# A Nyquist criterion for synchronization in networks of heterogeneous linear systems

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**Abstract:**
We study the synchronization of a set of SISO subsystems interconnected via a time-invariant Laplacian matrix. By synchronization we mean that the outputs of all subsystems must be asymptotically equal to each other and behave in the same manner. We assume that the subsystems can be represented as the sum of a common nonzero transfer function plus a perturbation. A Nyquist-type criterion is established which ensures synchronization provided that the convex hull of the frequency responses of the subsystems does not intersect a certain region defined by the spectrum of the interconnection matrix. The result is applied to a variety of examples of different nature for which synchronization takes place. A counter example for which complete synchronization is impossible is also presented.

## 1. INTRODUCTION

Over the past few years, scientists from several research communities have put much effort into the field of distributed networks, both for control and estimation purposes. This has been motivated by the interest in large scale plants, in which centralized control is computationally prohibitive. It is preferable to allow each agent to decide its own action based on the knowledge of a limited set of neighbors. Another natural application of distributed algorithms is formation control, in which a set of autonomous vehicles are coordinated to follow the same trajectory. In all such applications, the set of agents need not be identical. Hence, along with the coordination we need to ensure a certain degree of robustness to model imperfections. The purpose of this paper is to address the problem of robustness.

In this paper we consider the problem of synchronization in a network of $N$ dynamic agents, where each agent is modeled as a linear time invariant system described by its SISO transfer function $H_k(s)$, $k = 1, \ldots, N$. By synchronization, we mean that each output must asymptotically approach the synchronized state, in which they all share the same, possibly time-varying value.

The stabilization problem for a set of identical autonomous linear systems has been widely studied in the last decades. The paper Pecora and Carroll (1998) proposes a general way to prove the stability of the synchronized state by means of a suitable variational equation. Recently, the problem has been addressed in Fax and Murray (2004) for the scenario of vehicle control by using a Nyquist type criterion. In Scardovi and Sepulchre (2009) the interest is switched to state-space representation, showing how it is possible for a set of identical linear systems to synchronize their outputs. This is achieved by controlling the outputs to a manifold which is defined by a linear combination of the modes of the systems, as if all of them originated from a common, fictitious, initial condition. In this article we proceed a step further. Given a set of different linear systems interconnected via a matrix $\Gamma$, we provide a Nyquist type criterion which ensures that the interconnection synchronizes all the outputs in the sense already described.

This type of robustness result has to our knowledge previously only been addressed for the case of consensus networks in Lestas and Vinnicombe (2007); Jönsson and Kao (2010). Our main results extends the heterogeneous consensus result Jönsson and Kao (2010) to a more general class of systems. The new criterion is satisfactory from several points of view. First of all, it is capable of assuring synchronization for nonidentical systems. This is useful when the subsystems are subjected to perturbations, as in real plants, but also in situations in which they are nonidentical by construction. Moreover, the criterion provided is a computationally inexpensive test on the subsystems frequency responses that is easily verified in the frequency domain. The continuous and the discrete time cases are completely analogous.

A possible application of our criterion is in the field of clock synchronization. This is a topic which has attracted much attention due to the fact that in practical applications agents need to perform their actions in a restricted time interval in order to save energy consumption, and thus the necessity of a common time. The problem has been addressed via hierarchical algorithms such as leader election in a spanning tree (Ganeriwal et al. (2003), Maróti et al. (2004)), or clustering of the network (Elson et al. (2002)).

Despite the effectiveness of such algorithms, they suffer from bad scalability characteristics and from failure of

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nodes. Full distributed algorithms have been proposed (Schenato and Fiorentin (2009), Solis et al. (2006)). Since all the agents in such algorithms act in exactly the same way, there is no need for a leader and since the network remains connected it does not suffer from node failures. We proceed in this direction. The model we consider has been originally considered in Carli et al. (2008), where clocks are modeled as integrators with constant and identical disturbances, which model the skew. This simple and ideal case has been modified in Carli et al. (2010), where each clock is modeled as a double integrator whose skew is not constant, and where there is no leader. In order to synchronize the clocks the local times are exchanged over a network, and their relative differences are used to produce a local control acting both on the local time and on the local skew.

This is the model we use in our example. The main results in this paper are well suited to address the synchronization in networks of such clocks due to the inherent difference between each subsystem. Our results are also illustrated in examples where linear oscillators and unstable systems are synchronized.

