Robust Synchronisation of Heterogeneous Networks via Integral Quadratic Constraints

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***Abstract*— A general framework for the study of robust synchronisation in large-scale networks is provided. Agents are represented as a common nominal linear time-invariant (LTI) single-input-single-output (SISO) system with simple poles on the imaginary axis, subject to LTI SISO stable multiplicative perturbations. The agents exchange information in order to achieve output-synchronisation, namely steer their outputs to the same, possibly time-varying, signal. Such an information exchange is modeled through a sparse dynamical operator that maps the outputs of the agents into their inputs. The theory of integral quadratic constraints is used to capture the structural uncertainties of the perturbations, and to give certificates for robust synchronisation of the systems. Since the IQC theory is nominally applied to open-loop stable systems, the main idea is to introduce a new space of signals with respect to which the notion of feedback stability implies that of synchronisation under appropriate assumptions on the interconnection operator. The proposed criterion unifies and extends several results in the literature.**

***Index Terms*— Synchronisation, heterogeneous networks, robustness, integral quadratic constraints**

I. Introduction

Synchronisation in large-scale networks is a ubiquous phenomenon that takes place both in natural and engineered contexts. In the former, examples arise from biological or energy-exchanging networks [1]; in the latter, clock synchronisation or power network phase locking. One of the most studied synchronisation problems is consensus, or agreement, in which agents in a large-scale network exchange information in order to agree over time on some quantity of interest, usually the average of their initial conditions. The network's structure is captured by a *communication graph* $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ in which $\mathcal{V}$ is the set of nodes/agents, and an edge $(i, j) \in \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ exists if and only if node j can receive information from node i. The most popular consensus strategy is the linear consensus algorithm [2], [3], where each agent is an integrator whose input is given by a weighted difference between its neighbor's states and its own state. Despite its simplicity, the linear consensus algorithm has been employed as basic tool for a number of more complex tasks such as formation control, distributed estimation, load balancing, distributed optimisation, distributed demodulation [4], [5], [6], [7]. However, it is often the case that agents cannot be represented as simple integrators. For example, in formation control agents can be more realistically represented as second order systems, clocks can be modeled as double integrators, and power plants as systems with a couple of complex conjugate poles. For this reason, in the last decade there appeared an increasing number of studies on higher order consensus networks, in which agents can be represented as generic linear time-invariant (LTI) single-input-single-output (SISO) systems, and synchronisation is meant as output-synchronisation. These works address both the *homogeneous* case [8], [9], in which agents are represented by the same system, and the *heterogeneous* case, in which agents exhibit some degree of difference. A particularly studied scenario is when agents can be represented as additive or multiplicative stable perturbations of the same nominal system [10], [11], [12]. This is the scenario considered in this paper.

We examine a network of agents that are modeled as a common nominal LTI SISO system multiplicatively perturbed by stable LTI SISO operators. The nominal system has poles in the closed half-plane $\mathbb{C}_-$, of which those on the imaginary axis are assumed to be simple. This setting encompasses scenarios such as consensus networks or networks of oscillators. Agents are interconnected through a (possibly dynamical) sparse LTI interconnection operator Γ that maps their outputs into their inputs, and that is consistent with the communication graph in the sense that $[\Gamma]_{ij}$, from the j-th output to the i-th input, is not the zero operator if and only if $(j, i) \in \mathcal{E}$. We provide general synchronisation certificates that unify certain *ad hoc* methods in the literature by employing the powerful robustness analysis tool known as Integral Quadratic Constraint (IQC) theory [13].

The classical IQC theory aims at studying stability of feedback interconnections by capturing the characteristics and the structural uncertainties of open-loop stable systems via quadratic inequalities, where stability is defined in the classical sense of having no singularities in the closed right-half plane. As such, it cannot be directly applied to the case considered in this paper, in which agents can share marginally stable poles on the imaginary axis. A usual approach to circumventing this problem is by employing loop-trasformations to convert the system into one that can be cast into the IQC framework [14], [12]. The objective of this paper is to show that, instead, the IQC theory can be applied directly to the study of synchronisation of heterogeneous systems. This approach allows for a less constrained usage of the theory itself and for a clearer relation between the characteristics of the systems and the possibility to achieve synchronisation. Furthermore, since the interconnection operator is allowed to be a dynamical system, we recover scenarios such as time-delayed information communication [15].

