $0_{n\times m}$ are used to denote n-dimensional identity matrix and $n \times m$ zero matrix, respectively. The subscript and superscript are dropped when the dimension is clear from the context. Given a matrix M, the transposition and the conjugate transposition of M are denoted by M^T and M^*, respectively.

The Hilbert space $\mathcal{H}^n$ denotes either the continuous time signal space $\mathbf{L}_2^n[0,\infty)$ or the discrete time signal space $l_2^n(0,\infty)$, where n denotes the spatial dimension of the signals. The corresponding extended space $\mathcal{H}_e^n$ consists of signals for which $P_T v \in \mathcal{H}^n$, $\forall T \geq 0$, where P_T is the truncation operator defined as $(P_T v)(t) = v(t)$ when $t \leq T$ and $(P_T v)(t) = 0$ when $t > T$. The inner product of $\mathcal{H}$ is denoted by $\langle \cdot, \cdot \rangle$.

An operator $H : \mathcal{H}^n \mapsto \mathcal{H}^n$ is bounded if its gain $\gamma(H) := \sup_{v\in\mathcal{H}^n, v\neq 0} \|H(v)\|/\|v\|$ is bounded, where $\|\cdot\|$ is the norm on $\mathcal{H}^n$. When H is linear time-invariant (LTI), H has an equivalent representation in the frequency domain via Laplace/Z transforms, which is denoted by the same symbol. The adjoint of a bounded operator Π is denoted by Π^*. Π is called self-adjoint if $\Pi = \Pi^*$. A self-adjoint bounded LTI operator Π defines a quadratic form $\langle v, \Pi v \rangle$. Positive definiteness (positive semi-definiteness, negative definiteness, and negative semi-definiteness, respectively) of Π is denoted by $\Pi > 0$ ("$\geq$","$<$", and "$\leq$", respectively). It is well-known that a necessary and sufficient condition for $\Pi > 0$ is that the frequency representation of Π satisfies $\Pi(j\omega) = \Pi(j\omega)^* > 0$, $\forall \omega \in \mathbb{R} \cup \{\infty\}$, for the continuous-time case, or $\Pi(e^{j\omega}) = \Pi(e^{j\omega})^* > 0$, $\forall \omega \in [0, 2\pi]$, for the discrete-time case. We denote the set of bounded and causal LTI operators on $\mathcal{H}^n$ by

$\mathcal{A}^{n\times n}$ and the set of bounded self-adjoint LTI operators by $S_{\mathcal{A}}^{n\times n}$.

We denote by Ω the instability domain of the the Laplace/Z transform, and $\partial\Omega$ its boundary. In the continuous-time case we have $\Omega = \{s : \text{Re}s \geq 0\}$ and $\partial\Omega = j\mathbb{R}$, while in the discrete-time case we have $\Omega = \{z : |z| \geq 1\}$ and $\partial\Omega = e^{j[0,2\pi]}$.

Finally, the Kronecker product is denoted by $\otimes$, $\text{diag}\,(H_1, H_2) = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix}$ if H_1 and H_2 are operators, and the diagonal augmentation is defined as

$$\text{daug}\left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}\right) = \begin{bmatrix} A_{11} & 0 & A_{12} & 0 \\ 0 & B_{11} & 0 & B_{12} \\ A_{21} & 0 & A_{22} & 0 \\ 0 & B_{21} & 0 & B_{22} \end{bmatrix}$$

II. A MODEL FOR HETEROGENEOUS NETWORKS

In this section, we present a model for studying synchronization of *heterogeneous* networks of N agents. The model is aimed for a variety of different applications; it includes enough generality which allows us to recover many cases already presented in the literature. The model is aimed for a variety of different applications, and includes enough generality to recover many cases already appeared in the literature.

We are interested in heterogeneous networks, in which network which are heterogeneous in the sense that agents differ one each other. The agents are modeled as input-output operators with external outputs $y_k \in \mathcal{H}_e,\ k = 1, \ldots, N$, internal outputs $v_k \in \mathcal{H}_e,\ k = 1, \ldots, N$, interconnection inputs $u_k \in \mathcal{H}_e,\ k = 1, \ldots, N$, internal inputs $w_k \in \mathcal{H}_e,\ k = 1, \ldots, N$ and external inputs $r_k \in \mathcal{H}_e,\ k = 1, \ldots, N$, related through the *nominal dynamics*

$$y_k = h_{uy}u_k + h_{ry}r_k + h_{wy}w_k, \qquad v_k = h_{uv}u_k$$

where h_{uy}, h_{ry}, h_{wy}, h_{uv} are LTI operators and *common* for all the agents. The differences in dynamics among the agents are modeled through the internal input-output pair (w_k, v_k), which we assume to evolve according to $w_k = \Delta_k(v_k)$, where Δ_k is a bounded causal operator. Here the Δ_k operator represents the *perturbation* of k^{th} agent from the nominal dynamics; it also models nonlinearity and uncertainty in the dynamics of the agent. The following assumption will hold throughout the whole paper.

Assumption 1. *Operator h_{uv} is bounded and causal. The transfer functions representing the operators h_{uy}, h_{ry} and h_{wy} are such that*

$$h_{*y}(s) = \frac{b_{*y}(s)}{a(s)} f_{*y}(s), \quad * \in \{u, r, w\},$$

where $f_{*y} \in \mathcal{A}$, b_{*y} is a stable polynomial (i.e., no root of b_{*y} is in the instability domain Ω), and

$$a(s) = \prod_{i=1}^{m}(s - s_i)^{\rho_i}$$

where $s_i \in \Omega$ and $\deg(b_{*y}) < \deg(a)$. In other words, the agents are strictly proper and share the same unstable poles.

Remark 1. *In this paper agents are heterogeneous as they are possibly nonlinear perturbation of nominal linear systems, and as such nonlinearities, even common to all the systems, are modeled through the operator Δ only, differently from existing literature [28], [35] which considers instead the nonlinearity as part of the nominal system. Our choice is due to the fact that the IQC theory well fits with settings in which linear stable plants are in feedback with possibly nonlinear perturbations. We shall consider the more general scenario of nonlinear nominal plants for future research.*

The agents interact through the interconnection inputs $u_k, k = 1, \cdots, N$. For the k^{th} agent at time t, u_k is produced according to

$$u_k(t) = \Gamma_k(t,\, \mathbf{y}(t)) \tag{1}$$

where $\mathbf{y} := \begin{bmatrix} y_1, \cdots, y_N \end{bmatrix}^T$ and Γ_k is a bounded memoryless operator. The structure of Γ_k is determined according to the communication graph $\mathcal{G} = (V, \mathcal{E})$, as $\Gamma_k(t, \mathbf{y}(t))$ depends explicitly on $y_j(t)$ only if the edge (j, k) exists, i.e., if agent k can utilize information coming from agent j. Namely, for any $t \geq 0$, and for almost all $\mathbf{y} \in \mathbb{R}^N$,

$$\frac{\partial\Gamma_k(t,\, \mathbf{y})}{\partial y_j} \neq 0 \quad \Longleftrightarrow \quad (j, k) \in \mathcal{E}.$$

Let $\mathbf{u} := \begin{bmatrix} u_1, \cdots, u_N \end{bmatrix}^T$, $\mathbf{w} := \begin{bmatrix} w_1, \cdots, w_N \end{bmatrix}^T$, $\mathbf{r} := \begin{bmatrix} r_1, \cdots, r_N \end{bmatrix}^T$, $\mathbf{v} := \begin{bmatrix} v_1, \cdots, v_N \end{bmatrix}^T$. The complete set of equations describing our model for a heterogeneous network of interconnected agents is

$$\begin{cases} \mathbf{y} = H_{uy}\mathbf{u} + H_{ry}\mathbf{r} + H_{wy}\mathbf{w} \\ \mathbf{v} = H_{uv}\mathbf{u} \\ \mathbf{w} = \Delta(\mathbf{v}), \qquad \mathbf{u} = \Gamma(\cdot, \mathbf{y}), \end{cases} \tag{2}$$

where $H_{uy} := h_{uy}I_N$, $H_{ry} := h_{ry}I_N$, $H_{wy} := h_{wy}I_N$, $H_{uv} := h_{uv}I_N$, $\Delta := \text{diag}\,(\Delta_1, \cdots, \Delta_N)$ and $\Gamma(\cdot,\cdot) := \begin{bmatrix} \Gamma_1(\cdot,\cdot), \cdots, \Gamma_k(\cdot,\cdot) \end{bmatrix}^T$. We call Γ the *interconnection operator*.

Example 1. *In the perturbed higher–order consensus problem, already partially depicted in the Introduction, each agent is modeled as a SISO system with dynamics*

$$y_k = N_0(1 + \Delta_k)(u_k) + N_0 r_k$$

where N_0 represents a common LTI convolution operator, while Δ_k is a possibly nonlinear and dynamical perturbation operator. The consensus algorithm is $\mathbf{u}(t) = -L\mathbf{y}(t)$, where $L = I - P$ is the Laplacian of P and P is a row-stochastic matrix. The overall system can be expressed in the form of (2), with $h_{uy} = h_{ry} = h_{wy} = N_0$, $h_{uv} = I$, and $\Gamma(\cdot, \mathbf{y}) = \Upsilon\mathbf{y} = -L\mathbf{y}$.

III. SYNCHRONIZATION OVER HETEROGENEOUS NETWORKS

In this section we develop our main result for proving *synchronization* of the heterogeneous network we presented in the previous section. We define synchronization of such networks as follows.

Definition 1. *Consider the system in* (2) *and a subspace* $\mathcal{Z} \subset \mathbb{R}^N$. *Let* $\mathbf{y}_\perp = \mathcal{P}_{\mathcal{Z}_\perp}\mathbf{y}$ *be the projection of* $\mathbf{y}$ *onto the orthogonal complement of* $\mathcal{Z}$. *Let* $\mathcal{M} : \mathcal{H}_e^N \to \mathcal{H}_e^N$ *denote the causal system mapping* $\mathbf{r}$ *to* $\mathbf{y}_\perp$. *We say that the system synchronizes to* $\mathcal{Z}$ *if* $||\mathcal{M}||_{\mathcal{H}_e^N \to \mathcal{H}_e^N} < \infty$. *In this case,* $\mathcal{Z}$ *is called the synchronization subspace of the system.*

Under weak assumptions on the nominal dynamics, this definition of synchronization implies that if $\mathbf{r} \in \mathcal{H}$ then $\mathbf{y}$ asymptotically converges to $\mathcal{Z}$, which is the reason why $\mathcal{Z}$ is called the *synchronization subspace*. The typical case is $\mathcal{Z} = \operatorname{span}\{\mathbf{1}\}$, so that the term synchronization recovers its usual meaning: all the entries of $\mathbf{y}$ "synchronize" to the same trajectory. Following Definition 1, we will in this section develop a computational tool for checking the induced-norm of $\mathcal{M}$; to this end, we make the following assumptions on the operator Γ.

Assumption 2. *For any* $\mathbf{z}_1 \in \mathcal{Z}$ *and* $\mathbf{z}_2 \in \mathcal{Z}_\perp$, $\Gamma(t,\, \mathbf{z}_1 + \mathbf{z}_2) = \Gamma(t,\, \mathbf{z}_2)$ *and* $\mathbf{z}_1^*\Gamma(t, \mathbf{v}) = 0$, $\forall\, t \geq 0$, $\forall\, \mathbf{v} \in \mathbb{R}^N$.

This assumption is reasonable since it mimics the common strategy adopted in the linear consensus algorithms when the communication is bidirectional. In such cases, $\Gamma(\cdot, \mathbf{y}) = \Upsilon\mathbf{y}$ and $\Upsilon \in \mathbb{R}^{N\times N}$ can be chosen to be a symmetric matrix whose right kernel and left kernel are both spanned by $\mathbf{1}$. This means that the input should only depend on the part of the output which is not aligned to the synchronization subspace. The first part of Assumption 2 is mild, and requires the input to depend on the component of the outputs which is orthogonal to the synchronization subspace only. The second condition is stronger and implies that the input itself is othogonal to the synchronization space. The class of interconnection operators satisfying Assumption 2 covers the popular strategy of static multiplication by a matrix, $\Gamma(\cdot, \mathbf{y}) = \Upsilon\mathbf{y}$, where Υ's right kernel and left kernel are spanned by elements of $\mathcal{Z}$, both in case of directed and undirected communication. It also covers more general nonlinear odd interconnection operators in case of undirected communication, such as the example provided in Section V.

Remark 2. *The second condition in Assumption 2 is a technical condition that allows to project the system into the orthogonal of the synchronization subspace. Seeing its absence in the literature on synchronization (except in consensus when agreement is on the average of the initial conditions), and confirmed by numerical simulations not reported in this paper, we believe that it is a mere technicality which can be relaxed. We leave this issue for future research.*

Remark 3. *Assumptions 1 2 imply that the synchronization space is positively invariant in absence of external inputs* r_k. *Other works sppeared in the literature allow instead agents to be completely different one each other [29]. We notice however that in this more general case synchronization cannot be obtained with memoryless interconnection operators as those employed here. Future research will be aimed at relaxing this last assumption and, correspondingly, to extend the considered notion of heterogeneity.*

Let Z be a matrix whose columns form an orthonormal basis for $\mathcal{Z}$, and V be any orthonormal complement of Z. We have then

$$Z^*Z = I_p, \quad V^*V = I_{N-p}, \quad V^*Z = 0, \quad VV^* + ZZ^* = I_N$$

where $p = \dim \mathcal{Z}$. Note that VV^* and ZZ^* are projection operators onto $\mathcal{Z}_\perp$ and $\mathcal{Z}$, respectively. With Z and V, one can readily verify that Assumption 2 leads to the following equivalent conditions

$$\Gamma(t,\, \mathbf{y}) = \Gamma(t,\, VV^*\mathbf{y}), \quad \Gamma(t,\, \mathbf{y}) = VV^*\Gamma(t,\, \mathbf{y}), \forall t \geq 0\,.$$

Clearly in the case $\Gamma(t, \mathbf{y}) = \Upsilon\mathbf{y}$, where Υ is a constant matrix, the conditions mean that $\mathcal{Z}$ is the left and right kernel of Υ. Note that by Assumption 2, $\mathbf{u}(\cdot) = \Gamma(\cdot, \mathbf{y}(\cdot))$ implies that $\mathbf{u} = VV^*\mathbf{u}$ and therefore $\mathbf{u}(t) \in \mathcal{Z}_\perp, \forall t$. This is analogous to the linear consensus algorithm where the matrix P is double stochastic. In fact, its Laplacian $L = I - P$ satisfies $\mathbf{1}^T L = 0$, and thus $L = VV^*L$. Also notice that, since $\mathbf{v} = h_{uv}\mathbf{u}$ and $\mathbf{u} = VV^*\mathbf{u}$, we have $\mathbf{v} = VV^*\mathbf{v}$ and therefore $\mathbf{v}(t) \in \mathcal{Z}_\perp, \forall t$.