The paper is organized as follows. In Sect. 2 we state our main result on synchronization as well as some corollaries. The results are applied to several interesting cases in Sect. 3. We end with some conclusions and proposals for future work in Sect. 4.

## 2. MAIN RESULT

Our main results provide sufficient conditions for the synchronization of a finite set of linear SISO time invariant systems, represented by rational transfer functions $H_k$, $k = 1, \dots, n$, which are interconnected via a matrix $\Gamma$. Our standing assumption is that $\Gamma$ is normal and such that $\Gamma \mathbf{1} = 0$. It is thus diagonalizable via a unitary matrix $U$. It is no loss of generality to assume $U = [\mathbf{1}\ V]$ and thus $U^* \Gamma U = \operatorname{diag}(0, \lambda_2, \dots, \lambda_n)$. Note that since $U$ is unitary it follows $V^* \mathbf{1} = 0$.

If the interconnection is defined by a communication graph $\mathcal{G} = (V, \mathcal{E})$, then the matrix $\Gamma$ is a weighted Laplacian consistent with $\mathcal{G}$. Namely, $\Gamma_{ij} = 0$ if $(i, j) \notin \mathcal{E}$, $\Gamma_{ij} > 0$ if $i \neq j$ and $(i, j) \in \mathcal{E}$, and finally $\Gamma \mathbf{1} = 0$. If $\mathcal{G}$ is strongly connected, it is possible to show that $\dim \ker \Gamma = 1$, so $\ker \Gamma = \operatorname{span}\{\mathbf{1}\}$, and that all the nonzero eigenvalues of $\Gamma$ have negative real part.

Let us now define some sets which will be used in the main result. Given a set $\Lambda$, we denote by $\mathrm{co}\{\Lambda\}$ the convex hull of the elements of $\Lambda$, namely

$$\mathrm{co}\{\Lambda\} = \{\sum_{i=1}^{N} \alpha_i \lambda_i,\ \alpha \geq 0,\ \sum_{i=1}^{N} \alpha_i = 1,\ \lambda_i \in \Lambda\}.$$

We denote with $\mathcal{N}[H_1, \dots, H_m](\omega)$ the 3d-Nyquist polytope of the set of subsystems $H_1, \dots, H_m$

$$\mathcal{N}[H_1, \dots, H_m](\omega) := \mathrm{co}\{(\mathrm{Re} H_k(j\omega),\ \mathrm{Im} H_k(j\omega),\ |H_k(j\omega)|^2) : k = 1, \dots, n\}, \tag{1}$$

and with $\Omega_e$ the *instability region* defined by the spectrum of $\Gamma$ as

$$\Omega_e := (0,\, 0,\, \mathbb{R}^+) + \mathrm{co}\left\{\left(\mathrm{Re}\frac{1}{\lambda_k},\ \mathrm{Im}\frac{1}{\lambda_k},\ \frac{1}{|\lambda_k|^2}\right) : k = 2, \dots, n\right\}. \tag{2}$$

We are now ready to state and prove our main result.

*Theorem 1.* Consider the continuous time system

$$\begin{cases} y = Hu \\ u = \Gamma y + r \end{cases} \tag{3}$$

where $H(s) = \operatorname{diag}(H_k(s) : k = 1, \dots, n)$. Assume moreover that $H_k(s)$ can be decomposed as

$$H_k(s) = N_0(s) + N_k(s)$$

where $N_0(s)$ is a "nominal plant" and $N_k(s)$ is a small perturbation of it. Assume also $H_0(s) = N_0(s)$.

Let $\alpha \in \mathbb{R}$, $\alpha > 0$, and define $N_{\alpha 0}(s) = N_0(s - \alpha)$ and $N_{\alpha k}(s) = N_k(s - \alpha)$, and analogously for $H_k(s)$, $k = 0, \dots, n$.

Assume that $\alpha$ is chosen such that:

i) for every nonzero eigenvalue $\lambda_k$ of $\Gamma$ we have that $W_{\alpha 0}(s) = \frac{N_{\alpha 0}(s)}{1 - N_{\alpha 0}(s)\lambda_k}$ is a stable system and such that $1 - N_{\alpha 0}(s)\lambda_k$ is nonsingular on the imaginary axis;
ii) the transfer functions $\frac{N_{\alpha k}(s)}{N_{\alpha 0}(s)}$ are stable;
iii) it holds
$$\mathcal{N}[H_{\alpha 0}, \dots, H_{\alpha n}](\omega) \cap \Omega_e = \emptyset, \forall\, \omega \in \mathbf{R} \cup \{\infty\}.$$

Then the outputs of the system satisfy $e^{\alpha t} y(t) \longrightarrow \operatorname{span}\{\mathbf{1}\}$ as $t \to \infty$ for any input $r$ which satisfies $e^{\alpha t} r(t), e^{\alpha t} \dot{r}(t) \in \mathbf{L}_2[0, \infty)$.