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Differently from the usual generalised Nyquist criterion based approach in the literature, we shall introduce in this paper a non-classical frequency-domain $\mathbf{L}_2$ signal space with an integration contour that avoids via indentations the marginally stable poles, and apply IQC theory to the open-loop systems, which are 'stable' with respect to the new space. The IQC based closed-loop stability proof method differs from [13] in that graph-topological results of [16] are used to establish closed-loop well-posedness for LTI systems, thereby simplifying the IQC conditions for synchronisation.

As a final note, we distinguish the results presented here from the literature on heterogeneous networks in which agents are completely different, and as such the design of the interconnection operator is more involved [17]. This interesting problem is left for future research.

The paper is organised as follows. Notation is defined in the following section. In Section III the problem of synchronisation is formulated. The main result is given in Section IV. Finally, some concluding remarks are provided.

II. Notation and preliminaries

A. Matrices

Let $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex numbers respectively. $j\mathbb{R}$ denotes the imaginary axis, $\mathbb{C}_+$ (resp. $\bar{\mathbb{C}}_+$) the open (resp. closed) right half complex plane, and $|\cdot|$ the Euclidean norm. Given an $A \in \mathbb{C}^{m\times n}$ (resp. $\mathbb{R}^{m\times n}$), $A^* \in \mathbb{C}^{n\times m}$ (resp. $A^T \in \mathbb{R}^{n\times m}$) denotes its complex conjugate transpose (resp. transpose). A_{ij} denotes the (i,j) entry of A. The i^{th} row and j^{th} column of A are denoted respectively by $A_{i\bullet}$ and $A_{\bullet j}$. Given a vector $v \in \mathbb{C}^n$, $\operatorname{diag}(v) \in \mathbb{C}^{n\times n}$ denotes the diagonal matrix whose diagonal entires are $v_1, \ldots, v_n$. Let $\otimes$ denote the Kronecker product and $\oplus$ the direct sum of matrices. Define $\bigoplus_{i=1}^n A_i := A_1 \oplus A_2 \oplus \ldots \oplus A_n$. I_n denotes the identity matrix of dimensions $n \times n$.

B. Function spaces

Define the Lebesgue space

$$\mathbf{L}_\infty := \{\phi : j\mathbb{R} \to \mathbb{C} \,|\, \|\phi\|_\infty := \sup_{\omega\in\mathbb{R}} |\phi(j\omega)| < \infty\}$$

and the Hardy space

$$\mathbf{H}_\infty := \left\{ \phi \in \mathbf{L}_\infty \,\middle|\, \begin{array}{l} \phi \text{ has analytic continuation into } \mathbb{C}_+ \\ \text{with } \sup_{s\in\mathbb{C}_+} |\phi(s)| = \|\phi\|_\infty < \infty \end{array} \right\}.$$

Let $\mathbf{C}$ be the class of functions continuous on $j\mathbb{R}\cup\{\infty\}$, and $\mathbf{S} := \mathbf{H}_\infty \cap \mathbf{C}$. Note that $\mathbf{C} \subset \mathbf{L}_\infty$. An $H \in \mathbf{C}^{n\times n}$ is said to be Hermitian if $H(j\omega) = H(j\omega)^*$ for all $\omega \in \mathbb{R} \cup \{\infty\}$ and positive definite if in addition, $H(j\omega) > 0$.

Given an $\epsilon > 0$ and a point $jq \in j\mathbb{R}$, define the semi-circle of radius ϵ in the right-half plane as

$$\mathcal{S}_\epsilon(jq) := \{s \in \mathbb{C} : |s - jq| = \epsilon, \Re(s) > 0\}$$

and $\mathcal{S}_0(jq) := \{\}$. Given a finite ordered set $j\mathcal{Q} = \{jq_1, jq_2, \ldots, jq_K\} \subset j\mathbb{R}$ with $q_1 > q_2 > \ldots > q_K$, define a contour parameterised by $\epsilon \geq 0$ as