Let $\mathbf{y}_\perp := V^*\mathbf{y}$, $\mathbf{v}_\perp := V^*\mathbf{v}$, $\mathbf{u}_\perp := V^*\mathbf{u}$, $\mathbf{r}_\perp := V^*\mathbf{r}$, $\mathbf{w}_\perp := V^*\mathbf{w}$. Note that $V^*\mathbf{u}(\cdot) = V^*\boldsymbol{\Gamma}(\cdot, VV^*\mathbf{y}(\cdot))$ and $V^*\mathbf{w} = V^*\Delta(VV^*\mathbf{v})$. Thus, by defining $\Gamma_\perp(\cdot,\, \mathbf{y}_\perp(\cdot)) := V^*\Gamma(\cdot,\, V\mathbf{y}_\perp(\cdot))$ and $\Delta_\perp(\mathbf{v}_\perp) := V^*\Delta(V\mathbf{v}_\perp)$, we obtain the following reduced-dimension system mapping $\mathbf{r}_\perp$ to $\mathbf{y}_\perp$

$$\begin{cases} \mathbf{y}_\perp = H_{uy}\mathbf{u}_\perp + H_{ry}\mathbf{r}_\perp + H_{wy}\mathbf{w}_\perp \\ \mathbf{v}_\perp = H_{uv}\mathbf{u}_\perp \\ \mathbf{w}_\perp = \Delta_\perp(\mathbf{v}_\perp), \qquad \mathbf{u}_\perp = \Gamma_\perp(\cdot,\, \mathbf{y}_\perp) \end{cases} \tag{3}$$

It is worth to notice that the diagonal structure of the linear part H has been maintained after the dimension reduction at the price that the diagonal structure of the perturbation is lost.

Based on integral quadratic constraints (IQC), a theory is developed in the following subsections for verifying that the reduced-dimension system (3) has a bounded induced gain, which by definition implies the heterogeneous network system (2) synchronizes to subspace $\mathcal{Z}$.

A. A criterion for synchronization

Consider feedback configurations of the following form

$$q = Mp + e, \quad p = \Delta(q) \tag{4}$$

where p, q, $e \in \mathcal{H}_e^n$ and Δ has bounded gain. We will denote such feedback interconnections as $[M, \Delta]$. We say the interconnection $[M, \Delta]$ is *well-posed* if the map $q \mapsto e$ has a causal inverse on $\mathcal{H}_e^n$. The interconnection is *stable* if, in addition, the inverse is bounded; i.e., if there exists $c > 0$ such that $\|q\|^2 \leq c\|e\|^2$, $\forall e \in \mathcal{H}$. Note that the reduced-dimension system (3) is exactly in the form of (4), with $\Delta := \operatorname{diag}(\Gamma_\perp, \Delta_\perp)$ and

$$\begin{aligned} &M := \begin{bmatrix} H_{uy} & H_{ry} \\ H_{uv} & 0 \end{bmatrix}, \quad p := \begin{bmatrix} \mathbf{u}_\perp \\ \mathbf{w}_\perp \end{bmatrix}, \quad q := \begin{bmatrix} \mathbf{y}_\perp \\ \mathbf{v}_\perp \end{bmatrix}, \\ &e := \begin{bmatrix} H_{ry}\mathbf{r}_\perp \\ 0 \end{bmatrix}. \end{aligned} \tag{5}$$

Definition 2. *Let $\Pi \in S_{\mathcal{A}}^{2m\times 2m}$. A bounded causal operator $\Delta : \mathcal{H}_e^m \to \mathcal{H}_e^m$ is said to satisfy the IQC defined by Π (denoted as "$\Delta \in IQC(\Pi)$") if*

$$\left\langle \begin{bmatrix}\Delta(q)\\ q\end{bmatrix}, \Pi \begin{bmatrix}\Delta(q)\\ q\end{bmatrix} \right\rangle \le 0, \quad \forall\, q \in \mathcal{H}.$$

We are now ready to state our criterion for synchronization.

Theorem 1. *Consider the heterogeneous network (2) and its associated reduced-dimension system (3). Suppose there exist bounded and causal (i.e., "stable") LTI operators Q_1, Q_2, Q_3 and continuous (in the norm topology) (in the topology induced by a generalized ν-gap metric) PLEASE CHECK THIS SENTENCE, I FEAR IT IS WRONG parameterizations $\Gamma[\tau] := Q_1 + Q_2 \circ \tilde{\Gamma}[\tau] \circ Q_3$ and $\Delta[\tau]$, $\tau \in [0,1]$, such that*

- *(i) Q_3^{-1}, $(I - H_{uy}Q_1)^{-1}$, $(I - H_{uy}Q_1)^{-1}H_{uy}$, $Q_1(I - H_{uy}Q_1)^{-1}$, $Q_1(I - H_{uy}Q_1)^{-1}H_{uy}$ are stable*
- *(ii) $\Gamma[1] = \Gamma_\perp$, $\Delta[1] = \Delta_\perp$, and $\tilde{\Gamma}[\tau]$ is bounded and causal for all $\tau \in [0,1]$;*
- *(iii) the feedback interconnection $[M, \mathrm{diag}\,(\Gamma[\tau], \Delta[\tau])]$ is well-posed for all $\tau \in [0,1]$ and is stable for $\tau = 0$, where M is as defined in (5);*
- *(iv) $\Gamma[\tau] \in \mathrm{IQC}(\Pi_\Gamma)$ and $\Delta[\tau] \in \mathrm{IQC}(\Pi_\Delta)$ for all $\tau \in [0,1]$;*
- *(v) there exists $\varepsilon > 0$ such that, for all $s \in \partial\Omega$,*

$$\begin{bmatrix}\tilde{M}_1\\ \tilde{M}_2\end{bmatrix}^* \mathrm{daug}\,(\Pi_\Gamma, \Pi_\Delta) \begin{bmatrix}\tilde{M}_1\\ \tilde{M}_2\end{bmatrix}(s) \ge \varepsilon I \tag{6}$$

where

$$\tilde{M}_1 := \begin{bmatrix}(I - Q_1H_{uy})^{-1}Q_2 & Q_1(I - H_{uy}Q_1)^{-1}H_{wy}\\ 0 & I\end{bmatrix}$$
$$\tilde{M}_2 := \begin{bmatrix}(I - H_{uy}Q_1)^{-1}H_{uy}Q_2 & (I - H_{uy}Q_1)^{-1}H_{wy}\\ H_{uv}(I - Q_1H_{uy})^{-1}Q_2 & H_{uv}Q_1(I - H_{uy}Q_1)^{-1}H_{wy}\end{bmatrix} \tag{7}$$

Then the reduced-dimension system (3) is stable[2] *and the heterogeneous network* (2) *synchronizes to the subspace $\mathcal{Z}$ in the sense of Definition 1.*

Several remarks are in order.

Remark 4. *Parameterizations $\Gamma[\tau]$ and $\Delta[\tau]$ are required for applying the IQC stability theory [34], which we rely on to show stability of the reduced-dimension system (3). A common choice of such parameterizations is $\tau\Gamma_\perp$, $\tau\Delta_\perp$.*

Remark 5. *The purpose of expressing $\Gamma[\tau]$ as $Q_1 + Q_2 \circ \tilde{\Gamma}[\tau] \circ Q_3$ is to perform a loop transformation so that the feedback interconnection $[M, \mathrm{diag}\,(\Gamma[\tau], \Delta[\tau])]$ is equivalent to a feedback interconnection of two* stable *operators. Replacing $\Gamma[\tau]$ by $Q_1 + Q_2 \circ \tilde{\Gamma}[\tau] \circ Q_3$, one can readily verify that the interconnection $[M, \mathrm{diag}\,(\Gamma[\tau], \Delta[\tau])]$ is equivalent to $[G, \mathrm{diag}(\tilde{\Gamma}[\tau], \Delta[\tau])]$, where*

$$G = \begin{bmatrix}Q_3(I - H_{uy}Q_1)^{-1}H_{uy}Q_2 & Q_3(I - H_{uy}Q_1)^{-1}H_{wy}\\ H_{uv}(I - Q_1H_{uy})^{-1}Q_2 & H_{uv}Q_1(I - H_{uy}Q_1)^{-1}H_{wy}\end{bmatrix}$$
$$:= \begin{bmatrix}G_{11} & G_{12}\\ G_{21} & G_{22}\end{bmatrix} \tag{8}$$

This equivalent transformation is illustrated in Figure 1. Boundedness of $\mathrm{diag}(\tilde{\Gamma}[\tau], \Delta[\tau])$ *is by assumption, while stability of G will be explained in the next remark. Also note that, for given $\Gamma[\tau]$, Q_1, Q_2, and Q_3, if Q_2^{-1} and Q_3^{-1} are stable, one can always express $\Gamma[\tau]$ as $Q_1 + Q_2 \circ \tilde{\Gamma}[\tau] \circ Q_3$ by setting $\tilde{\Gamma}[\tau]$ to be $Q_2^{-1}(\Gamma[\tau] - Q_1)Q_3^{-1}$.*

[2]Note that the reduced-dimension system (3) is precisely the feedback interconnection $[M, \mathrm{diag}(\Gamma[1], \Delta[1])]$.

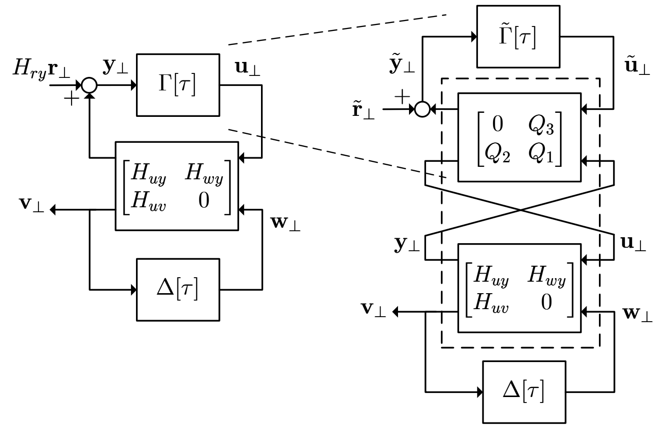

Fig. 1. Illustration of the loop transformation of the reduced-dimension system. By replacing $\Gamma[\tau]$ by $Q_1 + Q_2 \circ \tilde{\Gamma}[\tau] \circ Q_3$, the the reduced-dimension system in the left can be equivalently expressed as the system in the right, where $\tilde{\mathbf{r}}_\perp := Q_3(I - H_{uy}Q_1)^{-1}H_{ry}\mathbf{r}_\perp$. The LTI part in the dash box is G, as given in (8). Note that by condition (i) of Theorem 1, $Q_3(I - H_{uy}Q_1)^{-1}H_{ry}$ is stable and so is G as explained in Remark 6.

Remark 6. *Stability of $(I - H_{uy}Q_1)^{-1}H_{uy}$ implies stability of $(I - H_{uy}Q_1)^{-1}H_{wy}$ and $(I - H_{uy}Q_1)^{-1}H_{ry}$, since This is due to that h_{uy}, h_{wy}, and h_{ry} have the same unstable poles. Likewise, stability of $Q_1(I - H_{uy}Q_1)^{-1}H_{uy}$ implies stability of $Q_1(I - H_{uy}Q_1)^{-1}H_{wy}$, which also implies stability of $(I - Q_1H_{uy})^{-1}$ because $I + Q_1(I - H_{uy}Q_1)^{-1}H_{uy} = (I - Q_1H_{uy})^{-1}$. Therefore, by condition (i), all components of G, $\tilde{M}_1$ and $\tilde{M}_2$ are stable and so are the three operators. This means $\tilde{M}_1(s)$ and $\tilde{M}_2(s)$ have no singular points on $\partial\Omega$ and thus condition (v) is well-posed. This also means that the loop transformed system shown in the right-hand-side of Figure 1 is a feedback interconnection of two stable operators driven by $\mathbf{r}_\perp$ passing through a stable filter $Q_3(I - H_{uy}Q_1)^{-1}H_{ry}$.*

Remark 7. *By the expression $\Gamma_\perp = Q_1 + Q_2 \circ \tilde{\Gamma}[1] \circ Q_3$, one can intuitively view Q_1 as an approximation of $\Gamma_\perp$ with $Q_2 \circ \tilde{\Gamma}[1] \circ Q_3$ being the perturbation of $\Gamma_\perp$ from Q_1. Thus, the more information one has on the operator $\Gamma_\perp$, the better Q_1 can be selected. Furthermore, condition (i) requires Q_1 to* internally *stabilize H_{uy}. For a given H_{uy}, all stabilizing Q_1 can be found by the well-known Youla parametrization.*

Remark 8. *Condition* (6) *should be in principle checked for all frequencies. However, making use of the Kalman-Yakubovich-Popov lemma, we can readily see that, in continuous time, if* $\mathrm{daug}\,(\Pi_\Gamma(s), \Pi_\Delta(s)) = \begin{bmatrix}\Psi_1(s) & \Psi_2(s)\end{bmatrix}^* M \begin{bmatrix}\Psi_1(s) & \Psi_2(s)\end{bmatrix}$ *and* $\Psi_1(s)\tilde{M}_1(s) + \Psi_2(s)\tilde{M}_2(s)$ *is realized by* (A, B, C, D), *then* (6) *is equivalent to the existence of* $P = P^* > 0$ *such that*

$$\begin{bmatrix}PA + A'P & PB\\ B^TP & 0\end{bmatrix} + \begin{bmatrix}C^T\\ D^T\end{bmatrix} M \begin{bmatrix}C & D\end{bmatrix} < 0\,.$$

Even if this is a convex problem, it could be computationally *intractable for large-scale systems. However, such a computational burden can be greatly reduced in some special and important cases which will be addressed later in the paper.*