**Proof.** See detailed version of paper on ArXive.

The particular value of $\alpha$ is actually not important as long as the constraints in *ii*) and *iii*) are satisfied. It gives a lower bound on the rate of convergence to the synchronized state, namely the components of the states not aligned with $\mathbf{1}$ converge to zero exponentially with exponent al least $-\alpha$. Usually, we have $0 < \alpha \leq -\max_{k \neq 1} \mathrm{Re}\, \lambda_k$ so the second largest eigenvalue of the Laplacian is an upper bound on the rate of convergence.

Note that the input $r$ represents disturbances and the effect of initial conditions. If we assume that $H$ has a state space realization

$$y = Hu \ \Leftrightarrow \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

then provided that the pair $(A, B)$ is controllable we can generate arbitrary initial conditions using $r$. Indeed, for any $x_0$, there exists an input signal, $u^0$, defined over $-T_0 \leq t \leq 0$ such that the solution to $\dot{x} = Ax + Bu$, $x(-T_0) = 0$ is $x(0) = x_0$. Then the choice $r(t) = u^0(t) - \Gamma(Cx^0(t) + Du^0(t))$, for $-T_0 \leq t \leq 0$ and $r(t) = 0$ for $t > 0$ gives the desired initial condition, i.e. the system (3) could then be interpreted as the state space system

$$\dot{x} = (A + B(I - \Gamma D)^{-1} \Gamma C)x, \quad x(0) = x_0.$$

The result in Theorem 1 has a discrete time counterpart. For example, if we for the sake of simplicity use a unitary sample period, the discrete time formulation follows by using the Tustin transform (see Åström and Wittenmark (1997)): the Nyquist plots of $H_k(s)$ and of $H_k(z^{-1})$ are the same and one system is stable if, and only if, the other one is stable. The primary distinction is that now the frequency shift $s - \alpha$ with $\alpha > 0$ becomes the multiplication $\alpha z^{-1}$ with $|\alpha| > 1$.

We thus have the following corollary.

*Corollary 2.* Consider the discrete time system

$$\begin{cases} y = Hu \\ u = \Gamma y + r \end{cases} \tag{4}$$

where $H(z^{-1}) = \text{diag}\left(H_k(z^{-1}) : k = 1, \dots, n\right)$ [1]. Assume moreover that $H_k(z^{-1})$ can be decomposed as

$$H_k(z^{-1}) = N_0(z^{-1}) + N_k(z^{-1})$$

where $N_0(z^{-1})$ is a "nominal plant" and $N_k(z^{-1})$ is a small perturbation of it.

Let $\alpha \in \mathbb{R}$, $|\alpha| > 1$, and define $N_{\alpha 0}(z^{-1}) = N_0(\alpha z^{-1})$ and $N_{\alpha k}(z^{-1}) = N_k(\alpha z^{-1})$.

Assume that $\alpha$ is such that:

i) for every nonzero eigenvalue $\lambda_k$ of $\Gamma$ we have that $W_{\alpha 0}(z^{-1}) = \frac{N_{\alpha 0}(z^{-1})}{1 - N_{\alpha 0}(z^{-1})\lambda_k}$ is a stable system and such that $1 - N_{\alpha 0}(z^{-1})\lambda_k$ is nonsingular on the unit circle.

ii) the transfer functions $\frac{N_{\alpha k}(z^{-1})}{N_{\alpha 0}(z^{-1})}$ are stable;

iii) it holds

$$\mathcal{N}\left[H_{\alpha 0}, \dots, H_{\alpha n}\right](e^{j\omega}) \cap \Omega_e = \emptyset, \forall\, \omega \in \mathbf{R} \cup \{\infty\}.$$

Then the outputs of the system satisfy $\alpha^t y(t) \longrightarrow \text{span}\{\mathbf{1}\}$ as $t \to \infty$ for any input $r$ which satisfies $\alpha^t r(t) \in l_2[0, \infty)$.

The interpretation of (4) in state space domain is analogous to the continuous time case.