$$\begin{aligned} \mathcal{C}_\epsilon(j\mathcal{Q}) := j[&q_1 + \epsilon, \infty) \\ &\cup \mathcal{S}_\epsilon(jq_1) \cup j[q_2 + \epsilon, q_1 - \epsilon] \\ &\cup \mathcal{S}_\epsilon(jq_2) \cup j[q_3 + \epsilon, q_2 - \epsilon] \\ &\quad\vdots \\ &\cup \mathcal{S}_\epsilon(jq_K) \cup j(-\infty, q_K - \epsilon]. \end{aligned}$$

that is, a straight line on the imaginary axis indented to the right of every point in $j\mathcal{Q}$ by a semi-circle of radius ϵ. In particular, notice that $C_0(j\mathcal{Q}) = j\mathbb{R}$ for any $j\mathcal{Q} \subset j\mathbb{R}$. Denote by $\mathcal{C}_\epsilon^+(j\mathcal{Q})$ the open half plane that lies to the right of $\mathcal{C}_\epsilon(j\mathcal{Q})$, i.e.

$$\mathcal{C}_\epsilon^+(j\mathcal{Q}) := \{s = \sigma + j\omega \in \mathbb{C} \mid \bar{\sigma} + j\omega \in \mathcal{C}_\epsilon(j\mathcal{Q}) \implies \sigma > \bar{\sigma}\},$$

and $\bar{\mathcal{C}}_\epsilon^+(j\mathcal{Q})$ its closure. Let $\mathbf{C}_\epsilon(j\mathcal{Q})$ be the class of functions continuous on $\mathcal{C}_\epsilon(j\mathcal{Q}) \cup \{\infty\}$. Given $X \in \mathbf{C}_\epsilon(j\mathcal{Q})^{n\times m}$, define $\|X\|_\infty := \sup_{s\in\mathcal{C}_\epsilon(j\mathcal{Q})} \bar{\sigma}(X(s))$, where $\bar{\sigma}(\cdot)$ denotes the maximum singular value. An $H \in \mathbf{C}_\epsilon(j\mathcal{Q})^{n\times n}$ is said to be Hermitian if $H(s) = H(s)^*$ for all $s \in \mathcal{C}_\epsilon(j\mathcal{Q}) \cup \{\infty\}$.

Let the Lebesgue space $\mathbf{L}_2^n$ denote the class of functions $f : [0,\infty) \to \mathbb{R}^n$ with finite energy, i.e. square-integrable $\|f\|_2^2 := \int_0^\infty |f(t)|^2\, dt < \infty$. The Fourier transform of $f \in \mathbf{L}_2^n$ is denoted $\hat{f}(j\omega) := \int_0^\infty e^{-j\omega t} f(t)\, dt$. Note that $\|\hat{f}\|_2 = \|f\|_2$ and $\hat{f}$ has analytic continuation into $\mathbb{C}_+$ and $\sup_{\sigma>0} \|\hat{f}(\sigma + \cdot)\|_2 = \|\hat{f}\|_2 < \infty$. The set of Fourier transforms of functions in $\mathbf{L}_2^n$ is denoted $\mathbf{H}_2^n$. A linear operator mapping between Banach spaces $X : \mathcal{X} \to \mathcal{Y}$ is said to be bounded if

$$\|X\|_{\mathcal{X}\to\mathcal{Y}} := \sup_{f\in\mathcal{X}:\|f\|_\mathcal{X}=1} \|Xf\|_\mathcal{Y} < \infty.$$

Note that multiplication by a transfer function in $\mathbf{S}$ as an operator on $\mathbf{H}_2$ defines a corresponding causal and bounded LTI operator on $\mathbf{L}_2$ in the time domain via the Laplace transform isomorphism [18].