Remark 9 (Synchronization conditions for the original system)**.** *Suppose Q_2^{-1} is bounded. Then condition (v) can also be formulated as: there exists $\epsilon > 0$ such that*

$$\begin{bmatrix} I \\ M \end{bmatrix}^* \text{daug}\left(\Pi_\Gamma, \Pi_\Delta\right) \begin{bmatrix} I \\ M \end{bmatrix}(s) \geq \epsilon I, \quad s \in \partial\Omega \setminus \{s_i\}_{i=1}^m \tag{9}$$

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix}^* \text{daug}\left(\Pi_\Gamma, \Pi_\Delta\right) \begin{bmatrix} T_1 \\ T_2 \end{bmatrix}(s_i) > 0, \quad i = 1, \cdots, m \tag{10}$$

where

$$T_1(s) = \begin{bmatrix} 0 & -\tilde{h}_{wy}(s)\tilde{h}_{uy}(s)^{-1} \\ 0 & I \end{bmatrix},$$

$$T_2(s) = \begin{bmatrix} -Q_1(s)^{-1}Q_2(s) & -\tilde{h}_{wy}(s)\tilde{h}_{uy}(s)^{-1}Q_1(s)^{-1} \\ 0 & -h_{uv}(s)\tilde{h}_{wy}(s)\tilde{h}_{uy}(s)^{-1} \end{bmatrix},$$

$$\tilde{h}_{uy}(s) = \prod_{i=1}^m (s - s_i)^{\rho_i} h_{uy}(s), \quad \tilde{h}_{wy}(s) = \prod_{i=1}^m (s - s_i)^{\rho_i} h_{wy}(s)$$

and we recall $\{s_i\}_{i=1}^m$ denotes the set of singular points (poles) of h_{uy}, h_{ry}, h_{wy} on $\partial\Omega$. To see this, we note that $\tilde{M}_2(s) = M(s)\tilde{M}_1(s)$. Hence, for $s \in \partial\Omega \setminus \{s_i\}_{i=1}^m$,

$$\begin{aligned} &\begin{bmatrix} \tilde{M}_1 \\ \tilde{M}_2 \end{bmatrix}^* \text{daug}\left(\Pi_\Gamma, \Pi_\Delta\right) \begin{bmatrix} \tilde{M}_1 \\ \tilde{M}_2 \end{bmatrix}(s) \geq \varepsilon I \\ \Leftrightarrow \ &\begin{bmatrix} I \\ M \end{bmatrix}^* \text{daug}\left(\Pi_\Gamma, \Pi_\Delta\right) \begin{bmatrix} I \\ M \end{bmatrix}(s) \geq \varepsilon(\tilde{M}_1(s)\tilde{M}_1^*(s))^{-1} \end{aligned} \tag{11}$$

Boundedness of Q_2^{-1} ensures $\tilde{M}_1(s)^{-1}$ is well-defined on $\partial\Omega$; therefore the right-hand-side inequality of (11) is equivalent to inequality (9). Moreover, by Assumption 1, one can readily express $\tilde{M}_1(s)$ and $\tilde{M}_2(s)$ as in (12), where $p(s) = \prod_{i=1}^m (s - s_i)^{\rho_i}$. Since $p(s_i) = 0$ for $i = 1, \cdots, m$, we see that for $s \in \{s_i\}_{i=1}^m$, $\tilde{M}_1(s) = T_1(s)$ and $\tilde{M}_2(s) = T_2(s)$. For these s_i, inequality (6) is equivalent to inequality (10).

B. Proof of Theorem 1

The development of Theorem 1 is based on the IQC stability theory [34]. As explained in the previous subsection, the reduced-dimension system $[M, \text{diag}\left(\Gamma[\tau], \Delta[\tau]\right)]$ is equivalent to a feedback interconnection of two bounded and causal operators G and $\text{diag}(\tilde{\Gamma}[\tau], \Delta[\tau])$, for which the standard IQC stability theory can be applied. Note that the IQC theory is not applicable to system $[M, \text{diag}\left(\Gamma[\tau], \Delta[\tau]\right)]$ since the H_{uy}, H_{ry}, H_{wy} are not bounded operators. In the followings, we will show that the conditions stated in Theorem 1 implies that G and $\text{diag}(\tilde{\Gamma}[\tau], \Delta[\tau])$ satisfies all assumptions of the standard IQC stability theorem; therefore, by the theorem the system $[G, \text{diag}(\tilde{\Gamma}[\tau], \Delta[\tau])]$ is stable, which in turn implies stability of $[M, \text{diag}\left(\Gamma[\tau], \Delta[\tau]\right)]$.

First of all, since $[M, \text{diag}\left(\Gamma[\tau], \Delta[\tau]\right)]$ is equivalent to $[G, \text{diag}(\tilde{\Gamma}[\tau], \Delta[\tau])]$, condition (iii) implies that the interconnected system $[G, \text{diag}(\tilde{\Gamma}[\tau], \Delta[\tau])]$ is well-posed for all $\tau \in [0, 1]$ and $[G, \text{diag}(\tilde{\Gamma}[0], \Delta[0])]$ is stable.

Secondly, we note that $\Gamma[\tau] \in \text{IQC}(\Pi_\Gamma)$ implies that $\tilde{\Gamma}[\tau] \in \text{IQC}(\Phi^*\Pi_\Gamma\Phi)$, where

$$\Phi = \begin{bmatrix} Q_2 & Q_1Q_3^{-1} \\ 0 & Q_3^{-1} \end{bmatrix}.$$

To see this, let $(\tilde{u}, \tilde{y})$ satisfy $\tilde{u} = \tilde{\Gamma}[\tau](\cdot, \tilde{y})$ and $u = Q_1Q_3^{-1}\tilde{y} + Q_2\tilde{u}$. Clearly $u = Q_1(Q_3^{-1}\tilde{y}) + Q_2\tilde{\Gamma}[\tau](\cdot, Q_3(Q_3^{-1}\tilde{y}))$ and thus $(u, Q_3^{-1}\tilde{y})$ satisfies $u = \Gamma[\tau](\cdot, Q_3^{-1}\tilde{y})$. Since $\Gamma[\tau] \in \text{IQC}(\Pi_\Gamma)$, we have

$$\begin{aligned} \left\langle \begin{bmatrix} u \\ Q_3^{-1}\tilde{y} \end{bmatrix}, \Pi_\Gamma \begin{bmatrix} u \\ Q_3^{-1}\tilde{y} \end{bmatrix} \right\rangle &= \left\langle \Phi \begin{bmatrix} \tilde{u} \\ \tilde{y} \end{bmatrix}, \Pi_\Gamma \Phi \begin{bmatrix} \tilde{u} \\ \tilde{y} \end{bmatrix} \right\rangle \\ &= \left\langle \begin{bmatrix} \tilde{u} \\ \tilde{y} \end{bmatrix}, \Phi^*\Pi_\Gamma\Phi \begin{bmatrix} \tilde{u} \\ \tilde{y} \end{bmatrix} \right\rangle \leq 0 \end{aligned}$$

This implies $\tilde{\Gamma}[\tau] \in \text{IQC}(\Phi^*\Pi_\Gamma\Phi)$. Furthermore, since $\Delta[\tau] \in \text{IQC}(\Pi_\Delta)$, one can readily verify that $\text{diag}(\tilde{\Gamma}[\tau], \Delta[\tau]) \in \text{IQC}(\text{daug}\left(\Phi^*\Pi_\Gamma\Phi, \Pi_\Delta\right))$. By condition (iv), this holds for all $\tau \in [0, 1]$.

Finally, we show that condition (v) implies that there exists $\varepsilon > 0$ such that, for all $s \in \partial\Omega$,

$$\begin{bmatrix} I \\ G \end{bmatrix}^* \text{daug}\left(\Phi^*\Pi_\Gamma\Phi, \Pi_\Delta\right) \begin{bmatrix} I \\ G \end{bmatrix}(s) \geq \varepsilon I. \tag{13}$$

To see this, we note that

$$\begin{aligned} &\begin{bmatrix} I \\ G \end{bmatrix}^* \text{daug}\left(\Phi^*\Pi_\Gamma\Phi, \Pi_\Delta\right) \begin{bmatrix} I \\ G \end{bmatrix} = \begin{bmatrix} \tilde{G}_1 \\ \tilde{G}_2 \end{bmatrix}^* \text{diag}\left(\Phi^*\Pi_\Gamma\Phi, \Pi_\Delta\right) \begin{bmatrix} \tilde{G}_1 \\ \tilde{G}_2 \end{bmatrix} \\ &= \begin{bmatrix} \Phi\tilde{G}_1 \\ \tilde{G}_2 \end{bmatrix}^* \text{diag}\left(\Pi_\Gamma, \Pi_\Delta\right) \begin{bmatrix} \Phi\tilde{G}_1 \\ \tilde{G}_2 \end{bmatrix} \end{aligned}$$

where $\tilde{G}_1 := \begin{bmatrix} I & 0 \\ G_{11} & G_{12} \end{bmatrix}$ and $\tilde{G}_2 := \begin{bmatrix} 0 & I \\ G_{21} & G_{22} \end{bmatrix}$. One can readily verify that $\Phi\tilde{G}_1$ is equal to

$$\begin{bmatrix} Q_2 + Q_1(I - H_{uy}Q_1)^{-1}H_{uy}Q_2 & Q_1(I - H_{uy}Q_1)^{-1}H_{wy} \\ (I - H_{uy}Q_1)^{-1}H_{uy}Q_2 & (I - H_{uy}Q_1)^{-1}H_{wy} \end{bmatrix}$$

Since $Q_2 + Q_1(I - H_{uy}Q_1)^{-1}H_{uy}Q_2 = (I - Q_1H_{uy})^{-1}Q_2$, we see that $\begin{bmatrix} \Phi\tilde{G}_1 \\ \tilde{G}_2 \end{bmatrix}^* \text{diag}\left(\Pi_\Gamma, \Pi_\Delta\right) \begin{bmatrix} \Phi\tilde{G}_1 \\ \tilde{G}_2 \end{bmatrix}$ is equal to $\begin{bmatrix} \tilde{M}_1 \\ \tilde{M}_2 \end{bmatrix}^* \text{daug}(\Pi_\Gamma, \Pi_\Delta) \begin{bmatrix} \tilde{M}_1 \\ \tilde{M}_2 \end{bmatrix}$, where $\begin{bmatrix} \tilde{M}_1 \\ \tilde{M}_2 \end{bmatrix}$ is defined in (7). Thus, inequality (6) is equivalent to inequality inequality (13).

In summary, we have shown that by the conditions listed in Theorem 1,

(1) the interconnection $[G, \text{diag}(\tilde{\Gamma}[\tau], \Delta[\tau])]$ is well-posed for all $\tau \in [0, 1]$ and is stable for $\tau = 0$;
(2) $\text{diag}(\tilde{\Gamma}[\tau], \Delta[\tau]) \in \text{IQC}(\text{daug}\left(\Phi^*\Pi_\Gamma\Phi, \Pi_\Delta\right))$ for all $\tau \in [0, 1]$;
(3) G satisfies inequality (13) for all $s \in \partial\Omega$.

As such, G and $\text{diag}(\tilde{\Gamma}[\tau], \Delta[\tau])$ satisfy all the assumptions of the IQC stability of [34], and therefore by the theorem we conclude that $[G, \text{diag}(\tilde{\Gamma}[\tau], \Delta[\tau])]$ is stable for all $\tau \in [0, 1]$, in particular for $\tau = 1$. When $\tau = 1$, $[G, \text{diag}(\tilde{\Gamma}[1], \Delta[1])] = [M, \text{diag}\left(\Gamma_\perp, \Delta_\perp\right)]$, which is the reduced-dimension system (3). In this case, as illustrated in Figure 1, the external input to $[G, \text{diag}(\tilde{\Gamma}[1], \Delta[1])]$ is $Q_3(I - H_{uy}Q_1)^{-1}H_{ry}\mathbf{r}_\perp$. Stability of $[G, \text{diag}(\tilde{\Gamma}[1], \Delta[1])]$ and $Q_3(I - H_{uy}Q_1)^{-1}H_{ry}$ implies there exists $\tilde{c} > 0$ such that

$$\|\tilde{\mathbf{y}}_\perp\|^2 + \|\mathbf{v}_\perp\|^2 \leq \tilde{c}\|\mathbf{r}_\perp\|^2$$

where $\tilde{\mathbf{y}}_\perp = Q_3\mathbf{y}_\perp$. This in turn implies $\|\mathbf{y}_\perp\|^2 = \|Q_3^{-1}\tilde{\mathbf{y}}_\perp\|^2 \leq \|Q_3^{-1}\|^2\|\tilde{\mathbf{y}}_\perp\|^2 \leq \tilde{c}\|Q_3^{-1}\|^2\|\mathbf{r}_\perp\|^2$. Since Q_3^{-1} is stable and clearly $\|\mathbf{r}_\perp\| \leq \|\mathbf{r}\|$, we conclude that

$$\tilde{M}_1(s) := \begin{bmatrix} p(s)\left(p(s)I - \tilde{h}_{uy}(s)Q_1(s)\right)^{-1} Q_2(s) & \tilde{h}_{wy}(s)Q_1(s)\left(p(s)I - \tilde{h}_{uy}(s)Q_1(s)\right)^{-1} \\ 0 & I \end{bmatrix},$$
$$\tilde{M}_2(s) := \begin{bmatrix} \tilde{h}_{uy}(s)\left(p(s)I - \tilde{h}_{uy}(s)Q_1(s)\right)^{-1} Q_2(s) & \tilde{h}_{wy}(s)\left(p(s)I - \tilde{h}_{uy}(s)Q_1(s)\right)^{-1} \\ h_{uv}(s)p(s)\left(p(s)I - \tilde{h}_{uy}(s)Q_1(s)\right)^{-1} Q_2(s) & h_{uv}(s)\tilde{h}_{wy}(s)Q_1(s)\left(p(s)I - \tilde{h}_{uy}(s)Q_1(s)\right)^{-1} \end{bmatrix} \tag{12}$$

$\|\mathbf{y}_\perp\| \le c\|\mathbf{r}\|$ for some constant $c > 0$ and the reduced-dimension system has finite gain. Therefore, by Definition 1, the heterogeneous system 2 synchronizes to the subspace $\mathcal{Z}$.

C. An IQC for the structured operator $\Delta_\perp$

We end this section by a technical result which will be used several times in the remaining part of the paper. It shows that if all individual Δ_k satisfies IQC defined by the same multiplier, the operator $\Delta_\perp$ satisfies an IQC obtained by aggregating the IQC for the individual Δ_k.