### 2.1 Derived criteria

Condition *iii)* in Theorem 1 can be rewritten in an equivalent way, as explained in Jönsson and Kao (2010), by using the so called Inverse Nyquist polytope $\check{\mathcal{N}}\left[H_1, \dots, H_m\right](\omega)$, which is defined as

$$\text{co}\left\{\left(\text{Re}\frac{1}{H_k(j\omega)},\, \text{Im}\frac{1}{H_k(j\omega)},\, \frac{1}{|H_k(j\omega)|^2|}\right) : k = 1, \dots, n\right\} \tag{5}$$

and the corresponding instability region

$$\check{\Omega}_e := \text{co}\left\{\left(\text{Re}\lambda_k,\, \text{Im}\lambda_k,\, |\lambda_k|^2\right) : k = 1, \dots, n\right\}, \tag{6}$$

where we used that $\lambda_1 = 0$.

These criteria can sometimes be hard to visualize. In order to simplify them, we can project all the convex sets to the complex plane. If we do this, concerning the first criterion we have to impose that the sets

$$\mathcal{N}_{2d}\left[H_1, \dots, H_m\right](\omega) := \text{co}\{H_1(j\omega), \dots, H_n(j\omega)\} \tag{7}$$

and

$$\Omega_{2de} := \text{co}\left\{\frac{1}{\lambda_2}, \dots, \frac{1}{\lambda_n}\right\} \tag{8}$$

do not intersect.

For the Inverse criterion, the corresponding sets are

$$\check{\mathcal{N}}_{2d}\left[H_1, \dots, H_m\right](\omega) := \text{co}\left\{\frac{1}{H_1(j\omega)}, \dots, \frac{1}{H_n(j\omega)}\right\} \tag{9}$$

and

$$\check{\Omega}_{2de} := \text{co}\{0, \lambda_2, \dots, \lambda_n\}. \tag{10}$$

[1] We will often use the symbol $\oplus_{k=1}^{n} H_k = \text{diag}\,(H_k : k = 1, \dots, n)$.

## 3. EXAMPLES

In this section we are going to apply Theorem 1 to several examples: clock synchronization, unstable systems synchronization and synchronization of oscillators.

### 3.1 Clocks Synchronization

In this section we will apply our result to the clock model presented in Carli et al. (2010), which we will now briefly describe. We consider a synchronous version of a synchronization algorithm for clocks. This scenario is somewhat unrealistic, since synchrony is the objective of the algorithm, not the assumption. However, although the model considered for the clocks is ideal, it nonetheless captures some of the difficulties of the problem. We will assume that each clock is characterized by a two component vector $x_k$: the first component is the relative time for the clock, while the second is an estimate of its skew. We assume that each clock has an unknown parameter, the skew $\delta_k$, and that, relatively to an external fictitious clock, it behaves as a ramp with slope $\delta_k$. Observe that this parameter is intrinsically unknowable, because each clock could assume its skew to be the "right one". Let us assume to sample the time with period $\delta_\tau$: we obtain the following model for the clocks:

$$\begin{cases} x_k(t+1) = \begin{bmatrix} 1 & q_k \\ 0 & 1 \end{bmatrix} x_k(t) + F u_k(t), \\ y_k(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k(t), \end{cases} \tag{11}$$

where $q_k = \frac{\delta_\tau}{\delta_k}$ and where we assume that we can only sense the time of each clock, namely its first component. The input is a scalar feedback from the output of the interconnected system. In particular, we assume the following form for the input:

$$u_k(t) = \sum_{j=1}^{n} \Gamma_{kj} y_j(t)$$

where $\Gamma$ is the Laplacian corresponding to interconnection topology. We assume it to be normal and such that $\Gamma \mathbf{1} = 0$. Let us write $F = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$. We assume it is fixed for each subsystem.

The problem we want to deal with is the following: which constraints should we pose on $F$ and on $\Gamma$ in order for the system to *synchronize* in the sense that $y(t+1) - y(t) \longrightarrow \text{span}\{\mathbf{1}\}$? We shall use the tools developed in Sect. 2 in order to answer this question.