For $\epsilon \geq 0$ and finite subset $j\mathcal{Q} \subset j\mathbb{R}$, define $\mathbf{H}_{2\epsilon}^n(j\mathcal{Q})$ to be the set of functions $\hat{f} : \bar{\mathcal{C}}_\epsilon(j\mathcal{Q}) \to \mathbb{C}^n$ that are analytic on $\mathcal{C}_\epsilon^+(j\mathcal{Q})$ and square-integrable on $\mathcal{C}_\epsilon(j\mathcal{Q})$, i.e. $\|\hat{f}\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}^2 := \int_{\mathcal{C}_\epsilon(j\mathcal{Q})} |\hat{f}(s)|^2\, ds < \infty$. The $\hat{\ }$ notation is occasionally dropped when there is no need to distinguish between time and frequency-domain signals. Note that $\mathbf{H}_2^n = \mathbf{H}_{2\epsilon}^n(j\mathcal{Q})$ when $\epsilon = 0$ and multiplication by a transfer function in $\mathbf{C}_\epsilon(j\mathcal{Q})$ defines a bounded operator on $\mathbf{H}_{2\epsilon}$. Furthermore, $\mathbf{H}_{2\epsilon}$ is a Hilbert space with inner product $\langle u, v\rangle_{\mathcal{C}_\epsilon(j\mathcal{Q})} := \int_{\mathcal{C}_\epsilon(j\mathcal{Q})} u(s)^* v(s)\, ds$. It can be seen that multiplication by an $X \in \mathcal{S}$ is bounded on $\mathbf{H}_{2\epsilon}$ for all $\epsilon \geq 0$. One the other hand, given a $q \in \mathbb{R}$, multiplication by $\frac{1}{s-jq}$ is bounded on $\mathbf{H}_{2\epsilon}(\{jq\})$ for all $\epsilon > 0$ but not on $\mathbf{H}_2$.

C. Graph theory

A graph is denoted by $\mathcal{G} = (V, E)$, where $V = \{v_1, \ldots, v_n\}$ is the set of nodes and $E \subset V \times V$, $E = \{e_1, \ldots, e_m\}$ is the set of edges such that $e_k = \{v_i, v_j\} \in E$ if node i is connected to node j. A graph is undirected if $\{v_i, v_j\} \in E$ then $\{v_j, v_i\} \in E$. A path on $\mathcal{G}$ of length N is an ordered set of distinct vertices $\{v_0, v_1, \dots, v_N\}$ such that $\{v_i, v_{i+1}\} \in E$ for all $i \in \{0, 1, \dots, N-1\}$. An undirected graph is said to be *connected* if any two nodes in V is connected by a path. The adjacency matrix $A = [A_{ij}] \in \mathbb{R}^{n\times n}$ is defined by $A_{ij} = 1$ if $\{v_i, v_j\} \in E$ and $A_{ij} = 0$ otherwise. Note that A is symmetric for an undirected graph. In an undirected graph, let the neighbours of node $v_i \in V$ be defined as $N_i := \{v_j \in V : \{v_i, v_j\} \in E\}$ and denote its degree by $|N_i|$. The graph Laplacian is defined as $L := \mathrm{diag}(|N_i|) - A$. L has a zero eigenvalue corresponding to the vector of ones $1_n \in \mathbb{R}^n$. The multiplicity of the zero eigenvalue is one if the graph is connected [19]. The Laplacian matrix can be factorised as $L = DD^T$, where $D = [D_{ik}] \in \mathbb{R}^{n\times m}$ is the incidence matrix. It is defined by associating an orientation to every edge of the graph: for each $e_k = \{v_i, v_j\} = \{v_j, v_i\}$, one of v_i, v_j is defined to be the head and the other tail of the edge.

$$D_{ik} := \begin{cases} +1 & \text{if } v_i \text{ is the head of } e_k \\ -1 & \text{if } v_i \text{ is the tail of } e_k \\ 0 & \text{otherwise.} \end{cases}$$

Note that the Laplacian matrix is invariant to the choice of orientation.

III. Problem formulation

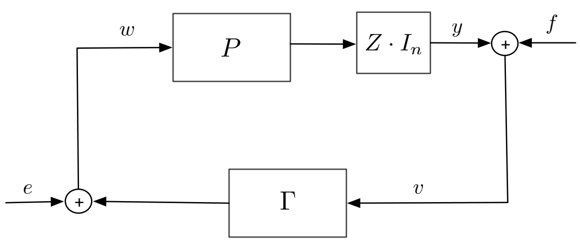

Fig. 1. Feedback setup for synchronisation.