Lemma 1. *Consider $\Delta := \mathrm{diag}(\Delta_1, \cdots, \Delta_N)$, where $\Delta_k : \mathcal{H} \mapsto \mathcal{H}$, $k = 1, \cdots, N$ are bounded and causal operators. Let $\Delta_\perp(\cdot) := V^*\Delta(V\cdot)$ as defined in (3), where the columns of V form a set of orthornormal basis of some p-dimensional subspace of $\mathbb{R}^N$. Suppose $\Delta_k \in \mathrm{IQC}(\Pi)$, $k = 1, \cdots, N$, and $\Pi_{11} \ge 0$. Then $\Delta_\perp \in \mathrm{IQC}(\Pi \otimes I_p)$.*

Proof. Let $v \in \mathcal{H}^p$ and $\tilde{v} = Vv$. Since $\Delta_\perp(v) = V^*\Delta(Vv) = V^*\Delta(\tilde{v})$, we observe

$$\begin{aligned} &\left\langle \begin{bmatrix}\Delta_\perp(v)\\ v\end{bmatrix}, \Pi \otimes I_p \begin{bmatrix}\Delta_\perp(v)\\ v\end{bmatrix}\right\rangle \\ &= \langle V^*\Delta(\tilde{v}), \Pi_{11}V^*\Delta(\tilde{v})\rangle + 2\langle V^*\Delta(\tilde{v}), \Pi_{12}v\rangle + \langle v, \Pi_{22}v\rangle \\ &= \langle \Delta(\tilde{v}), \Pi_{11}VV^*\Delta(\tilde{v})\rangle + 2\langle \Delta(\tilde{v}), \Pi_{12}Vv\rangle + \langle Vv, \Pi_{22}Vv\rangle \\ &= \langle \Delta(\tilde{v}), \Pi_{11}VV^*\Delta(\tilde{v})\rangle + 2\langle \Delta(\tilde{v}), \Pi_{12}\tilde{v}\rangle + \langle \tilde{v}, \Pi_{22}\tilde{v}\rangle \end{aligned}$$

Here we are using the fact that Π_{11}, Π_{12}, Π_{22} are scalar, $V^*V = I_p$, and that $\langle A^*w_1, w_2\rangle = \langle w_1, Aw_2\rangle$. Thus,

$$\begin{aligned} \left\langle \begin{bmatrix}\Delta_\perp(v)\\ v\end{bmatrix}, \Pi \otimes I_p \begin{bmatrix}\Delta_\perp(v)\\ v\end{bmatrix}\right\rangle &= \left\langle \begin{bmatrix}\Delta(\tilde{v})\\ \tilde{v}\end{bmatrix}, \Pi \otimes I_N \begin{bmatrix}\Delta(\tilde{v})\\ \tilde{v}\end{bmatrix}\right\rangle \\ &+ \langle \Delta(\tilde{v}), \Pi_{11}(VV^* - I_N)\Delta(\tilde{v})\rangle \end{aligned}$$

Since $VV^* - I_N \le 0$ and $\Pi_{11} \ge 0$, we have

$$\begin{aligned} &\left\langle \begin{bmatrix}\Delta_\perp(v)\\ v\end{bmatrix}, \Pi \otimes I_p \begin{bmatrix}\Delta_\perp(v)\\ v\end{bmatrix}\right\rangle \le \left\langle \begin{bmatrix}\Delta(\tilde{v})\\ \tilde{v}\end{bmatrix}, \Pi \otimes I_N \begin{bmatrix}\Delta(\tilde{v})\\ \tilde{v}\end{bmatrix}\right\rangle \\ &= \sum_{k=1}^{N} \left\langle \begin{bmatrix}\Delta_k(\tilde{v}_k)\\ \tilde{v}_k\end{bmatrix}, \Pi \begin{bmatrix}\Delta_k(\tilde{v}_k)\\ \tilde{v}_k\end{bmatrix}\right\rangle \le 0 \end{aligned}$$

The last inequality follows from $\Delta_k \in \mathrm{IQC}(\Pi)$. This concludes the proof. ☐

IV. Synchronization with Normal Interconnection Matrices

Theorem 1 in Section III provides a general tool to study the synchronization properties of a network of heterogeneous agents. In this section we consider the simplest choice for the interconnection operator; namely $\Gamma(\cdot, \mathbf{y}) := \Upsilon\mathbf{y}$, where Υ is a constant matrix, as in the linear consensus algorithm. Furthermore, we will make the following assumptions for Υ.

Assumption 3. *The matrix Υ satisfies Assumption 2, is normal (i.e., $\Upsilon\Upsilon^* = \Upsilon^*\Upsilon$), and synchronizes the nominal system to the subspace $\mathcal{Z} := \ker\Upsilon$.*

Normality of Υ implies that it is orthogonally diagonalizable, and this allows us to characterize the synchronization properties by making use of its spectral properties only. Let $U := \begin{bmatrix}V & Z\end{bmatrix} \in \mathbb{R}^{N\times N}$ be an unitary matrix that diagonalizes Υ, where the columns of Z form an orthonormal basis for $\mathcal{Z}$. Then $\Upsilon = V\Upsilon_\perp V^*$, where $\Upsilon_\perp = \mathrm{diag}\{\lambda_1, \ldots, \lambda_p\}$ and $\lambda_1, \ldots, \lambda_p$ are the non-zero eigenvalues of Υ. Clearly, the columns of V form an orthonormal basis for $\mathcal{Z}_\perp$ and $\Upsilon_\perp = V^*\Upsilon V$. We are now ready to state and prove the main result of this section.

Theorem 2. *Consider the heterogeneous network* (2) *and its associated reduced-dimension system (3), in which $\Gamma(t, \mathbf{y}) = \Upsilon\mathbf{y}$, where Υ respects Assumption 3. Suppose*

(i) $[M, \mathrm{diag}(\Upsilon_\perp, \tau\Delta_\perp)]$ is well-posed for all $\tau \in [0, 1]$, where M is defined in (5);

(ii) $\Delta_k \in \mathrm{IQC}(\Pi)$, for all $k = 1, \cdots, N$, where $\Pi \in S^{2\times 2}_{\mathcal{A}}$ with $\Pi_{11} \ge 0$ and $\Pi_{22} \le 0$;

(iii) there exists an $\varepsilon > 0$ such that, for all $s \in \partial\Omega$,

$$\begin{bmatrix} 1 \\ \frac{\lambda_k h_{uv} h_{wy}}{1 - \lambda_k h_{uy}} \end{bmatrix}^* \Pi \begin{bmatrix} 1 \\ \frac{\lambda_k h_{uv} h_{wy}}{1 - \lambda_k h_{uy}} \end{bmatrix}(s) \ge \varepsilon, \quad k = 1, \cdots, p \tag{14}$$

where $\lambda_1, \ldots, \lambda_p$ are nonzero eigenvalues of Υ.

Then system (2) *synchronizes to the subspace $\mathcal{Z}$.*

Remark 10. *We note that conditions stated in Theorem 2 exhibit a* scalability *property. In contrast to condition (6) of Theorem 1 where the dynamics of the* collective *of agents are involved, condition (14) involves only the dynamics of* one single agent *(to satisfy p constraints). The computational burden required to verify the condition is thus tremendously reduced. Even more so, as already pointed out in [2], a "plug-and-play" strategy can be employed, in which if a new agent is added to a pre-existing network, synchronization is maintained as long as conditions involving the eigenvalues of the interconnection matrix Υ are satisfied. There are classes of communication graphs for which this argument is particularly appealing. For example, consider an undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, and consider its unweighted Laplacian matrix*

$$L_{ij} = \begin{cases} -1, & (i,j) \in \mathcal{E} \\ |\mathcal{N}_i|, & i = j \\ 0, & \text{otherwise.} \end{cases}$$

and let the interconnection matrix be $\Upsilon = -\varepsilon L$. Since Υ is symmetric all its eigenvalues lie within an interval which can be bounded using geometric properties of the graph - see e.g. [36] for a construction using Poincaré and Cheeger bounds. Thus, when adding a new agent, one only has to check whether simple geometric properties of the graph are maintained, for the eigenvalues to lie in a given interval. In turn, the frequency-wise inequality ensuring synchronization is then required to be verified in this interval only.

Remark 11. *The strategy proposed in Remark 8 can be profitably used in the case of a normal interconnection matrix. Indeed, instead of a single problem with dimension $2(N-1)$, we have $N-1$ problems (one for each nonzero eigenvalue of Υ) of dimension 2, which is computationally much more favorable.*

Proof. To apply Theorem 1, we choose $Q_1 := \Upsilon_\perp$, $Q_2 := I$, $Q_3 := I$, and $\tilde{\Gamma}[\tau] := \tau \cdot 0 \equiv 0$. That is, $\Gamma[\tau] := \Upsilon_\perp + \tau \cdot 0 \equiv \Upsilon_\perp$. With this selection, condition (i) of Theorem 1 is satisfied, as a consequence of the assumption that $\mathbf{u} = \Upsilon \mathbf{y}$ synchronizes the nominal system to the $\mathcal{Z}$. Indeed, as proven in [6], a necessary and sufficient condition for the synchronization of the nominal system to $\mathcal{Z}$ is that the transfer functions $\frac{h_{ry}}{1-\lambda_k h_{uy}}$ are stable for any nonzero eigenvalue $\lambda_1, \cdots, \lambda_p$ of Υ. This implies that the four systems $(I - H_{uy}Q_1)^{-1}$, $(I - H_{uy}Q_1)^{-1}H_{uy}$, $Q_1(I - H_{uy}Q_1)^{-1}$, $Q_1(I - H_{uy}Q_1)^{-1}H_{uy}$ stated in condition (i) of Theorem 1 must be stable. Moreover, let $\Delta[\tau] := \tau\Delta_\perp$. By assumption (i), we have $[M, \mathrm{diag}(\Gamma[\tau], \Delta[\tau])]$ is well-posed for all $\tau \in [0, 1]$. Moreover, we have $[M, \mathrm{diag}(\Gamma[0], \Delta[0])]$ being stable.

Since $\Gamma[\tau] \equiv \Upsilon_\perp$, one can readily verify that, for any τ, $\Gamma[\tau]$ satisfies IQC defined by the multiplier

$$\Pi_\Gamma := \begin{bmatrix} \nu I_p & -\nu\Upsilon_\perp \\ -\nu\Upsilon_\perp^* & \nu\Upsilon_\perp^*\Upsilon_\perp \end{bmatrix}$$

where ν is a real number. By Lemma 1, all Δ_k satisfy IQC defined by Π implies $\Delta_\perp$ satisfies IQC defined by $\Pi_\Delta := \Pi \otimes I_p$. Furthermore, since $\Pi_{11} \geq 0$ and $\Pi_{22} \leq 0$, one can readily verifies $\Delta_\perp \in \mathrm{IQC}(\Pi \otimes I_p)$ implies $\Delta[\tau] := \tau\Delta_\perp \in \mathrm{IQC}(\Pi \otimes I_p)$ for all $\tau \in [0, 1]$.

It only remains to check condition (v) of Theorem 1. Notice that, for the given Q_1, Q_2, Q_3, Π_Γ, and Π_Δ, the corresponding equation (6) reduces to $\hat{M}_1^*\Pi_\Gamma\hat{M}_1 + \hat{M}_2^*\Pi_\Gamma\hat{M}_2 \geq \varepsilon I$, where

$$\hat{M}_1 = \begin{bmatrix} (I - \Upsilon_\perp H_{uy})^{-1} & \Upsilon_\perp(I - H_{uy}\Upsilon_\perp)^{-1}H_{wy} \\ (I - H_{uy}\Upsilon_\perp)^{-1}H_{uy} & (I - H_{uy}\Upsilon_\perp)^{-1}H_{wy} \end{bmatrix}$$

$$\hat{M}_2 = \begin{bmatrix} 0 & I \\ H_{uv}(I - \Upsilon_\perp H_{uy})^{-1} & H_{uv}\Upsilon_\perp(I - H_{uy}\Upsilon_\perp)^{-1}H_{wy} \end{bmatrix}$$

One can readily verify that $\hat{M}_1^*\Pi_\Gamma\hat{M}_1 + \hat{M}_2^*\Pi_\Gamma\hat{M}_2$ are in the form of

$$\begin{bmatrix} \nu I + (\star) & (\star) \\ (\star) & (\diamondsuit) \end{bmatrix}$$

where $(\star)$ denotes bounded terms of no significance and $(\diamondsuit)$ is equal to

$$\begin{bmatrix} I \\ \hat{M}_{2,22} \end{bmatrix}^* (\Pi \otimes I_p) \begin{bmatrix} I \\ \hat{M}_{2,22} \end{bmatrix}$$

Here, $\hat{M}_{2,22} = H_{uv}\Upsilon_\perp(I - H_{uy}\Upsilon_\perp)^{-1}H_{wy}$. Since ν can be any real number, by Schur's complement, $\hat{M}_1^*\Pi_\Gamma\hat{M}_1 + \hat{M}_2^*\Pi_\Gamma\hat{M}_2$ is strictly positive definite if and only if $(\diamondsuit)$ is strictly positive definite. Finally, note that $(\diamondsuit)$ is diagonal, with the k^{th} entry being

$$\begin{bmatrix} 1 \\ \frac{\lambda_k h_{uv} h_{wy}}{1-\lambda_k h_{uy}} \end{bmatrix}^* \Pi \begin{bmatrix} 1 \\ \frac{\lambda_k h_{uv} h_{wy}}{1-\lambda_k h_{uy}} \end{bmatrix}.$$

Therefore, $(\diamondsuit)(s) > \varepsilon I, \forall s \in \partial\Omega$ if and only if (14) holds. This concludes the proof. $\square$

In the remaining of this section, we consider two specific types of consensus problems for which Theorem 2 is applied to obtain conditions for synchronization. The first problem concerns a heterogeneous network with nonlinear agents, while the second problem concerns the so-called "reversible interconnection".