First of all, we must obtain the $z$-transfer function for our model: it is straightforward to obtain

$$H_k(z^{-1}) = \frac{z^{-2}(f_2 q_k - f_1) + f_1 z^{-1}}{(1 - z^{-1})^2}.$$

Let us now assume that $q_k$ can be modeled as $q_k = q + \varepsilon_k$, where $q$ is some "nominal value" of the skew. We can then decompose each subsystem as $H_k(z^{-1}) = N_0(z^{-1}) + N_k(z^{-1})$ with

$$N_0(z^{-1}) = \frac{z^{-2}(f_2 q - f_1) + f_1 z^{-1}}{(1 - z^{-1})^2}$$

$$N_k(z^{-1}) = \frac{f_2 \varepsilon_k z^{-2}}{(1 - z^{-1})^2}.$$

Let $\alpha > 1$ be a value such that the two roots of the polynomial

$$D_\alpha(z^{-1}) = (1 - \alpha z^{-1})^2 - [(f_2 q - f_1)\alpha^2 z^{-2} + f_1 \alpha z^{-1}]\lambda_k$$

have absolute value smaller that 1 for each $\lambda_k$, $k = 2, \dots, n$. Then assumption $i)$ is satisfied. Regarding assumption $ii)$, observe that

$$\frac{N_{\alpha k}(z^{-1})}{N_{\alpha 0}(z^{-1})} = \frac{f_2 \varepsilon_k}{\alpha^2 z^{-2}(f_2 q - f_1) + \alpha f_1 z^{-1}}$$

and it turns out that we need to impose $|1 - \frac{f_2}{f_1} q| < \alpha$.

Once this is done, if the perturbations are small enough such that also assumption $iii)$ is satisfied, then the system is stable in the sense that $\alpha^t y(t) \to \text{span}\{\mathbf{1}\}$, as $t \to \infty$. We want to show something more, namely that the system synchronizes to the behavior of a clock, which is a ramp.

This can be done if we analyze the state space form of our system. Let us define $Q = \text{diag}\,(q_1, \dots, q_n)$ and $x(t) = \begin{bmatrix} x'(t) \\ x''(t) \end{bmatrix} \in \mathbb{R}^{2N}$ a vector whose first $N$ entries $x'(t)$ are the first states of the subsystems, and whose $N$ entries $x''(t)$ are the second states of the subsystems. It is easy to observe that the entire system evolves according to

$$x(t+1) = \begin{bmatrix} I_N - f_1\Gamma & Q \\ -f_2\Gamma & I_N \end{bmatrix} x(t) = Ax(t). \qquad (12)$$

It is easy to see that the vector $v = \begin{bmatrix} \mathbf{1}^T & 0 \end{bmatrix}^T$ is the (unique) eigenvector of $A$ relative to 1, and that $w = \begin{bmatrix} 0 & \mathbf{1}^T Q^{-1} \end{bmatrix}^T$ is the generalized eigenvector. The above argument assures that $y(t) = x'(t)$ converges to span $\{\mathbf{1}\}$, and this implies also $x''(t) \to cQ^{-1}\mathbf{1}$. We just have to show that, if the initial condition of the second state of each subsystem is positive, then $c$ is positive too. To do this, we can observe that

$$\mathbf{1}^T x''(t+1) = \mathbf{1}^T(-f_2\Gamma x'(t) + x''(t)) = \mathbf{1}^T x''(t),$$

namely the sum of all the second states is an invariant of the system. So $\mathbf{1}^T x''(0) = c\mathbf{1}^T Q^{-1}\mathbf{1}$, and $c$ is positive if the entries of $x''(0)$ are all positive. It follows that

$$y(t+1) - y(t) = -f_1\Gamma y(t) + Qx''(t) \to c\mathbf{1},$$

as $t \to \infty$, which is the desired synchronization.

Note that the proof is based on the fact that $\mathbf{1}^T\Gamma = 0$. This can be proved as follows. By assumption, $\ker\Gamma = \text{span}\,\{\mathbf{1}\}$ and $\Gamma$ is normal, hence

$$\Gamma\Gamma^T = \Gamma^T\Gamma.$$

Multiplying on the right by $\mathbf{1}$ both sides of such relation, we obtain $\Gamma\Gamma^T\mathbf{1} = \Gamma^T\Gamma\mathbf{1} = 0$, which means that $\Gamma^T\mathbf{1}$ is in the kernel of $\Gamma$, and thus that there exists some real $k$ such that $\Gamma^T\mathbf{1} = k\mathbf{1}$. Multiplying on the left by $\mathbf{1}^T$ both sides of this last equation, we obtain $\mathbf{1}^T\Gamma^T\mathbf{1} = k\mathbf{1}^T\mathbf{1} = kN$. Since $\mathbf{1}^T\Gamma^T = 0$, we obtain $k = 0$, and thus the thesis $\Gamma^T\mathbf{1} = 0$.

