Robust synchronisation of unstable linear time-invariant systems

Sei Zhen Khong, Chung-Yao Kao, and Enrico Lovisari

***Abstract*— A framework based on the gap metric and integral quadratic constraints (IQCs) is developed for analysing robust synchronisation of heterogenous linear time-invariant networks. Both the agents and the communication channels are allowed to be dynamic and unstable. Structural properties of the uncertainty are described by IQCs and exploited in synchronisation analysis as a means to reduce conservatism. The homotopy employed in IQC analysis is defined with respect to the graph topology as induced by the gap metric, whereby open-loop unstable dynamics are accommodated. The results in this paper extend recent developments, which have been shown to unify several existing synchronisation analysis methods in the literature.**

***Index Terms*— Synchronisation, consensus, multi-agent networks, unstable dynamics, integral quadratic constraints**

I. INTRODUCTION

The problem of synchronisation of heterogeneous multi-agent systems is widely studied in various engineering applications ranging from power system networks to biological cells. In [1], [2], a unifying framework for the analysis of synchronisation of heterogenous open-loop stable linear time-invariant (LTI) agents is proposed. The agents communicate through an interconnection transfer matrix with stable dynamics to reach synchronisation. The developed framework incorporates the problem of consensus as a special case. The approach relies on the use of integral quadratic constraints (IQCs) [3] to encapsulate structural properties or uncertainty of the dynamics in order to reduce conservatism in synchronisation analysis.

In this paper, the framework of [1] is generalised to accommodate open-loop unstable dynamics in both the agents as well as the interconnection transfer matrix. Heterogenous distributed-parameter LTI systems are considered. These include, for instance, time-delay operations. The gap metric [4], [5] is employed to define a homotopy in the IQC analysis, reminiscing the main results in [6], [7]. Sufficient conditions for synchronisation based on the blended IQC/gap metric are provided. These find applications in situations where the agents exchange information through unstable dynamic communication channels, for example.

The paper evolves along the following lines. In the next section, the notation used in the paper is defined and some preliminaries on linear analysis and graph theoretic concepts stated. The problem of synchronisation is formulated in Section III. The blended IQC/gap metric based analysis framework for synchronisation involving unstable dynamics is delineated in Section IV. Illustrative examples are provided in Section V. Section VI contains some concluding remarks.

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II. NOTATION AND PRELIMINARIES

A. Matrices

Let $\mathbb{R}$ and $\mathbb{C}$ denote the real and complex numbers respectively. $j\mathbb{R}$ denotes the imaginary axis, $\mathbb{C}_+$ (resp. $\bar{\mathbb{C}}_+$) the open (resp. closed) right half complex plane, and $|\cdot|$ the Euclidean norm. Given an $A \in \mathbb{C}^{m\times n}$ (resp. $\mathbb{R}^{m\times n}$), $A^* \in \mathbb{C}^{n\times m}$ (resp. $A^T \in \mathbb{R}^{n\times m}$) denotes its complex conjugate transpose (resp. transpose). A_{ij} denotes the (i,j) entry of A. The i^{th} row and j^{th} column of A are denoted respectively by $A_{i\bullet}$ and $A_{\bullet j}$. Given a vector $v \in \mathbb{C}^n$, $\operatorname{diag}(v) \in \mathbb{C}^{n\times n}$ denotes the diagonal matrix whose diagonal entires are $v_1, \dots, v_n$. Let $\otimes$ denote the Kronecker product and $\oplus$ the direct sum of matrices. Define $\bigoplus_{i=1}^n A_i := A_1 \oplus A_2 \oplus \dots \oplus A_n$. I_n denotes the identity matrix of dimensions $n \times n$.

B. Function spaces

Define the Lebesgue space

$$\mathbf{L}_\infty := \{\phi : j\mathbb{R} \to \mathbb{C} \,|\, \|\phi\|_\infty := \sup_{\omega\in\mathbb{R}} |\phi(j\omega)| < \infty\}$$

and the Hardy space

$$\mathbf{H}_\infty := \left\{ \phi \in \mathbf{L}_\infty \;\middle|\; \begin{array}{l} \phi \text{ has analytic continuation into } \mathbb{C}_+ \\ \text{with } \sup_{s\in\mathbb{C}_+} |\phi(s)| = \|\phi\|_\infty < \infty \end{array} \right\}.$$

Let $\mathbf{C}$ be the class of functions continuous on $j\mathbb{R}\cup\{\infty\}$, and $\mathbf{S} := \mathbf{H}_\infty \cap \mathbf{C}$. Note that $\mathbf{C} \subset \mathbf{L}_\infty$. An $H \in \mathbf{C}^{n\times n}$ is said to be Hermitian if $H(j\omega) = H(j\omega)^*$ for all $\omega \in \mathbb{R}\cup\{\infty\}$ and positive semidefinite if in addition, $H(j\omega) \geq 0$ and positive definite if $H(j\omega) \geq \gamma$ for some $\gamma > 0$.