A. Quasi-saturation in the Interconnection Inputs

We consider a higher-order consensus problem in the continuous-time setting, where $h_{uy} = h_{ry} = h_{wy} = h$, $h_{uv} = 1$, and Δ_k is a nonlinear function of the following form

$$\Delta_k(u)(t) = \begin{cases} 0, & |u(t)| \leq \bar{u} \\ -\phi_k(t,\, u(t) - \mathrm{sgn}(u(t))\bar{u}), & |u(t)| > \bar{u} \end{cases}$$

where $\phi_k(\cdot,\, v)$ is an odd memoryless nonlinearity such that $\phi_k(t,\, 0) = 0$, $\forall\, t \geq 0$. We assume the function ϕ_k is such that Δ_k's satisfy the following slope condition

$$-\alpha_{\min} \leq \frac{\Delta_k(x_1)(t) - \Delta_k(x_2)(t)}{x_1(t) - x_2(t)} \leq 0, \quad \text{for all } x_1(t) \neq x_2(t) \tag{15}$$

where $0 \leq \alpha_{\min} < 1$. This setup leads to the higher–order consensus problem of the form

$$\mathbf{y} = h(1 + \Delta)\mathbf{u} + h\mathbf{r}, \quad \mathbf{u} = \Upsilon\mathbf{u} \tag{16}$$

where the term $(1 + \Delta)$ constitutes what we call "quasi-saturation"; i.e., $(1 + \Delta_k)(u_k)(t) = u_k(t)$ only when $|u_k(t)|$ is smaller than the threshold value $\bar{u}$. When $|u_k(t)|$ is larger than $\bar{u}$, $(1 + \Delta_k)(u_k)(t)$ is reduced from $u_k(t)$ by an amount governed by ϕ_k. An example of this quasi-saturation function is depicted in Figure 2. Notice that the sector condition imposed on Δ_k excludes the "standard saturation". In fact, it is possible to show that system (16) does not always synchronize to $\ker\Upsilon$ should $(1+\Delta)$ be allowed to be a standard saturation function.

By the sector condition imposed on Δ_k, one can show that all Δ_k satisfy IQC defined by the following multiplier [34]

$$\Pi_{\mathrm{sp}}(j\omega) = \begin{bmatrix} 2 & \alpha_{\min}(1 - \sigma \cdot j\omega) \\ \alpha_{\min}(1 + \sigma \cdot j\omega) & 0 \end{bmatrix}, \tag{17}$$

where σ is any real number. This is a special case of the so-called Zames-Falb multiplier for slope constrained operators. See, e.g. [37], for details. With this multiplier, Theorem 2 immediately leads to the following result.

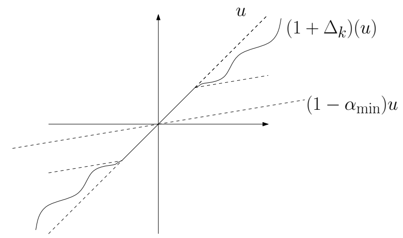

Fig. 2. A memoryless example of quasi–saturation of the input. Dashed lines represent $f_1(u) = u$ and $f_2(u) = (1 - \alpha_{\min})u$, which are the extremes of the section in which $(1 + \Delta_k)(u)$ lies. The dashed line in the middle is parallel to the lower one, and depicts the sector condition for Δ_k.

Corollary 1. *Consider the system in* (16) *where* Δ_k *satisfies* (15) *and* Υ *satisfies Assumption 3. Then the system synchronizes to* $\ker\Upsilon$ *if there exists* $\sigma \in \mathbb{R}$ *and* $\varepsilon > 0$ *such that*

$$\begin{bmatrix} 1 \\ \frac{\lambda_k h(j\omega)}{1-\lambda_k h(j\omega)} \end{bmatrix}^* \Pi_{\mathrm{sp}}(j\omega) \begin{bmatrix} 1 \\ \frac{\lambda_k h(j\omega)}{1-\lambda_k h(j\omega)} \end{bmatrix} \geq \varepsilon, \quad \forall \omega \in [0, \infty]. \tag{18}$$

Inequality (18) can be verified graphically using a Popov plot. Let $G_r(j\omega) = \frac{\lambda_k h(j\omega)}{1-\lambda_k h(j\omega)}$ and

$$\mathcal{P} = \{z \in \mathbb{C} : z = \mathrm{Re}G_r(j\omega) - j\omega\mathrm{Im}G_r(j\omega),\ \omega \in [0, \infty]\}.$$

Then system (16) synchronizes to $\ker\Upsilon$ if $\mathcal{P}$ entirely lies on the right to the line with slope $\frac{1}{\sigma}$ and crossing the x–axis in the point $-\frac{1}{\alpha_{\min}}$.

Note that if $\alpha_{\min} \to 0$ (i.e., $\Delta_k \to 0$), then the above criterion is always satisfied, as expected since the nominal system without Δ is assumed to reach synchronization by the interconnection $\mathbf{u} = \Upsilon\mathbf{y}$.

Example: As an example, let us consider continuous-time clock synchronization problem [19]. A clock is modeled as a double-integrator with state space model

$$\dot{x}_k(t) = \begin{bmatrix} 0 & q \\ 0 & 0 \end{bmatrix} x_k(t) + F u_k(t), \quad y_k(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k(t), \tag{19}$$

The state is the 2-dimensional vector $x_k = [x_{k1}, x_{k2}]$. The first component gives the relative time of the clock. The value q represents the "skew" of the clock, the rate at which the clock measures the absolute time. For simplicity we assume q to be the same for all the clocks. If $q = 1$, the clock is measuring the "real" time. The clock is slower than the real time if $q < 1$, while it is faster if $q > 1$. The second component x_{k2} can be interpreted as an "estimate" of $1/q$. The clocks synchronize if $q x_{k2}(t) = \kappa$, where $\kappa \in \mathbb{R}^+$ is a constant shared by all the clock.

We assume that each clock is allowed to modify its pair of states by making use of the (shared) matrix $F = \begin{bmatrix} f_1 & f_2 \end{bmatrix}^T$, and $\mathbf{u} := \begin{bmatrix} u_1, \cdots, u_N \end{bmatrix}^T$ is determined by the rule $\mathbf{u}(t) = (I + \Delta)(\Upsilon\mathbf{y}(t))$, where $I + \Delta$ is a quasi-saturation operator defined as follows: $(1 + \Delta_k)(u_k)(t)$ is equal to

$$\begin{cases} u_k(t), & |u_k(t)| \leq \bar{u} \\ \mathrm{sgn}(u_k(t))\bar{u} + \nu_k(u_k(t) - \mathrm{sgn}(u_k(t))\bar{u}), & |u_k(t)| > \bar{u} \end{cases}$$

Parameter ν_k models how much the input is reduced outside the linear region. For the simulation results shown below, the actual values of ν_k are selected randomly from the interval $[0.2, 1]$. It is easy to see that in this case $\phi(t, u - \mathrm{sgn}(u)\bar{u}) = (1 - \nu_k)(u_k - \mathrm{sgn}(u_k)\bar{u})$, and that $\alpha_{\min} = .8 < 1$.

Let $N = 9$, $q = 1$, $f_1 = 1.7$, and $f_2 = 1$. The transfer function of the clocks is

$$h(s) = \frac{f_1 s + f_2 q}{s^2} = \frac{1.7s + 1}{s^2},$$

Assume the clocks are positioned in a cyclic formation, in which a clock can exchange information with its immediately left and right neighbors, as well as the fifth neighbors from the right and the left. A weighted interconnection matrix coherent with this communication graph is

$$\Upsilon = -I_9 + 0.15(\mathcal{C} + \mathcal{C}^{-1}) + 0.30\mathcal{C}^5 + 0.40\mathcal{C}^{-5},$$

where $\mathcal{C}$ is the 9×9 circulant matrix whose first row is equal to $\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \end{bmatrix}$. It can be verified that $\frac{h(s)}{1-\lambda_k h(s)}$ is a stable transfer function for all k, and that $\ker\Upsilon = \mathrm{span}\{\mathbf{1}\}$, so that synchronization has the usual meaning $|y_i - y_j| \xrightarrow{t\to\infty} 0$.

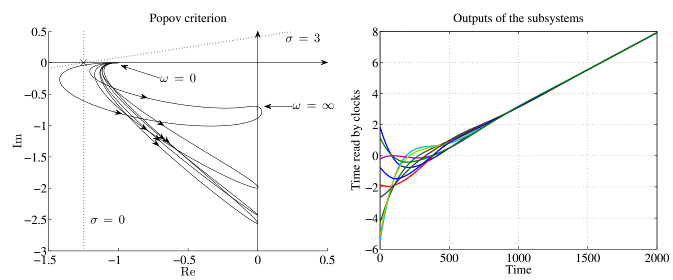

Fig. 3. Clock synchronization with quasi–saturation. Left panel: the Popov plot in the case of quasi saturation. The thin lines are the functions $G_r(j\omega)$, the big cross marks the point $-\frac{1}{\alpha_{\min}} + j0$. Right panel: a typical trajectory of the outputs.

To rigorously prove the synchronization of this network, we apply Corollary 1. It can be verified that the condition is satisfied for $\sigma = 3$ by using the Popov plot, which is illustrated in Figure 3 (the left figure). The line with $\sigma = 0$ corresponds to the more conservative circle criterion. A typical trajectory of the synchronizing clocks is shown in the right figure of Figure 3. It can be also checked that the Popov multiplier yields a criterion which is less conservative of both the small-gain and the passivity criterion. Indeed, the small-gain criterion can be put into the IQC framework with $\Pi_{SM} = \begin{bmatrix} 1 & 0 \\ 0 & -\gamma^2 \end{bmatrix}$. It is not difficult to check that $\Delta \in IQC(\Pi_{SM})$ with $\gamma = 1 - \alpha_{\min} = 0.8$, but the LHS of (14) is $1 - \gamma \frac{h(j\omega)}{1-\lambda_k h(j\omega)}^* \gamma \frac{h(j\omega)}{1-\lambda_k h(j\omega)}$, and it can be easily checked numerically that for low frequencies the inequality in (14) is not satisfied. Analogously, the passivity criterion corresponds to the multiplier $\Pi_P = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and again while easily $\Delta \in IQC(\Pi_P)$, (14) corresponds to $\mathrm{Re}\{\frac{h(j\omega)}{1-\lambda_k h(j\omega)}\} \geq \varepsilon > 0$, which is again not true at low frequencies. In summary, for the considered double integrators system and the given nonlinearity, the criterion corresponding to the Popov multiplier yields a criterion which is less conservative then the circle criterion, the small-gain theorem and the passivity theorem.

B. Reversible Interconnection in Leader-Following Networks

Consider a communication graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ and a row-stochastic matrix P associated with $\mathcal{G}$. We partition $\mathcal{V}$ into $\mathcal{V}_{\mathrm{L}} \bigcup \mathcal{V}_{\mathrm{F}}$, where $\mathcal{V}_{\mathrm{L}} := \{1, \cdots, q\}$ and $\mathcal{V}_{\mathrm{F}} := \{q+1, \cdots, N\}$, and let $\mathcal{E}_{\mathrm{L}} := \{(i,j) : (i,j) \in \mathcal{E}, i \in \mathcal{V}_{\mathrm{L}}, j \in \mathcal{V}_{\mathrm{L}}\} \subseteq \mathcal{E}$. Define subgraph $\mathcal{G}_{\mathrm{L}}$ to be $(\mathcal{V}_{\mathrm{L}}, \mathcal{E}_{\mathrm{L}})$. We make the following assumptions for the graph $\mathcal{G}$:

(i) $\mathcal{G}_{\mathrm{L}}$ is strongly connected and aperiodic;
(ii) for any agent $j \in \mathcal{V}_{\mathrm{F}}$, there exists at least one *directed* path from an agent in $\mathcal{V}_{\mathrm{L}}$ to j;
(iii) there are no *directed* paths from any agent in $\mathcal{V}_{\mathrm{F}}$ to any agent in $\mathcal{V}_{\mathrm{L}}$.

Since information flows from the agents in $\mathcal{V}_{\mathrm{L}}$ to the agents in $\mathcal{V}_{\mathrm{F}}$ but not viceversa, we call the former *leaders* and the latter *followers*. In this section, we consider the higher–order consensus problem,

$$\mathbf{y} = h(I + \Delta)\mathbf{u} + h\mathbf{r}, \quad \mathbf{u} = \Upsilon \mathbf{y} \tag{20}$$

where $\Delta := \oplus_{k \in \mathcal{V}} \Delta_k$ and $\Upsilon := -\nu(I - P)$. By assumptions (ii) and (iii), it is easy to see that P and Υ must have the low-triangular structure

$$P = \begin{bmatrix} P_1 & 0 \\ P_3 & P_2 \end{bmatrix} \qquad \Upsilon = \begin{bmatrix} \Upsilon_1 & 0 \\ \Upsilon_3 & \Upsilon_2 \end{bmatrix} \tag{21}$$

where P_3 and Υ_3 are non-zero matrices. Hence, by partitioning the agents into the two groups (i.e.,"leaders" and "followings"), (20) can be expressed as

$$\begin{cases} \begin{bmatrix} \mathbf{y}_{\mathrm{L}} \\ \mathbf{y}_{\mathrm{F}} \end{bmatrix} = \begin{bmatrix} h(I + \Delta_{\mathrm{L}}) & 0 \\ 0 & h(I + \Delta_{\mathrm{F}}) \end{bmatrix} \begin{bmatrix} \mathbf{u}_{\mathrm{L}} \\ \mathbf{u}_{\mathrm{F}} \end{bmatrix} + h \begin{bmatrix} \mathbf{r}_{\mathrm{L}} \\ \mathbf{r}_{\mathrm{F}} \end{bmatrix}, \\ \begin{bmatrix} \mathbf{u}_{\mathrm{L}} \\ \mathbf{u}_{\mathrm{F}} \end{bmatrix} = \begin{bmatrix} \Upsilon_1 & 0 \\ \Upsilon_{12} & \Upsilon_2 \end{bmatrix} \begin{bmatrix} \mathbf{y}_{\mathrm{L}} \\ \mathbf{y}_{\mathrm{F}} \end{bmatrix}, \end{cases} \tag{22}$$

where $\Delta_{\mathrm{L}} = \oplus_{k \in \mathcal{V}_{\mathrm{L}}} \Delta_k$ and $\Delta_{\mathrm{F}} = \oplus_{k \in \mathcal{V}_{\mathrm{F}}} \Delta_k$.