*Numerical example:* In this example we consider a system with $N = 9$ clocks whose skews are given by $q_k = 1 + \varepsilon_k$, $\varepsilon_k \sim \mathcal{N}(0, 10^{-2})$. The gains are constant for every clock and given by $f_1 = 1.7$ and $f_2 = 1$. The interconnection matrix is given by

$$\Gamma = -I_N + 0.22(\mathcal{C}_N + \mathcal{C}_N^{-1}) + 0.31\mathcal{C}_N^5 + 0.25\mathcal{C}_N^{-5}$$

where $\mathcal{C}_N$ is the $N \times N$ circulant matrix generated by $c = [0\ 1\ 0\ \dots\ 0]$.

In Fig. 1 is depicted a possible trajectory for the clocks. In Fig. 2 and Fig. 3 are shown respectively the $3d$ and the $2d$ Nyquist criteria for $\alpha = e^{0.1}$: by computation one can see that such an $\alpha$ respects the assumptions.

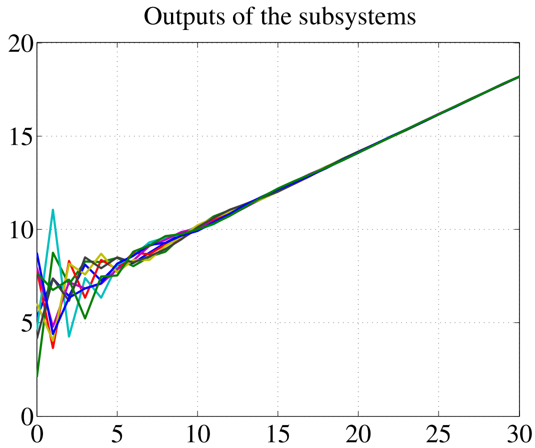


Fig. 1. Clocks synchronization. Trajectories of the output of the systems for the given network.

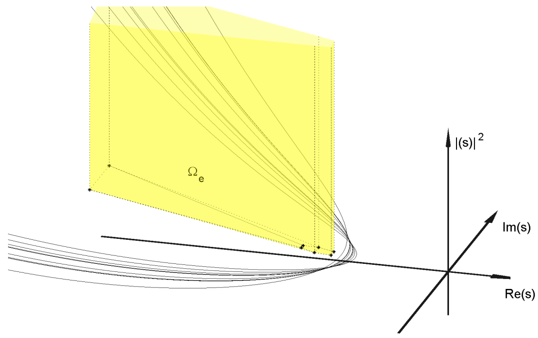


Fig. 2. Clocks synchronization. 3D Nyquist criterion applied to our subsystems destabilized by $\alpha = e^{0.1}$. In yellow the instability region. For sake of clarity the Nyquist polytope is not drawn.

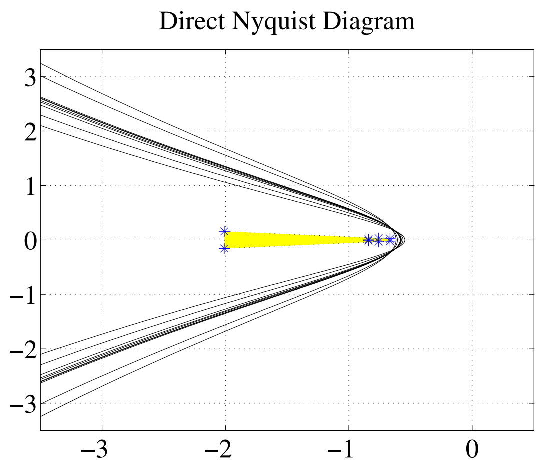


Fig. 3. Clocks synchronization. 2D Nyquist criterion applied to our subsystems destabilized by $\alpha = e^{0.1}$. In yellow the instability region. For sake of clarity the Nyquist polytope is not drawn.

### 3.2 *Synchronization of unstable subsystems*

In this example we show that if the interconnection network is well designed, synchronization can take place even if the subsystems are not stable. Consider in fact the situation in which

$$H_k(s) = \frac{1 + \Delta_k(s)}{s - \tau}$$

where $\tau > 0$ and $\Delta_k(s)$ are stable filters. Assume $\mathrm{Re}\,\lambda_k < -\eta$ for every nonzero eigenvalue $\lambda_k$ of $\Gamma$. It is easy to see that, if we take $0 < \alpha < \eta - \tau$, then assumptions *i)* and *ii)* are all satisfied provided that $\Delta_k(s - \alpha)$ are stable. If we consider "benign" $\Delta_k(s)$ such that *(iii)* in Theorem 1 is satisfied then we have the synchronization to the unstable mode $e^{\tau t}$.