Given an $\epsilon > 0$ and a point $jq \in j\mathbb{R}$, define the semi-circle of radius ϵ in the right-half plane as

$$\mathcal{S}_\epsilon(jq) := \{s \in \mathbb{C} : |s - jq| = \epsilon, \Re(s) > 0\}$$

and $\mathcal{S}_0(jq) := \{\}$. Given a finite ordered set $j\mathcal{Q} = \{jq_1, jq_2, \dots, jq_K\} \subset j\mathbb{R}$ with $q_1 > q_2 > \dots > q_K$, define a contour parameterised by $\epsilon \geq 0$ as

$$\begin{aligned}\mathcal{C}_\epsilon(j\mathcal{Q}) := j[q_1+\epsilon, \infty) \cup \mathcal{S}_\epsilon(jq_1) \cup j[q_2+\epsilon, q_1-\epsilon] \\ \cup\, \mathcal{S}_\epsilon(jq_2) \cup j[q_3+\epsilon, q_2-\epsilon] \\ \vdots \\ \cup\, \mathcal{S}_\epsilon(jq_K) \cup j(-\infty, q_K-\epsilon].\end{aligned}$$

that is, a straight line on the imaginary axis indented to the right of every point in $j\mathcal{Q}$ by a semi-circle of radius ϵ. In particular, notice that $C_0(j\mathcal{Q}) = j\mathbb{R}$ for any $j\mathcal{Q} \subset j\mathbb{R}$. Denote by $\mathcal{C}_\epsilon^+(j\mathcal{Q})$ the open half plane that lies to the right of $\mathcal{C}_\epsilon(j\mathcal{Q})$, i.e.

$$\mathcal{C}_\epsilon^+(j\mathcal{Q}) := \{s = \sigma + j\omega \in \mathbb{C} \mid \bar{\sigma} + j\omega \in \mathcal{C}_\epsilon(j\mathcal{Q}) \implies \sigma > \bar{\sigma}\},$$

and $\bar{\mathcal{C}}_\epsilon^+(j\mathcal{Q})$ its closure. Let $\mathbf{C}_\epsilon(j\mathcal{Q})$ be the class of functions continuous on $\mathcal{C}_\epsilon(j\mathcal{Q}) \cup \{\infty\}$. Given $X \in \mathbf{C}_\epsilon(j\mathcal{Q})^{n\times m}$, define $\|X\|_{\mathbf{C}_\epsilon(j\mathcal{Q})} := \sup_{s\in\mathcal{C}_\epsilon(j\mathcal{Q})} \bar{\sigma}(X(s))$, where $\bar{\sigma}(\cdot)$ denotes the maximum singular value. An $H \in \mathbf{C}_\epsilon(j\mathcal{Q})^{n\times n}$ is said to be Hermitian if $H(s) = H(s)^*$ for all $s \in \mathcal{C}_\epsilon(j\mathcal{Q}) \cup \{\infty\}$. Let $\mathbf{S}_\epsilon(j\mathcal{Q})$ be the subclass of $\mathbf{C}_\epsilon(j\mathcal{Q})$ containing functions that have analytic continuation into $\mathcal{C}_\epsilon^+(j\mathcal{Q})$. Note that $\mathbf{S} \subset \mathbf{S}_\epsilon(j\mathcal{Q})$ for all $\epsilon \geq 0$.

Let the Lebesgue space $\mathbf{L}_2^n$ denote the class of functions $f : [0, \infty) \to \mathbb{R}^n$ with finite energy, i.e. square-integrable, satisfying $\|f\|_2^2 := \int_0^\infty |f(t)|^2 \, dt < \infty$. The Fourier transform of $f \in \mathbf{L}_2^n$ is denoted $\hat{f}(j\omega) := \int_0^\infty e^{-j\omega t} f(t) \, dt$. Note that $\|\hat{f}\|_2 = \|f\|_2$ and $\hat{f}$ has analytic continuation into $\mathbb{C}_+$ and $\sup_{\sigma>0} \|\hat{f}(\sigma + \cdot)\|_2 = \|\hat{f}\|_2 < \infty$. The set of Fourier transforms of functions in $\mathbf{L}_2^n$ is denoted $\mathbf{H}_2^n$. A linear operator mapping between Banach spaces $X : \mathcal{X} \to \mathcal{Y}$ is said to be bounded if the induced norm

$$\|X\|_{\mathcal{X}\to\mathcal{Y}} := \sup_{f\in\mathcal{X}:\|f\|_\mathcal{X}=1} \|Xf\|_\mathcal{Y} < \infty.$$

Note that multiplication by a transfer function in $\mathbf{S}$ as an operator on $\mathbf{H}_2$ defines a corresponding causal and bounded LTI operator on $\mathbf{L}_2$ in the time domain via the Laplace transform isomorphism [8].