Also notice that assumptions (i) and (ii) imply that the directed communication graph is weakly connected and that there is only one strongly connected component. By Perron-Frobenius theorem, this implies that P has a single dominant eigenvalue at 1 associated with eigenvector $\mathbf{1}$ and all the other are strictly inside the unit circle. Correspondingly, Υ has a single eigenvalue at 0 and all the others have strictly negative real part. Moreover, there exists a vector ξ such that $\xi^T \mathbf{1} = 1$ and $\xi^T P = \xi^T$. Let $\xi_{\mathrm{L}} := \left[\xi_1, \cdots, \xi_q\right]^T$, $\xi_{\mathrm{F}} := \left[\xi_{q+1}, \cdots, \xi_N\right]^T$, $D_{\mathrm{L}} := \mathrm{diag}\,(\xi_{\mathrm{L}})$, $D_{\mathrm{F}} := \mathrm{diag}\,(\xi_{\mathrm{F}})$, and $D := \mathrm{diag}\,(D_{\mathrm{L}}, D_{\mathrm{F}})$. By assuming P_1 is *reversible*; i.e., $D_{\mathrm{L}} P_1 = P_1^T D_{\mathrm{L}}$, and P_2 is *normal*, we have the following result regarding consensus of the leader-following network (22). Note that reversibility of P_1 and normality of P_2 imply reversibility of Υ_1 (i.e., $D_{\mathrm{L}} \Upsilon_1 = \Upsilon_1^T D_{\mathrm{L}}$) and normality of Υ_2, respectively.

Proposition 1. *Consider the leader-following network* (22)*, where* Υ_1 *is reversible and* Υ_2 *is normal. Suppose that the transfer functions* $\frac{\lambda_k h(s)}{1 - \lambda_k h(s)}$ *are stable for any nonzero eigenvalue of* Υ*, and*

(i) *with* Δ *replaced by* $\tau \cdot \Delta$*, system* (22) *is well-posed for all* $\tau \in [0, 1]$*;*
(ii) *all* Δ_k *satisfy IQC defined by* $\Pi \in S_{\mathcal{A}}^{2\times 2}$*, with* $\Pi_{11} \geq 0$ *and* $\Pi_{22} \leq 0$*;*
(iii) *there exists* $\varepsilon > 0$ *such that for all* $s \in \partial\Omega$

$$\begin{bmatrix} I \\ \frac{\lambda_k h}{1 - \lambda_k h} \end{bmatrix}^* \Pi \begin{bmatrix} I \\ \frac{\lambda_k h}{1 - \lambda_k h} \end{bmatrix} (s) \geq \varepsilon,$$

for any nonzero eigenvalue λ_k *of* Υ.

Then

(a) *the "leaders" synchronize to the subspace* $\mathcal{Z} = \mathrm{span}\,\{\mathbf{1}\}$*;*
(b) *the "followers" synchronize to the "leaders"; i.e., there exists a constant* $c > 0$ *such that*

$$||y_i - y_j|| \leq c\|\mathbf{r}\|, \forall i \in \mathcal{V}_{\mathrm{L}}, j \in \mathcal{V}_{\mathrm{F}}$$

Proof. To prove claim (a), let's consider the "leader" sub-network

$$\mathbf{y}_{\mathrm{L}} = h(I + \Delta_{\mathrm{L}})\mathbf{u}_{\mathrm{L}} + h\mathbf{r}_{\mathrm{L}}, \quad \mathbf{u}_{\mathrm{L}} = \Upsilon_1 \mathbf{y}_{\mathrm{L}} \tag{23}$$

Note that the lower-triangular structure of Υ implies that $\xi_{\mathrm{F}}^T \Upsilon_2 = 0$, which in turn implies $\xi_{\mathrm{F}} = 0$. Furthermore, the assumption that $\mathcal{G}_{\mathrm{L}}$ is strongly connected and aperiodic implies that P_{L} is *primitive* (i.e., there exists an m such that P_{L}^m is a positive matrix), which in turn implies that the entries of ξ_{L} are strictly positive and $D_{\mathrm{L}} > 0$. Since $D_{\mathrm{L}} \Upsilon_1 = \Upsilon_1^T D_{\mathrm{L}}$, we have $D_{\mathrm{L}}^{1/2} \Upsilon_1 D_{\mathrm{L}}^{-1/2} = D_{\mathrm{L}}^{-1/2} \Upsilon_1^T D_{\mathrm{L}}^{1/2}$. Thus, by defining $\bar{\mathbf{y}}_{\mathrm{L}} = D^{1/2} \mathbf{y}_{\mathrm{L}}$, $\bar{\mathbf{u}}_{\mathrm{L}} = D^{1/2} \mathbf{u}_{\mathrm{L}}$, $\bar{\mathbf{r}}_{\mathrm{L}} = D^{1/2} \mathbf{r}_{\mathrm{L}}$, $\bar{\Delta}_{\mathrm{L}}(\bar{\mathbf{u}}_{\mathrm{L}}) = D^{1/2} \Delta_{\mathrm{L}}(D^{-1/2} \bar{\mathbf{u}}_{\mathrm{L}})$, and $R = D_{\mathrm{L}}^{1/2} \Upsilon_1 D_{\mathrm{L}}^{-1/2}$, (23) can be equivalently expressed as

$$\bar{\mathbf{y}}_{\mathrm{L}} = h(I + \bar{\Delta}_{\mathrm{L}})\bar{\mathbf{u}}_{\mathrm{L}} + h\bar{\mathbf{r}}_{\mathrm{L}}, \quad \bar{\mathbf{u}}_{\mathrm{L}} = R\bar{\mathbf{y}}_{\mathrm{L}} \tag{24}$$

Notice that R is symmetric and therefore normal, and $\ker R$ is spanned by $D_{\mathrm{L}}^{1/2}\mathbf{1}$. Furthermore, since R and Υ_1 have the same eigenvalues, thus the assumption that $\frac{\lambda_k h(s)}{1 - \lambda_k h(s)}$ are stable for any nonzero eigenvalue of Υ ensures the nominal system of (24) (i.e., without $\bar{\Delta}_{\mathrm{L}}$) synchronizes to the subspace $\mathrm{span}\left\{D^{1/2}\mathbf{1}\right\}$. Hence R satisfies Assumption 3.

Moreover, by definition $\bar{\Delta}_{\mathrm{L},k}(\bar{\mathbf{u}}_{\mathrm{L},k}) = \xi_k^{1/2} \Delta_k(u_k)$, $k = 1, \cdots, q$, and therefore one can verify that

$$\begin{aligned} &\left\langle \begin{bmatrix} \bar{\Delta}_{\mathrm{L},k}(\bar{\mathbf{u}}_{\mathrm{L},k}) \\ \bar{\mathbf{u}}_{\mathrm{L},k} \end{bmatrix}, \Pi \begin{bmatrix} \bar{\Delta}_{\mathrm{L},k}(\bar{\mathbf{u}}_{\mathrm{L},k}) \\ \bar{\mathbf{u}}_{\mathrm{L},k} \end{bmatrix} \right\rangle \\ &= \xi_k \left\langle \begin{bmatrix} \Delta_k(u_k) \\ u_k \end{bmatrix}, \Pi \begin{bmatrix} \Delta_k(u_k) \\ u_k \end{bmatrix} \right\rangle \end{aligned}$$

Since $\xi_k > 0$ for $k = 1, \cdots, q$, therefore Δ_k satisfies IQC defined by Π implies $\bar{\Delta}_{\mathrm{L},k}$ also satisfies IQC defined by Π. Thus, applying Theorem 2, we conclude that system (24) synchronizes to $\mathrm{span}\left\{D_{\mathrm{L}}^{1/2}\mathbf{1}\right\}$. Finally, since $D^{1/2}$ is an invertible constant matrix, $\bar{\mathbf{y}} = D^{1/2}\mathbf{y}$, and $\bar{\mathbf{r}} = D^{1/2}\mathbf{r}$, we conclude that the system (23) synchronizes to $\mathrm{span}\,\{\mathbf{1}\}$.

To prove claim (b), we first decompose $\mathbf{y}_{\mathrm{L}}$ to be $y_\ell \mathbf{1} + V\mathbf{y}_{\mathrm{L}}^{\perp}$. Define $w = \mathbf{1}^T(I + \Delta_{\mathrm{L}})(\mathbf{u}_{\mathrm{L}}) + \mathbf{1}^T \mathbf{r}_{\mathrm{L}}$. We have $y_\ell = h(w/q)$, where q is the dimension of $\mathbf{y}_{\mathrm{L}}$. This is due to the fact that $\mathbf{1}^T V \mathbf{y}_{\mathrm{L}}^{\perp}(t) = 0, \forall t$. Notice that claim (a) implies there exists a constant $\tilde{c}_1 > 0$ such that $\|V\mathbf{y}_{\mathrm{L}}^{\perp}\| \leq \tilde{c}_1 \|\mathbf{r}_{\mathrm{L}}\|$. Since $\mathbf{u}_{\mathrm{L}} = \Upsilon_1 \mathbf{y}_{\mathrm{L}} = \Upsilon_1 V \mathbf{y}_{\mathrm{L}}^{\perp}$, we have $\|w\| \leq \sqrt{q}(\tilde{c}_1 \|I + \Delta_{\mathrm{L}}\| \|\Upsilon_1\| + 1)\|\mathbf{r}_{\mathrm{L}}\| := \sqrt{q}\tilde{c}_2 \|\mathbf{r}_{\mathrm{L}}\|$.

Now consider the "followers" subsystem. We have

$$\begin{aligned}\mathbf{u}_{\rm F} &= \Upsilon_3\mathbf{y}_{\rm L} + \Upsilon_2\mathbf{y}_{\rm F} = y_\ell\Upsilon_3\mathbf{1} + \Upsilon_3 V\mathbf{y}_{\rm L}^\perp + \Upsilon_2\mathbf{y}_{\rm F}\\ &= \Upsilon_2(\mathbf{y}_{\rm F} - y_\ell\mathbf{1}) + \Upsilon_3 V y_{\rm L}^\perp\end{aligned}$$

Here for the last equality, we use $\Upsilon_2\mathbf{1} + \Upsilon_3\mathbf{1} = 0$. Let $\tilde{\mathbf{y}}_{\rm F} = \mathbf{y}_{\rm F} - y_\ell\mathbf{1}$. We have

$$\begin{aligned}\tilde{\mathbf{y}}_{\rm F} &= h(I+\Delta_{\rm F})(\mathbf{u}_{\rm F}) + h\mathbf{r}_{\rm F} - y_\ell\mathbf{1} = h\Upsilon_2\tilde{\mathbf{y}}_{\rm F}\\ &\quad + h\Upsilon_3 V\mathbf{y}_{\rm L}^\perp + h\Delta_{\rm F}(\mathbf{u}_{\rm F}) + h\mathbf{r}_{\rm F} - h(w/q)\mathbf{1}\end{aligned}$$

and we obtain the following feedback system that governs $\mathbf{u}_{\rm F}$:

$$\mathbf{u}_{\rm F} = Mp + e, \quad p = \Delta_{\rm F}(\mathbf{u}_{\rm F})$$

where $M := \Upsilon_2(I - h\Upsilon_2)^{-1}h$, $e := M(\Upsilon_3 V\mathbf{y}_{\rm L}^\perp + \mathbf{r}_{\rm F} - (w/q)\mathbf{1}) + \Upsilon_3 V\mathbf{y}_{\rm L}^\perp$. Since Υ_2 is normal, Υ_2 can be unitarily diagonalized. One can readily verify that, by the assumption that $\frac{\lambda_k h(s)}{1-\lambda_k h(s)}$ are stable for any nonzero eigenvalue of Υ, M is stable. Moreover, one can also verify that

- assumption (ii) of the proposition implies $\Delta_{\rm F} \in IQC(\Pi\otimes I)$;
- assumption (iii) of the proposition implies
$$\begin{bmatrix} I \\ M\end{bmatrix}^* (\Pi\otimes I)\begin{bmatrix} I \\ M\end{bmatrix}(s) \geq \varepsilon.$$

Hence, there exists a constant $\tilde{c}_3 > 0$ such that $\|\mathbf{u}_{\rm F}\| \leq \tilde{c}_3\|e\|$. Since $\tilde{\mathbf{y}}_{\rm F} = \Upsilon_2^{-1}(\mathbf{u}_{\rm F} - \Upsilon_3 V\mathbf{y}_{\rm L}^\perp)$, one can readily verify, via applying a series triangular inequalities, that $\|\tilde{\mathbf{y}}_{\rm F}\| \leq \tilde{c}_4\|\mathbf{r}_{\rm L}\| + \tilde{c}_5\|\mathbf{r}_{\rm F}\|$, where $\tilde{c}_4 := (\tilde{c}_1(\tilde{c}_3\|M\| + \tilde{c}_3 + 1)\|\Upsilon_3\| + \tilde{c}_2\tilde{c}_3\|M\|)\,\|\Upsilon_2^{-1}\|$ and $\tilde{c}_5 := \tilde{c}_3\|M\|\|\Upsilon_2^{-1}\|$. Now given agent $i \in \mathcal{V}_1$ and agent $j \in \mathcal{V}_2$, we have

$$\begin{aligned}\|y_i - y_j\| &= \|y_i - y_\ell + y_\ell - y_j\| \leq \|y_i - y_\ell\| + \|y_\ell - y_j\|\\ &\leq \|\mathbf{y}_{\rm L} - y_\ell\mathbf{1}\| + \|y_\ell\mathbf{1} - \mathbf{y}_{\rm F}\| = \|V\mathbf{y}_{\rm L}^\perp\| + \|\tilde{\mathbf{y}}_{\rm F}\|\\ &\leq (\tilde{c}_1 + \tilde{c}_4)\|\mathbf{r}_{\rm L}\| + \tilde{c}_5\|\mathbf{r}_{\rm F}\| \leq (\tilde{c}_1 + \tilde{c}_4 + \tilde{c}_5)\|\mathbf{r}\|\end{aligned}$$

which proves synchronization of the "followers" to the "leaders". □

Example: Consider a network of $N = 20$ interconnected oscillators. The nominal dynamics of each agent is represented by transfer function $h(s) = \frac{0.2(1+s)}{s^2+1}$. The network is divided in two subsets, $\mathcal{V}_{\rm L}$ the set of leaders and $\mathcal{V}_{\rm F}$ the set of followers. Each set is made of 10 agents. The leaders are interconnected according to a random geometric graph: their positions have been picked randomly in $[-1,1]^2$, and two agents can communicate if their Euclidean distance is less than $d = 0.3$. Each follower is associated with a leader, which it receives information from. Moreover, the communication graph of the followers is the same as the communication graph of the leaders. The network is depicted in Fig. 4.