*Numerical example:* Assume we have $N = 11$ agents, $\tau = 0.3$, $N_0(s) = \frac{1}{s-\tau}$ and $N_k(s) = \frac{s+\varepsilon_k}{(s+2\nu_k)(s-\tau)}$, where $\varepsilon_k \in \mathcal{U}[0,\,1]$ and $\nu_k \in \mathcal{U}[.25,\,1]^{\,2}$. The matrix $\Gamma$ is $\Gamma = -I_N + 0.2 * (\mathcal{C}_N + \mathcal{C}_N^{-1}) + 0.35 * \mathcal{C}_N^5 + 0.25 * \mathcal{C}_N^6$, where $\mathcal{C}_N$ is as before. Given that $\eta \sim 0.329$, we can take $\alpha = 0.02$. In Fig. 4 and Fig. 5 are shown a possible trajectory of the subsystems (initial conditions are uniformly taken in $[0,\,100]$) and the $2d$ Nyquist criterion (see Eq. 7 and Eq. 8).

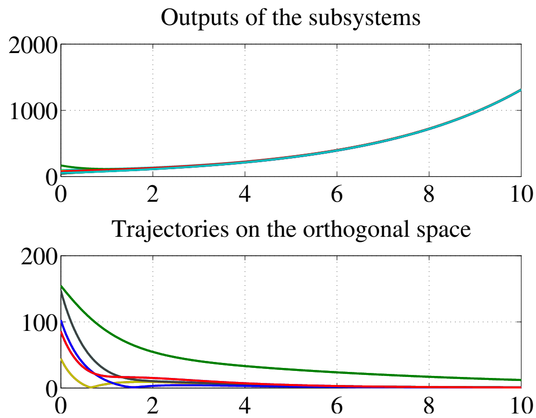


Fig. 4. Unstable synchronization. Trajectories of the output of the systems for the given network.

### 3.3 *A counterexample: the synchronization of different linear oscillators*

In this example we will show that assumption *(ii)* of the theorems is necessary. Assume our subsystems to be composed by a nominal plant of some type, $N_0(s)$, to which is added a perturbation with two complex conjugate purely imaginary poles:

$$H_k(s) = N_0(s) + N_k(s) = N_0(s) + \frac{P_k(s)}{s^2 + \omega_k^2}.$$

In this case, if $N_0(s)$ and the $H_k(s)$ are suitably chosen, it is not difficult to satisfy assumptions *i)* and *iii)*. The problem here is that assumption *ii)* surely cannot be satisfied: the transfer functions $\frac{N_{\alpha k}}{N_{\alpha 0}}$ will always have the two unstable poles $\pm j\omega_k + \alpha$. Hence, the theorem cannot be applied and in fact, in general we do not have synchronization.

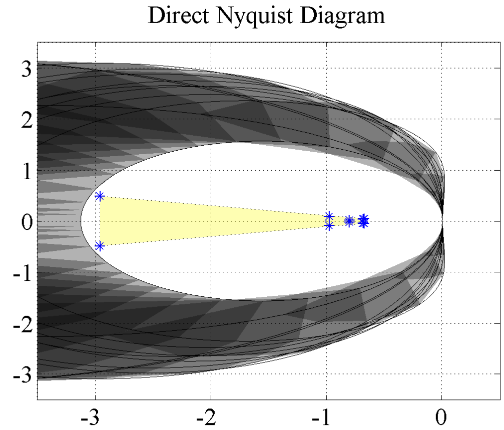


Fig. 5. Unstable synchronization. Direct 2d Nyquist criterion for the system destabilized by $\alpha = 0.02$. The black shaded region is the Nyquist polytope.

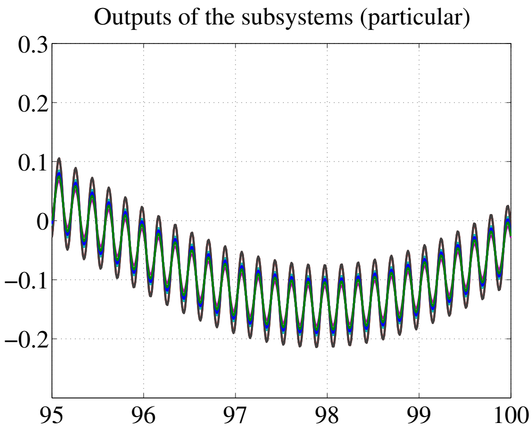


Fig. 6. Synchronization of oscillators: bad case. Zoom showing the "weak" synchronization. The low frequency carrier corresponds to the sinusoid at frequency 0.1 Hz.