For $\epsilon \geq 0$ and finite subset $j\mathcal{Q} \subset j\mathbb{R}$, define $\mathbf{H}_{2\epsilon}^n(j\mathcal{Q})$ to be the set of functions $\hat{f} : \bar{\mathcal{C}}_\epsilon(j\mathcal{Q}) \to \mathbb{C}^n$ that are analytic on $\mathcal{C}_\epsilon^+(j\mathcal{Q})$ and square-integrable on $\mathcal{C}_\epsilon(j\mathcal{Q})$, i.e. $\|\hat{f}\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}^2 := \int_{\mathcal{C}_\epsilon(j\mathcal{Q})} |\hat{f}(s)|^2 \, ds < \infty$. For notational simplicity, the spatial dimension n and the set of imaginary-axis poles $(j\mathcal{Q})$ are often dropped from $\mathbf{H}_{2\epsilon}^n(j\mathcal{Q})$. Note that $\mathbf{H}_2^n = \mathbf{H}_{2\epsilon}^n(j\mathcal{Q})$ when $\epsilon = 0$. Moreover, for all $\epsilon \geq 0$, multiplication by a transfer function $X \in \mathbf{S}_\epsilon(j\mathcal{Q})$ defines a bounded operator on $\mathbf{H}_{2\epsilon}$ with its induced norm equals to $\|X\|_{\mathbf{C}_\epsilon(j\mathcal{Q})}$. $\mathbf{H}_{2\epsilon}$ is a Hilbert space with inner product $\langle u, v\rangle_{\mathcal{C}_\epsilon(j\mathcal{Q})} := \int_{\mathcal{C}_\epsilon(j\mathcal{Q})} u(s)^* v(s) \, ds$. It can be seen that multiplication by an $X \in \mathbf{S}$ is bounded on $\mathbf{H}_{2\epsilon}$ for all $\epsilon \geq 0$. One the other hand, given a $q \in \mathbb{R}$, multiplication by $\frac{1}{s-jq}$ is bounded on $\mathbf{H}_{2\epsilon}(\{jq\})$ for all $\epsilon > 0$ but not on $\mathbf{H}_2$. In the following, we will not notationally distinguish between a transfer function and its associated multiplication operator. For instance, an $X \in \mathbf{S}$ defines a bounded operator $X : \mathbf{H}_{2\epsilon} \to \mathbf{H}_{2\epsilon}$ for all $\epsilon \geq 0$.

Given an $\epsilon \geq 0$, define the graph of the linear operator $X : \mathrm{dom}\,(X) \subset \mathbf{H}_{2\epsilon}^m(j\mathcal{Q}) \to \mathbf{H}_{2\epsilon}^n(j\mathcal{Q})$ to be

$$\mathscr{G}_\epsilon\,(X) := \left\{ \begin{bmatrix} u \\ y \end{bmatrix} \in \mathbf{H}_{2\epsilon}^{n+m}(j\mathcal{Q}) : y = Xu \right\}.$$

Similarly, define the (inverse) graph

$$\mathscr{G}_\epsilon'(X) := \left\{ \begin{bmatrix} u \\ y \end{bmatrix} \in \mathbf{H}_{2\epsilon}^{n+m}(j\mathcal{Q}) : u = Xy \right\}.$$

Denote by $\mathbf{F}^{m\times n}$ the class of linear operators which admit strong right graph representations in $\mathbf{S}$, i.e. for every $X \in \mathbf{F}^{m\times n}$, there exists $Y \in \mathbf{S}^{(m+n)\times m}$ such that Y has a left inverse in $\mathbf{S}^{m\times(m+n)}$ and $\mathscr{G}_\epsilon\,(X) = Y\mathbf{H}_{2\epsilon}^m$ for all $\epsilon \geq 0$. In particular, $\mathbf{F}^{m\times n}$ includes the class of proper rational transfer matrices and the Callier-Desoer algebra [9], where strong right graph representations can be constructed from right coprime factorisations.