The matrix Υ_1 is chosen according to the following "random-walk" scheme

$$[\Upsilon_1]_{ij} = \begin{cases}\frac{\kappa}{\mathcal{N}_i+1}, & (i,j)\in\mathcal{E}_{\rm L}\\ -\kappa\sum_k[\Upsilon_1]_{ik}, & i = j\\ 0, & (i,j)\notin\mathcal{E}_{\rm L}\end{cases}$$

where $\mathcal{E}_1$ is the set of edges among agents in $\mathcal{V}_1$ and κ is a tuning parameter which is set to 5 in the simulations.

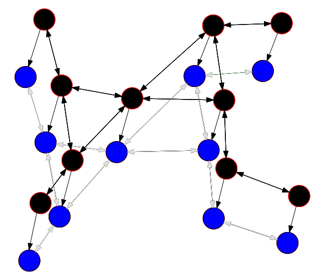

Fig. 4. Leader following for a network of perturbed oscillators: communication graph. In black the leaders. In blue the followers. The networks are the same among leaders and followers (in black and grey line, respectively), and each follower also receives information from the associated leader.

The matrix Υ_2 is instead constructed using the following popular uniform weight strategy

$$[\Upsilon_2]_{ij} = \begin{cases}\varepsilon, & (i,j)\in\mathcal{E}_{\rm F}\\ 1-\eta-\sum_{k\neq i}[\Upsilon_2]_{ik}, & i = j\\ 0, & (i,j)\notin\mathcal{E}_{\rm F}\end{cases}$$

where $\mathcal{E}_{\rm F}\subseteq\mathcal{E}$ is the subset of edges (i,j) such that $i\in\mathcal{V}_{\rm F}$ and $j\in\mathcal{V}_{\rm F}$, $\eta = 0.1$, and ε is a suitably chosen small constant. In this case, we select $\varepsilon = \frac{1}{\max_{i\in\mathcal{V}_2}|\mathcal{N}_i|+1}$. Finally, $\Upsilon_3 = \eta I$; namely, as already said, each follower receives information from an associated leader. It is rather easy to prove that the resulting Υ is a reversible matrix whose right kernel is spanned by $\mathbf{1}$, and that $\xi_i = \frac{|\mathcal{N}_i|+1}{\sum_{j\in\mathcal{V}_1}(|\mathcal{N}_j|+1)}$ if $i\in\mathcal{V}_1$ and $\xi_i = 0$ if $i\in\mathcal{V}_2$. The perturbation at each agent is a quasi-saturation operator as discussed in Section IV-A, with $\alpha_{\min} = 0.1$. The simulation is executed with random initial conditions.

The results are summarized in Fig. 5. As one can see, the leaders - depicted in thick solid black line in the left panel - tend to align along the same sinusoid, while the followers progressively forget their initial conditions and just align to the trajectory of the leaders. This can be formally proven by applying Proposition 1 with the multiplier proposed in (17). The resulting Popov plot with $G_r(j\omega) = \frac{\lambda_k h(j\omega)}{1-\lambda_k h(j\omega)}$ is shown in right panel of Fig. 5. As one can see, the Popov plot always lies on the right to a line crossing the real axis at the point $-\frac{1}{\alpha_{\min}}$ and with slope $1/\sigma = 1$; therefore the conditions of Proposition 1 is satisfied, which prove the synchronization. Again, using the circle criterion leads to a vertical line which turns out to be conservative.

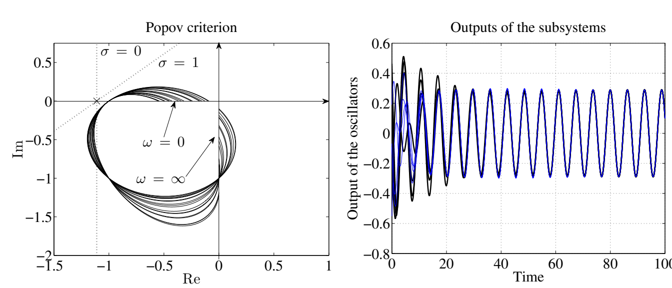

Fig. 5. Leader following for a network of perturbed oscillators. Left panel: The Popov criterion used to check that the system synchronizes. Right panel: a typical trajectory of the system.

V. Synchronization of a Nonlinearly Interconnected Network

In this section we analyze a particular type of nonlinearly interconnected networks, for which Theorem 1 is applied to study the synchronization property.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be an undirected graph with vertices $\{1, \ldots, N\} := \mathcal{V}$ and the set of edges $\mathcal{E}$. Note that since the graph is undirected, the pairs $\{i, j\}$ and $\{j, i\}$ indicate the same edge, and both imply there is information flow from agent i to agent j and vice versa. The corresponding adjacency matrix $A := [a_{ij}]_{i,j=1}^N$ is defined as

$$a_{ij} = \begin{cases} 1, & \text{if } \{i, j\} \text{ or } \{j, i\} \in \mathcal{E}, \quad i \neq j \\ 0, & \text{otherwise} \end{cases}$$

Consider the unweighted graph Laplacian $L = D - A$, where D is the diagonal node degree matrix, i.e., $d_{ii} = \sum_{j=1}^N a_{ij}$. Clearly $L\mathbf{1} = 0$ and L is symmetric since the graph is undirected. We assume that the graph is connected, which implies that the eigenvalues of L are distributed as $0 = \lambda_1 < \lambda_2 \leq \ldots \leq \lambda_N$.

We consider the following rule for producing the input at the i^{th} agent

$$\Gamma_i(\mathbf{y}) = \sum_{j=1}^N a_{ij} \phi_{[\eta,\beta]}(y_j - y_i), \tag{25}$$

where $\phi_{[\eta,\beta]}$ is an odd (i.e., $\phi_{[\eta,\beta]}(-x) = -\phi_{[\eta,\beta]}(x)$) memoryless nonlinearity satisfying the slope restriction condition

$$\eta \leq \frac{\phi_{[\eta,\beta]}(x_1) - \phi_{[\eta,\beta]}(x_2)}{x_1 - x_2} \leq \beta, \ \forall x_1 \neq x_2$$

where $0 < \eta < \beta < \infty$. The dynamics of the nonlinearly interconnected network we consider in this section can be formulated as

$$\mathbf{y} = h(I + \Delta)\mathbf{u}, \quad \mathbf{u} = \Gamma(\mathbf{y}) \tag{26}$$

where h is a scalar LTI system representing the nominal dynamics of all agents, and $\Delta := \oplus_{k=1}^N \Delta_k$ captures the differences of dynamics among the agents.

A concise expression of Γ can be obtained by introducing the oriented incidence matrix $B \in \{0, \pm 1\}^{|N \times \mathcal{E}}$. To this end, let us without loss of generality assume the edges in $\mathcal{E}$ are all distinct and label them e_m, $m = 1, \cdots, |\mathcal{E}|$. Define B such that

$$b_{im} = \begin{cases} 1, & \text{if } e_m = \{i, *\} \\ -1, & \text{if } e_m = \{*, i\} \\ 0, & \text{otherwise.} \end{cases}$$

One can readily verify that $L = BB^T$ and the interconnection operator Γ can be expressed as

$$\Gamma(\mathbf{y}) = -B\Phi_{[\eta,\beta]}(B^T\mathbf{y}) \tag{27}$$

where $\Phi_{[\eta,\beta]}(B^T\mathbf{y}) = \begin{bmatrix} \phi_{[\eta,\beta]}(B_1^T\mathbf{y}) & \cdots & \phi_{[\eta,\beta]}(B_{|\mathcal{E}|}^T\mathbf{y}) \end{bmatrix}^T$, and B_m denotes the m^{th} column of B. Clearly, Γ satisfies Assumption 2 for $\mathcal{Z} := \text{span}\{\mathbf{1}\}$. Let the Laplacian matrix L be decomposed as $V\Lambda V^T$, where $\Lambda := \text{diag}(\lambda_2, \cdots, \lambda_N)$, and the i^{th} column of V is the normalized eigenvector of L associated with λ_i. We note that the columns of V span $\mathcal{Z}_\perp$ and the $\Gamma_\perp$ operator associated with Γ is defined as $-V^TB\Phi_{[\eta,\beta]}(B^TV\mathbf{y}_\perp)$, where $\mathbf{y}_\perp := V^T\mathbf{y}$.

In order to apply Theorem 1, let us introduce the parameterized interconnection operator $\Gamma[\tau] := -\eta\Lambda + \tilde{\Gamma}[\tau]$, where $\tilde{\Gamma}[\tau](\mathbf{y}_\perp) := \tau \cdot (\Gamma_\perp(\mathbf{y}_\perp) + \eta\Lambda\mathbf{y}_\perp)$. That is, we select $Q_1 := -\eta\Lambda$ and $Q_2 = Q_3 := I_{N-1}$. Note that by the equality $BB^T = V\Lambda V^T$, we have $\Lambda = V^TBB^TV$ and therefore $\tilde{\Gamma}[\tau](\mathbf{y}_\perp)$ can be expressed as $\tilde{\Gamma}[\tau](\mathbf{y}_\perp) = -\tau \cdot V^TB\tilde{\Phi}_{[0,\beta-\eta]}(B^TV\mathbf{y}_\perp)$, where $\tilde{\Phi}_{[0,\beta-\eta]} := \Phi_{[\eta,\beta]} - \eta I$; i.e., $\tilde{\Phi}_{[0,\beta-\eta]}(\mathbf{y}) = \begin{bmatrix} \phi_{[\eta,\beta]}(y_1) - \eta y_1, \cdots, \phi_{[\eta,\beta]}(y_N) - \eta y_N \end{bmatrix}^T$. Denote the operator $\phi_{[\eta,\beta]}(\cdot) - \eta\cdot$ as $\tilde{\phi}_{[0,\beta-\eta]}(\cdot)$. Clearly, $\tilde{\phi}_{[0,\beta-\eta]}$ is odd and slope restricted, which satisfies

$$0 \leq \frac{\tilde{\phi}_{[0,\beta-\eta]}(x_1) - \tilde{\phi}_{[0,\beta-\eta]}(x_2)}{x_1 - x_2} \leq \beta - \eta,$$

for all $x_1 \neq x_2$. For the continuous-time setting, we have the following IQC characterization for $\Gamma[\tau]$.

Lemma 2. *For all $\tau \in [0, 1]$, the operator $\Gamma[\tau]$ satisfies the IQC defined by the multiplier $\Pi_\Gamma := \Psi^T(\Pi \otimes I_{N-1})\Psi$, where*

$$\Psi = \begin{bmatrix} I_{N-1} & \eta\Lambda \\ 0 & I_{N-1} \end{bmatrix},$$

Π is an LTI operator with the frequency domain representation

$$\begin{aligned} \Pi(j\omega) &:= \begin{bmatrix} \Pi_1(j\omega) & \Pi_2(j\omega) \\ \Pi_2(j\omega)^* & 0 \end{bmatrix} \\ &= \begin{bmatrix} \frac{2}{\lambda_N(\beta-\eta)}(1 + \operatorname{Re} M(j\omega)) & (1 + M(j\omega)) \\ (1 + \overline{M(j\omega)}) & 0 \end{bmatrix} \end{aligned} \tag{28}$$

and M is any LTI operator of which the $\mathbf{L}_1$-norm of the impulse response is no larger than 1.

Proof. One can readily verify that

$$\begin{aligned} &\left\langle \begin{bmatrix} \Gamma[\tau](\mathbf{y}) \\ \mathbf{y} \end{bmatrix}, \Psi^T(\Pi \otimes I_{N-1})\Psi \begin{bmatrix} \Gamma[\tau](\mathbf{y}) \\ \mathbf{y} \end{bmatrix} \right\rangle \\ =& \left\langle \begin{bmatrix} \tilde{\Gamma}[\tau](\mathbf{y}) \\ \mathbf{y} \end{bmatrix}, (\Pi \otimes I_{N-1}) \begin{bmatrix} \tilde{\Gamma}[\tau](\mathbf{y}) \\ \mathbf{y} \end{bmatrix} \right\rangle \\ =& \left\langle \begin{bmatrix} \tau\mathbf{v} \\ \mathbf{x} \end{bmatrix}, \begin{bmatrix} \Pi_1 B^TVV^TB & -\Pi_2 I_{N-1} \\ -\Pi_2^* I_{N-1} & 0 \end{bmatrix} \begin{bmatrix} \tau\mathbf{v} \\ \mathbf{x} \end{bmatrix} \right\rangle \end{aligned} \tag{29}$$

where $\mathbf{v} = \tilde{\Phi}_{[0,\beta-\eta]}(\mathbf{x})$. Let $\hat{\mathbf{v}}$ be the Fourier transform of $\mathbf{v}$. Note that $\langle \tau\mathbf{v}, (\Pi_1 B^TVV^TB)\tau\mathbf{v} \rangle$ is equal to

$$\begin{aligned} &\frac{\tau^2}{2\pi} \int_{-\infty}^{\infty} \Pi_1(j\omega) \|V^TB\hat{\mathbf{v}}(j\omega)\|^2 d\omega \\ \leq& \frac{\tau}{2\pi} \int_{-\infty}^{\infty} \lambda_N \Pi_1(j\omega) \|\hat{\mathbf{v}}(j\omega)\|^2 d\omega \\ =& \tau \left\langle \tilde{\Phi}_{[0,\beta-\eta]}(\mathbf{x}), (\lambda_N\Pi_1)\tilde{\Phi}_{[0,\beta-\eta]}(\mathbf{x}) \right\rangle \end{aligned}$$

Here for the inequality we use that $0 \leq \tau \leq 1$, $\Pi_1(j\omega) \geq 0$ for all ω, and $B^TVV^TB \leq \lambda_N I_{N-1}$. Thus, the inner product in (29) is bounded above by

$$\begin{aligned} &\tau \left\langle \begin{bmatrix} \tilde{\Phi}_{[0,\beta-\eta]}(\mathbf{x}) \\ \mathbf{x} \end{bmatrix}, \begin{bmatrix} \lambda_N\Pi_1 I_{N-1} & -\Pi_2 I_{N-1} \\ -\Pi_2^* I_{N-1} & 0 \end{bmatrix} \begin{bmatrix} \tilde{\Phi}_{[0,\beta-\eta]}(\mathbf{x}) \\ \mathbf{x} \end{bmatrix} \right\rangle \\ \leq& \tau \sum_{k=1}^{|\mathcal{E}|} \left\langle \begin{bmatrix} \tilde{\phi}_{[0,\beta-\eta]}(x_k) \\ x_k \end{bmatrix}, \begin{bmatrix} \lambda_N\Pi_1 & -\Pi_2 \\ -\Pi_2^* & 0 \end{bmatrix} \begin{bmatrix} \tilde{\phi}_{[0,\beta-\eta]}(x_k) \\ x_k \end{bmatrix} \right\rangle \leq 0 \end{aligned}$$

where x_k is the k^{th} component of $\mathbf{x}$. The last inequality holds since the odd and slope restricted nonlinearity $\tilde{\phi}_{[0,\beta-\eta]}$ satisfies the Zames-Falb IQC. See e.g. [37] and [34]. □

Applying Theorem 1, we have the following result regarding synchronization of the nonlinear network (26).