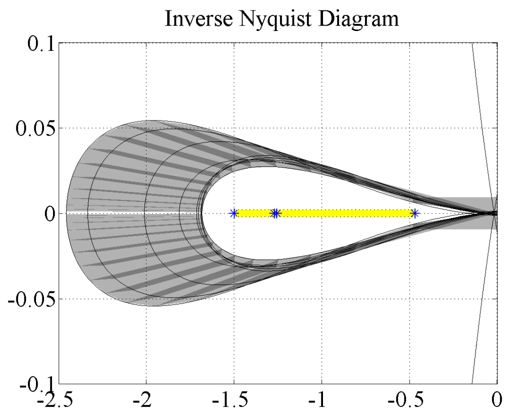


Fig. 7. Synchronization of oscillators: bad case. Zoom showing that the nonzero eigenvectors of $\Gamma$ respect the assumptions. Here $\alpha = 0.01$. The black shaded region is the Nyquist polytope.

*Numerical examples:* Let us consider the case in which $N = 9$ and

$$N_0(s) = \frac{s+1}{s^2 + (2\pi 0.1)^2}$$

and

$$N_k(s) = -\frac{1}{5}\frac{(s-1)((2+\varepsilon_k)s + 1 + \nu_k)}{s^2 + \omega_k^2}$$

where $\varepsilon_k,\, \nu_k \in \mathcal{U}[0,\, 1]$ and $\omega_k \in 10\pi + 2\pi\mathcal{U}[0,\, 1]$. The interconnection matrix is $\Gamma = -I_N + 0.2(\mathcal{C}_N + \mathcal{C}_N^{-1}) + 0.3(\mathcal{C}_N^4 + \mathcal{C}_N^5)$ ($\mathcal{C}_N$ as before).

Because the subsystems do not satisfy the assumptions, the system does not synchronize. Nonetheless, as we can see in Fig. 6, some sort of "weak" synchronization is approached. This is maybe due to the fact that the subsystems actually satisfy the Nyquist Criterion, as can be seen in Fig. 7.

As a second example, consider the case in which

$$H_k(s) = -\frac{1}{5}\frac{(s-1)(2s+1)}{s^2 + \omega_0^2} - \frac{1}{5}\frac{(s-1)(\varepsilon_k s + \nu_k)}{s^2 + \omega_0^2}$$

where $\varepsilon_k,\, \nu_k \in \mathcal{U}[0,\, 1]$ and $\omega_0 = 2\pi$. The interconnection matrix is the same as in the previous example.

In this case, taking $\alpha = 0.01$, the assumptions are all satisfied, and in fact, as we can see in Fig. 8, the subsystems synchronize.

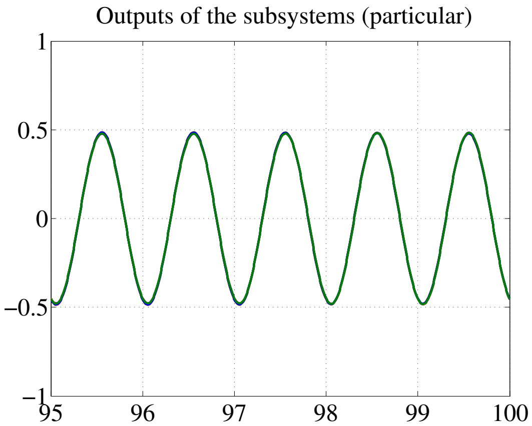


Fig. 8. Synchronization of oscillators: good case. Zoom showing the synchronization.

## 4. CONCLUSION AND FUTURE WORK

We derived a scalable and simple Nyquist criterion for checking the synchronization of SISO systems interconnected via a Laplacian matrix. We successfully applied this tool to the problem of clock synchronization. Future research on this topic can follow numerous directions. For example, $\Gamma$ can be generalized to be frequency dependent and the algorithm could be modified in order to let the outputs follow not only the natural modes of the subsystems, but also some controlled ones. On the application side, one could imagine to project distributed observers which can only sense some part of the whole state, allowing all the estimates to converge thanks to the proposed algorithm.

[2] Where $\mathcal{U}[l,\, u]$ means uniformly taken in $[l,\, u]$.


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