Consider the feedback interconnection in Figure 1. There, $P := \bigoplus_{i=1}^n P_i$ with the dynamical agents $P_i \in \mathbf{S}$ and $\Gamma \in \mathbf{S}^{n\times n}$ denotes the interconnection matrix. Z is a scalar proper rational transfer function analytic on $\mathbb{C}_+$ and has a finite number of non-repeated poles on $j\mathbb{R}$. These poles on the imaginary axis describe the trajectory of the output signal y under synchronisation. The interactions between the agents is determined by an underlying undirected and connected graph $\mathcal{G} = (V, E)$, where each node $v_i \in V$ is associated with a corresponding P_i and the edges describe the communication/connections between the agents. Figure 1 models the problem of synchronisation of a network of heterogeneous agents interconnected through a dynamic matrix.

The following standing assumption is made throughout the paper, where $j\mathcal{Q}$ denotes the set of poles of Z on $j\mathbb{R}$.

Assumption 3.1: For every $jq \in j\mathcal{Q}$, $\Gamma(jq)$ has a simple zero eigenvalue corresponding to the eigenvector 1_n.

In the simplest case, Γ can be equal to L, the graph Laplacian matrix for the graph $\mathcal{G}$. Dynamics can be included via the expression $\Gamma = D\,\mathrm{diag}(\Gamma_i)\,D^T$, where D denotes the incidence matrix and $\Gamma_i \in \mathcal{S}$ for $i = 1, \dots, m$; see Figure 2. Note that for both cases Γ satisfies Assumption 3.1 by the connectedness of the graph $\mathcal{G}$.

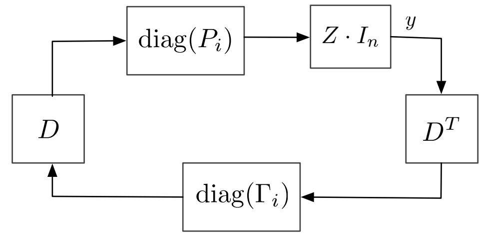

Fig. 2. A synchronisation setup with dynamical interconnection matrix.

Definition 3.2: The interconnection in Figure 1 is said to reach synchronisation if $|y_i(t) - y_j(t)| \to 0$ as $t \to \infty$ for all $i, j \in \{1, 2, \dots, n\}$. In other words, $\lim_{t\to\infty} y(t)$ lies in the subspace spanned by 1_n, i.e. $\mathrm{span}\{1_n\}$. This means the output y_i of each of the agent P_i synchronises to the same trajectory defined by the imaginary-axis poles of Z.

Remark 3.3: By defining $Z(s) := 1/s$, one recovers the standard consensus problem where all y_i's are to asymptotically converge to the same constant value. By contrast, if $Z(s) := \frac{\omega_0}{s^2+\omega_0^2}$ and synchronisation takes place, then each y_i will converge to a sinusoid of frequency ω_0 and the same phase/magnitude.

IV. Integral quadratic constraint based analysis of synchronisation

This section introduces a unified framework within which to analyse the problem of synchronisation using integral quadratic constraints (IQCs) [13]. First an IQC-based stability result is in order. Throughout this section, $j\mathcal{Q} = \{jq_1, jq_2, \dots, jq_K\}$ is taken to be the finite set of poles of Z on $j\mathbb{R}$.

Definition 4.1: Given $\epsilon \geq 0$, $P : \mathbf{H}^n_{2\epsilon}(j\mathcal{Q}) \to \mathbf{H}^n_{2\epsilon}(j\mathcal{Q})$ and $\Gamma : \mathbf{H}^n_{2\epsilon}(j\mathcal{Q}) \to \mathbf{H}^n_{2\epsilon}(j\mathcal{Q})$ in $\mathbf{S}^{n\times n}$, and a proper rational transfer function $Z : \mathbf{H}_{2\epsilon}(j\mathcal{Q}) \to \mathbf{H}_{2\epsilon}(j\mathcal{Q})$ that is analytic on $\mathbb{C}_+$, the feedback interconnection in Figure 1:

$$\begin{cases} v = ZPw + f \\ w = \Gamma v + e \end{cases} \tag{1}$$

is said to be $\mathbf{H}_{2\epsilon}$-*stable* if the map $(v, w) \mapsto (f, e)$ has a bounded inverse on $\mathbf{H}^{2n}_{2\epsilon}$, where the shorthand notation ZP has been used to denote $(Z \cdot I_n)P$.