Corollary 2. *Consider the nonlinear network (26), where the nonlinear interconnection operator* Γ *is defined as in (25). Suppose*

(i) *the transfer function* $\frac{1}{1+\eta\lambda_k h(s)}$ *is stable for any nonzero eigenvalue* λ_k *of* L*;*

(ii) *for* $\tau \in [0,1]$, $\tau\Delta_k$ *satisfies IQC defined by LTI multiplier* Π_Δ*;*

(iii) *there exists* $\varepsilon > 0$ *such that for all* $\omega \in [0,\infty]$,

$$\begin{bmatrix} 1 & 0 \\ \frac{h}{1+\eta\lambda_k h} & \frac{h}{1+\eta\lambda_k h} \end{bmatrix}^* \Pi \begin{bmatrix} 1 & 0 \\ \frac{h}{1+\eta\lambda_k h} & \frac{h}{1+\eta\lambda_k h} \end{bmatrix} (j\omega) + \begin{bmatrix} 0 & 1 \\ \frac{1}{1+\eta\lambda_k h} & \frac{-\eta\lambda_k h}{1+\eta\lambda_k h} \end{bmatrix}^* \Pi_\Delta \begin{bmatrix} 0 & 1 \\ \frac{1}{1+\eta\lambda_k h} & \frac{-\eta\lambda_k h}{1+\eta\lambda_k h} \end{bmatrix} (j\omega) \geq \varepsilon I$$

Then the system synchronizes to the subspace $\mathcal{Z} := \operatorname{span}\{\mathbf{1}\}$.

Proof. This corollary immediately follows Theorem 1, where the corresponding $Q_1 := -\eta\Lambda$, $Q_2 = Q_3 := I_N$, $H_{uy} = H_{wy} = hI_N$, $H_{uv} = I_N$. The corresponding $\Gamma[\tau]$ satisfies IQC defined by $\Psi^T(\Pi \otimes I_{N-1})\Psi$ as discussed in Lemma 2. The corresponding $\Delta[\tau]$ is chosen as $\tau\Delta_\perp$, where $\Delta_\perp$ is defined as $\Delta_\perp(\mathbf{v}) := V^T\Delta(V\mathbf{v})$. As discussed in Lemma 1, $\Delta_\perp$ satisfies IQC defined by $\Pi_\Delta \otimes I_{N-1}$ when each Δ_k satisfies IQC defined by Π_Δ. □

Example: As an example, let us consider a network of eight agents deployed in a circle in which each agent can exchange information with its left- and right-neighbors. In this case, $\mathcal{V} := \{1,\cdots,8\}$ and $\mathcal{E}$ contains edges $\{k,k+1\}$, $k = 1,\cdots,7$, and $\{8,1\}$. The unweighted Laplacian of this communication graph $\mathcal{G} = (\mathcal{V},\mathcal{E})$ is thus

$$L = 2I_8 - \mathcal{C}_8 - \mathcal{C}_8^{-1}$$

where $\mathcal{C}_8$ is a 8×8 circulant matrix whose first row is $\begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \end{bmatrix}$. It is easy to verify that the largest eigenvalue of L is $\lambda_8 = 4$.

Let the nominal dynamics of all agent be a simple integrator; i.e., $h(s) = 1/s$. Assume that each Δ_k is a quasi-saturation operator as in Section IV. The parameters for Δ_k are similar as those in the numerical examples of Section IV, with the only difference that $\alpha_{\min} = 0.5$. A Popov-like multiplier

$$\Pi_\Delta = \begin{bmatrix} 2 & \alpha_{\min}(1-\sigma\cdot j\omega) \\ \alpha_{\min}(1+\sigma\cdot j\omega) & 0 \end{bmatrix}$$

is thus applicable for analysis, and in particular, we set $\sigma = 0.1$. The interconnection operator $\Gamma(\cdot)$ is built as in (25), where $\phi_{[\eta,\beta]}(x) = x + \frac{1}{4}\sin x$. For this selection of $\phi_{[\eta,\beta]}$, it is easy to verify that $\eta = \frac{3}{4}$ and $\beta = \frac{5}{4}$, and the following Zames-Falb multiplier is applicable for analysis

$$\Pi = \begin{bmatrix} 1 + \frac{1}{1+\omega^2/100} & 1 + \frac{1}{1+j10\omega} \\ 1 + \frac{1}{1-j10\omega} & 0 \end{bmatrix}.$$

With these choices, it is possible to show that all the conditions in Theorem 1 are satisfied with $\varepsilon \approx 0.5$, and hence we can predict that synchronization will take place. This is indeed the case. In Figure 6 we show typical trajectories of states and inputs, where initial conditions for the states are taken randomly with $x_0 \sim \mathcal{N}(0, 25I_8)$. We can see that agents synchronize to $\operatorname{span}\{\mathbf{1}\}$ despite the nonlinear perturbation in the dynamics and the nonlinear interconnection.

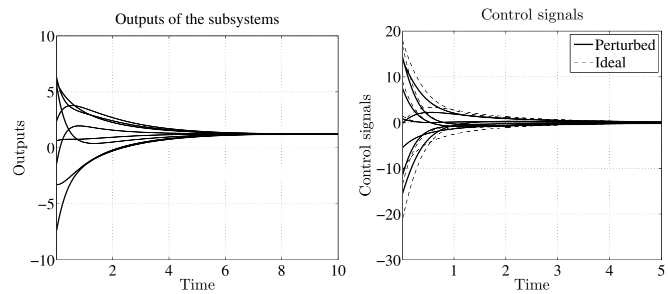

Fig. 6. Perturbed consensus with nonlinear interconnection. Left panel: a typical trajectory of the states. Right panel: ideal and perturbed inputs.

VI. Conclusions

In this paper we present a general framework for studying synchronization of heterogeneous multi–agents systems, where the dynamics of the agents are modeled as possibly nonlinear perturbed versions of a common nominal LTI operator, and they are interconnected via a sparse memoryless interconnection operator. The proposed synchronization criterion, which ensures robust synchronization, is derived based on the theory of Integral Quadratic Constraints. Applying this general criterion to the cases where the interconnection is governed by constant normal matrices yields scalable conditions. The results are applied to study clock synchronization and consensus of leader-following networks. Future research includes extending the results to dynamic interconnection operators and to stochastic systems.

VII. Acknowledgments

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References

[1] S. Hara and H. Tanaka, "$\mathcal{D}$-stability and robust stability conditions for lti systems with generalized frequency variables," in *Proc. of 49th Conf. on Decision and Control, CDC'10*, Dec. 2010.

[2] U. Jönsson and C.-Y. Kao, "A scalable robust stability criterion for systems with heterogeneous lti components," *IEEE Tran. Aut. Cont.*, vol. 55, no. 10, pp. 2219 – 2234, 2010.

[3] R. Pates and G. Vinnicombe, "Stability certificates for networks of heterogeneous linear systems," in *Proc. of 51st Conf. on Decision and Control, CDC'12*, Dec. 2012.

[4] S. Low, F. Paganini, and J. Doyle, "Internet congestion control," *IEEE Cont. Sys. Magazine*, vol. 22, no. 1, pp. 28 – 43, 2002.

[5] J. Cortes, S. Martinez, and F. Bullo, “Robust rendezvous for mobile autonomous agents via proximity graphs in arbitrary dimensions,” *IEEE Tran. Aut. Cont.*, vol. 51, no. 8, pp. 1289 – 1298, 2006.
[6] J. Fax and R. Murray, “Information flow and cooperative control of vehicle formations,” *IEEE Tran. Aut. Cont.*, vol. 49, no. 9, pp. 1465–1476, 2004.
[7] R. Olfati Saber, “Distributed kalman filter with embedded consensus filters,” in *Proc. of joint 44th Conf. on Decision and Control and Eur. Cont. Conf., CDC-ECC'05*, Dec. 2005.
[8] D. Bertsekas and T. J.N., *Parallel and Distributed Computation: Numerical Methods*. Prentice Hall, 1989.
[9] G. Cybenko, “Dynamic load balancing for distributed memory multiprocessors,” *J. Par. Dist. Comp.*, vol. 7, no. 2, pp. 279–301, 1989.
[10] S. Ganeriwal, P. R. Kumar, and M. B. Srivastava, “Timing-sync protocol for sensor networks,” in *Proc. of 1st Int. Conf. on Embedded Networked Sensor Systems. SenSys '03*, 2003.
[11] L. Schenato and G. Gamba, “A distributed consensus protocol for clock synchronization in wireless sensor network,” in *Proc. of 46th IEEE Conf. on Decision and Control, CDC'07*, Dec. 2007.
[12] B. Johansson, M. Rabi, and M. Johansson, “A simple peer-to-peer algorithm for distribured optimization in sensor networks,” in *Proc. of 46th IEEE Conf. on Decision and Control, CDC'07*, Dec. 2007, pp. 4705–4710.
[13] Z. Hao, A. Cano, and G. Giannakis, “Distributed demodulation using consensus averaging in wireless sensor networks,” in *Proc. of 42nd Asilomar Conf. on Signals, Systems and Computers*, 2008.
[14] R. Olfati Saber, J. Fax, and R. Murray, “Consensus and cooperation in multi-agent networked systems,” *Proc. of IEEE*, vol. 95, no. 1, pp. 215–233, 2007.
[15] F. Gantmacher, *The theory of matrices*. Chelsea publisher, 1959.
[16] A. Jadbabaie, J. Lin, and A. Morse, “Coordination of groups of mobile autonomous agents using nearest neighbor rules,” *IEEE Tran. Aut. Cont.*, vol. 48, no. 6, pp. 988–1001, June 2003.
[17] F. Fagnani and S. Zampieri, “Average consensus with packet drop communication,” *SIAM J. Cont. and Opt.*, vol. 48, pp. 102–133, 2009.
[18] S. Boyd, A. Ghosh, B. Prabhakar, and D. Shah, “Randomized gossip algorithms,” *IEEE Tran. Inf. Theory*, vol. 52, no. 6, pp. 2508 – 2530, 2006.
[19] R. Carli, A. Chiuso, L. Schenato, and S. Zampieri, “Optimal synchronization for networks of noisy double integrators,” *IEEE Tran. Aut. Cont.*, vol. 56, no. 5, pp. 1146–1152, 2011.
[20] F. Dörfler and F. Bullo, “Synchronization and transient stability in power networks and non-uniform Kuramoto oscillators,” *SIAM J. Cont. and Opt.*, vol. 50, no. 3, pp. 1616–1642, 2012.
[21] L. Scardovi and R. Sepulchre, “Synchronization in networks of identical linear systems,” *Automatica*, vol. 45, no. 11, pp. 2557–2562, 2009.
[22] K. You and L. Xie, “Network topology and communication data rate for consensusability of discrete-time multi-agent systems,” *IEEE Tran. Aut. Cont.*, vol. 56, no. 10, pp. 2262 – 2275, 2011.
[23] U. Münz, A. Papachristodoulou, and F. Allgöwer, “Delay robustness in consensus problems,” *Automatica*, vol. 46, no. 8, pp. 1252 – 1265, 2010.
[24] I. Lestas and G. Vinnicombe, “Scalable robustness for consensus protocols with heterogeneous dynamics,” in *Proc. of 16th IFAC World Congress*, 2005.
[25] ——, “Heterogeneity and scalability in group agreement protocols: Beyond small gain and passivity approaches,” *Automatica*, vol. 46, no. 7, pp. 1141–1151, 2010.
[26] M. Arcak, “Passivity as a design tool for group coordination,” *IEEE Tran. Aut. Cont.*, vol. 52, pp. 1380–1390, 2007.
[27] I. Lestas, “Large scale heterogeneous networks, the Davis-Wielandt shell and graph separation,” *SIAM J. Cont. and Opt.*, vol. 50, no. 4, pp. 1753–1774, 2012.
[28] L. Scardovi, M. Arcak, and E. Sontag, “Synchronization of interconnected systems with applications to biochemical networks: An input-output approach,” *IEEE Tran. Aut. Cont.*, vol. 55, pp. 1367–1379, 2010.
[29] K. Takaba, “Robust synchronization of multiple agents with uncertain dynamics,” in *Proc. of 18th IFAC World Congress*, Sep. 2011.
[30] U. Jönsson, C.-Y. Kao, and H. Fujioka, “Low dimensional stability criteria for large-scale interconnected systems,” in *Proc. Eur. Cont. Conf., ECC'07.*, 2007.
[31] U. Jönsson, “Primal and dual criteria for robust stability and their application to systems interconnected over a bipartite graph,” in *Proc. American Cont. Conf., ACC'10.*, 2010.
[32] I. Lestas, “On network stability, graph separation, interconnection structure and convex shells.” in *Proc. of 50th Conf. on Decision and Control, CDC'11.*, 2011.
[33] E. Lovisari and U. Jönsson, “A framework for robust synchronization in heterogeneous multi–agent networks,” in *Proc. of 50th IEEE Conf. on Decision and Control, CDC'11*, Dec. 2011.
[34] A. Megretski and A. Rantzer, “System analysis via integral quadratic constraints,” *IEEE Tran. Aut. Cont.*, vol. 42, no. 6, pp. 819–830, 1997.
[35] E. Nuño, R. Ortega, L. Basanez, and D. Hill, “Synchronization of networks of nonidentical euler-lagrange systems with uncertain parameters and communication delays,” *IEEE Tran. Aut. Cont.*, vol. 4, pp. 935 – 941, 2011.
[36] P. Diaconis and D. Stroock, “Geometric bounds for eigenvalues of Markov chains,” *Ann. App. Prob.*, vol. 1, no. 1, pp. 36–61, 1991.
[37] G. Zames and P. Falb, “Stability conditions for systems with monotone and slope–restricted nonlinearities,” *SIAM J. Cont.*, vol. 6, no. 1, pp. 89–108, 1968.