Given an $\epsilon \geq 0$, define the graph of $X \in \mathbf{C}_\epsilon(j\mathcal{Q})^{n\times m}$ to be

$$\mathcal{G}_\epsilon(X) := \begin{bmatrix} I_m \\ X \end{bmatrix} \mathbf{H}^m_{2\epsilon}(j\mathcal{Q}) = \left\{ \begin{bmatrix} u \\ y \end{bmatrix} \in \mathbf{H}^{n+m}_{2\epsilon}(j\mathcal{Q}) : y = Xu \right\}$$

Similarly, define the (inverse) graph

$$\mathcal{G}'_\epsilon(X) := \begin{bmatrix} X \\ I_m \end{bmatrix} \mathbf{H}^m_{2\epsilon}(j\mathcal{Q}) = \left\{ \begin{bmatrix} u \\ y \end{bmatrix} \in \mathbf{H}^{n+m}_{2\epsilon}(j\mathcal{Q}) : u = Xy \right\}$$

Theorem 4.2: Given $\epsilon > 0$, $P, \Gamma \in \mathbf{S}^{n\times n}$, the feedback interconnection of ZP and Γ in Figure 1 is $\mathbf{H}_{2\epsilon}$-stable if there exists a $\Pi \in \mathbf{C}_\epsilon(j\mathcal{Q})^{2n\times 2n}$ such that the following IQC conditions hold:

(i) $\langle v, \Pi v\rangle_{\mathcal{C}_\epsilon(j\mathcal{Q})} \geq 0$ for all $v \in \mathcal{G}_\epsilon(ZP)$;

(ii) there exists a $\gamma > 0$ for which $\langle w, \Pi w\rangle_{\mathcal{C}_\epsilon(j\mathcal{Q})} \leq -\gamma\|w\|^2_{\mathcal{C}_\epsilon(j\mathcal{Q})}$ for all $w \in \mathcal{G}'_\epsilon(\tau\Gamma)$ and $\tau \in [0,1]$.

Proof: Using an argument in the proof of [20, Lem. 5.1], let $\Psi := 2\Pi + \gamma I$, the IQC conditions thus become

$$\langle v, \Psi v\rangle_{\mathcal{C}_\epsilon(j\mathcal{Q})} \geq \gamma\|v\|^2_{\mathcal{C}_\epsilon(j\mathcal{Q})} \ \forall v \in \mathcal{G}_\epsilon(ZP)$$

and

$$\langle w, \Psi w\rangle_{\mathcal{C}_\epsilon(j\mathcal{Q})} \leq -\gamma\|w\|^2_{\mathcal{C}_\epsilon(j\mathcal{Q})} \ \forall w \in \mathcal{G}'_\epsilon(\tau\Gamma), \tau \in [0,1].$$

It follows that for any $v \in \mathcal{G}_\epsilon(ZP)$, $w \in \mathcal{G}'_\epsilon(\tau\Gamma)$ and $\tau \in [0,1]$,

$$\begin{aligned}
&\gamma(\|v\|^2_{\mathcal{C}_\epsilon(j\mathcal{Q})} + \|w\|^2_{\mathcal{C}_\epsilon(j\mathcal{Q})}) \\
&\leq \langle v, \Psi v\rangle_{\mathcal{C}_\epsilon(j\mathcal{Q})} - \langle w, \Psi w\rangle_{\mathcal{C}_\epsilon(j\mathcal{Q})} \\
&= \langle v+w, \Psi(v+w)\rangle_{\mathcal{C}_\epsilon(j\mathcal{Q})} - 2\langle w, \Psi(v+w)\rangle_{\mathcal{C}_\epsilon(j\mathcal{Q})} \\
&\leq \|\Psi\|_\infty\|v+w\|^2_{\mathcal{C}_\epsilon(j\mathcal{Q})} + 2\|\Psi\|_\infty\|w\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}\|v+w\|_{\mathcal{C}_\epsilon(j\mathcal{Q})} \\
&\leq \|\Psi\|_\infty\|v+w\|^2_{\mathcal{C}_\epsilon(j\mathcal{Q})} \\
&\qquad + \frac{2\|\Psi\|^2_\infty\|v+w\|^2_{\mathcal{C}_\epsilon(j\mathcal{Q})}}{\gamma} + \frac{\gamma}{2}\|w\|^2_{\mathcal{C}_\epsilon(j\mathcal{Q})},
\end{aligned}$$

where the last inequality holds since $2xy \leq \frac{x^2}{\beta} + \beta y^2$ for any $x, y, \beta \in \mathbb{R}$. This implies

$$\begin{aligned}
&\left(1 + \frac{2}{\gamma}\|\Psi\|_\infty\right)\|\Psi\|_\infty\|v+w\|^2_{\mathcal{C}_\epsilon(j\mathcal{Q})} \qquad (2)\\
&\qquad \geq \gamma\|v\|^2_{\mathcal{C}_\epsilon(j\mathcal{Q})} + \frac{\gamma}{2}\|w\|^2_{\mathcal{C}_\epsilon(j\mathcal{Q})} \\
&\qquad \geq \frac{\gamma}{2}\|w\|^2_{\mathcal{C}_\epsilon(j\mathcal{Q})} \\
&\implies \|v+w\|^2_{\mathcal{C}_\epsilon(j\mathcal{Q})} \geq \eta^2\|w\|^2_{\mathcal{C}_\epsilon(j\mathcal{Q})}, \qquad (3)
\end{aligned}$$

for any positive $\eta \leq \frac{\gamma}{\sqrt{2\|\Psi\|_\infty(\gamma+2\|\Psi\|_\infty)}}$.

Now observe that $\tau \in [0,1] \mapsto \tau\Gamma$ is continuous in the graph topology induced by the gap metric with the ambient space taken to be $\mathbf{H}_{2\epsilon}(j\mathcal{Q})$ [16]. To be specific, the gap distance between two systems $\Delta_1 : \mathbf{H}^n_{2\epsilon}(j\mathcal{Q}) \to \mathbf{H}^n_{2\epsilon}(j\mathcal{Q})$ and $\Delta_2 : \mathbf{H}^n_{2\epsilon}(j\mathcal{Q}) \to \mathbf{H}^n_{2\epsilon}(j\mathcal{Q})$ is given by

$$\delta(\Delta_1, \Delta_2) := \|\Pi_{\mathcal{G}_\epsilon(\Delta_1)} - \Pi_{\mathcal{G}_\epsilon(\Delta_2)}\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}.$$

Since the feedback interconnection is $\mathbf{H}_{2\epsilon}$-stable for $\tau = 0$, inequality (3) shows that the corresponding robust stability margin $b_{\mathcal{M},\mathcal{N}}$ in [16, Section 5] is bounded away from zero with $\mathcal{N} = \mathcal{G}_\epsilon(ZP)$ and $\mathcal{M} = \mathcal{G}'_\epsilon(0)$; see [21, Lem. 3.2.8]. In particular, $b_{\mathcal{M},\mathcal{N}} \geq \eta > 0$. By continuity in the graph topology, there exists an $\zeta > 0$ such that $\delta(ZP, \tau\Gamma) < \eta \leq b_{\mathcal{M},\mathcal{N}}$ for all $\tau \in [0, \zeta]$. Application of [16, Thm. 3] then leads to the feedback interconnection of ZP and $\tau\Gamma$ being $\mathbf{H}_{2\epsilon}$-stable for $\tau \in [0, \zeta]$. Repetitively applying the aforementioned arguments yields feedback stability for $\tau \in [\zeta, 2\zeta], [2\zeta, 3\zeta], \ldots$ in succession, and eventually for $\tau = 1$, as required. ∎

The main result on synchronisation is stated next.

Theorem 4.3: The feedback configuration in Figure 1 with $P := \bigoplus_{i=1}^n P_i : P_i \in \mathbf{S}$, a proper rational transfer function Z with non-repeated poles on $j\mathbb{R}$ and is analytic on $\mathbb{C}_+$, and $\Gamma \in \mathbf{S}^{n\times n}$ that satisfies Assumption 3.1 reaches synchronisation if there exists a $\Pi \in \mathbf{C}^{2n\times 2n}$ such that for all $\omega \in \mathbb{R}\backslash\mathcal{Q} = (q_1, \infty)\cup(q_2, q_1)\cup\ldots\cup(q_K, q_{K-1})\cup(-\infty, q_K)$,

(i) $\begin{bmatrix} I_n \\ Z(j\omega)P(j\omega)\end{bmatrix}^* \Pi(j\omega)\begin{bmatrix} I_n \\ Z(j\omega)P(j\omega)\end{bmatrix} \geq 0;$

(ii) $\begin{bmatrix} \tau\Gamma(j\omega) \\ I_n\end{bmatrix}^* \Pi(j\omega)\begin{bmatrix} \tau\Gamma(j\omega) \\ I_n\end{bmatrix} \leq -\gamma \ \forall\tau \in [0,1]$, where γ is some strictly positive constant.

Proof: The conditions of the theorem imply that the IQC conditions in Theorem 4.2 are satisfied for arbitrarily small $\epsilon > 0$, whereby the feedback configuration is $\mathbf{H}_{2\epsilon}$-stable. In turn, this implies that

$$\begin{aligned}
&Z(s)P(s)\left(I - \Gamma(s)Z(s)P(s)\right)^{-1} \\
&\qquad = P(s)\left(\frac{1}{Z(s)}I - \Gamma(s)P(s)\right)^{-1}
\end{aligned}$$

has no poles on $\bar{\mathbb{C}}_+\backslash j\mathcal{Q}$, i.e. $\det(\frac{1}{Z(s)}I - \Gamma(s)P(s))$ has no zeros on $\bar{\mathbb{C}}_+\backslash j\mathcal{Q}$. Moreover, by Assumption 3.1, $\det(\frac{1}{Z(s)}I - \Gamma(s)P(s))$ has a simple zero at every $s \in j\mathcal{Q}$ corresponding to the null space N satisfying $P(s)N \subset \operatorname{span}\{1_n\}$.

Now note that for any $e, f \in \mathbf{L}_2$, it can be derived from (1) and Figure 1 that

$$\begin{aligned}
\hat{y} &= ZP(I - \Gamma ZP)^{-1}(\hat{e} + \Gamma\hat{f}) \\
&= P\left(\frac{1}{Z}I - \Gamma P\right)^{-1}(\hat{e} + \Gamma\hat{f}),
\end{aligned}$$

which has a simple pole at every point in $j\mathcal{Q}$. As such, it follows from the above that $\hat{y} = \hat{v} + \hat{w}$, for some $\hat{v} \in \mathbf{H}_2$ and $\hat{w} \in \bar{Z}\operatorname{span}\{1_n\}$, where $\bar{Z}$ has only simple poles on $j\mathbb{R}$ at the same locations as those of Z. Taking the inverse Laplace transform yields that $y = v + w$, where $\lim_{t\to\infty} v(t) = 0$ because $v \in \mathbf{L}_2$ and $w(t)$ is a vector with equal entries corresponding to the trajectory defined by the imaginary-axis poles of Z for all $t \geq 0$. In other words, the feedback interconnection reaches synchronisation as time approaches infinity. ∎

Remark 4.4: It can be seen from the proof of Theorem 4.2 that the conditions of Theorem 4.3 may also be written as

(i) $\begin{bmatrix} I_n \\ Z(j\omega)P(j\omega)\end{bmatrix}^* \Pi(j\omega)\begin{bmatrix} I_n \\ Z(j\omega)P(j\omega)\end{bmatrix} \geq \gamma > 0;$

(ii) $\begin{bmatrix} \tau\Gamma(j\omega) \\ I_n\end{bmatrix}^* \Pi(j\omega)\begin{bmatrix} \tau\Gamma(j\omega) \\ I_n\end{bmatrix} \leq 0 \ \forall\tau \in [0,1],$

for all $\omega \in \mathbb{R}\backslash\mathcal{Q}$.

V. Conclusions

The paper demonstrates that it is possible to apply integral quadratic constraint based analysis to the study of synchronisation problems for heterogeneous networks without the use of loop transformations. Future research may involve developing synchronisation results for systems with imaginary-axis poles of order greater than 1 and for nonlinear systems in a similar spirit.

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