C. Graph theory

A graph is denoted by $\mathcal{G} = (V, E)$, where $V = \{v_1, \ldots, v_n\}$ is the set of nodes and $E \subset V \times V$, $E = \{e_1, \ldots, e_m\}$ is the set of edges such that $e_k = \{v_i, v_j\} \in E$ if node i is connected to node j. A graph is undirected if $\{v_i, v_j\} \in E$ then $\{v_j, v_i\} \in E$. A path on $\mathcal{G}$ of length N is an ordered set of distinct vertices $\{v_0, v_1, \ldots, v_N\}$ such that $\{v_i, v_{i+1}\} \in E$ for all $i \in \{0, 1, \ldots, N-1\}$. An undirected graph is said to be *connected* if any two nodes in V is connected by a path. The adjacency matrix $A = [A_{ij}] \in \mathbb{R}^{n\times n}$ is defined by $A_{ij} = 1$ if $\{v_i, v_j\} \in E$ and $A_{ij} = 0$ otherwise. Note that A is symmetric for an undirected graph. In an undirected graph, let the neighbours of node $v_i \in V$ be defined as $N_i := \{v_j \in V : \{v_i, v_j\} \in E\}$ and denote its degree by $|N_i|$. The graph Laplacian is defined as $L := \mathrm{diag}(|N_i|) - A$. L has a zero eigenvalue corresponding to the vector of ones $1_n \in \mathbb{R}^n$. The multiplicity of the zero eigenvalue is one if the graph is connected [10]. The Laplacian matrix can be factorised as $L = DD^T$, where $D = [D_{ik}] \in \mathbb{R}^{n\times m}$ is the oriented incidence matrix. It is defined by associating an orientation to every edge of the graph: for each $e_k = \{v_i, v_j\} = \{v_j, v_i\}$, one of v_i, v_j is defined to be the head and the other tail of the edge:

$$D_{ik} := \begin{cases} +1 & \text{if } v_i \text{ is the head of } e_k \\ -1 & \text{if } v_i \text{ is the tail of } e_k \\ 0 & \text{otherwise.} \end{cases}$$

Note that the Laplacian matrix is invariant to the choice of orientation. Define also the unoriented incidence matrix $\bar{D} \in \mathbb{R}^{n\times m}$ whose entries are the absolute value of those of D.

III. Synchronisation problem formulation

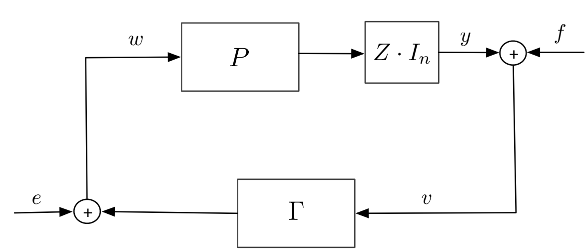

Fig. 1. Feedback setup for synchronisation.

Consider the feedback interconnection in Figure 1. There, $P := \bigoplus_{i=1}^n P_i = \mathrm{diag}(P_i)$ with the SISO dynamical agents $P_i \in \mathbf{F}$ and $\Gamma \in \mathbf{F}^{n\times n}$ denotes the interconnection matrix. Z is a SISO proper rational transfer function that has a finite number of poles on $j\mathbb{R}$. Throughout the paper, $j\mathcal{Q} = \{jq_1, jq_2, \ldots, jq_K\}$ is used to denote the set of poles of Z on the imaginary axis. These poles/modes describe the trajectory of the output signal y under synchronisation. The interactions between the agents is determined by an underlying undirected and connected graph $\mathcal{G} = (V, E)$, where each node $v_i \in V$ is associated with a corresponding P_i and the edges describe the communication/connections between the agents. Figure 1 models the problem of synchronisation of a network of heterogeneous agents interconnected through a dynamic matrix.

The following standing assumption is made throughout the paper.

Assumption 3.1: For every $jq \in j\mathcal{Q}$,

$$\lim_{s\to jq} \frac{1}{s^{m_q-1}}\Gamma(s)1_n = 0,$$

where m_q denotes the multiplicity of the pole jq of Z. Furthermore, there exists no $x \in \mathbb{C}^n$, $x \neq 1_n$ such that $\Gamma(jq)x = 0$. In other words, $\det(\Gamma(s))$ has a zero at every $s = jq \in j\mathcal{Q}$ of multiplicity m_q corresponding to the null space $\operatorname{span}\{1_n\}$.

In the case where Z has non-repeated poles on $j\mathbb{R}$, Γ can be set to L, the graph Laplacian matrix for the graph $\mathcal{G}$. Dynamics can be included via the expression $\Gamma = D\,\operatorname{diag}(\Gamma_i)\,D^T$, where D denotes the incidence matrix and $\Gamma_i \in \mathbf{F}$ for $i = 1, \ldots, m$; see Figure 2. This models a heterogeneous network configuration of agents interconnected via dynamically weighted matrices. Note that for both cases Γ satisfies Assumption 3.1 by the connectedness of the graph $\mathcal{G}$.

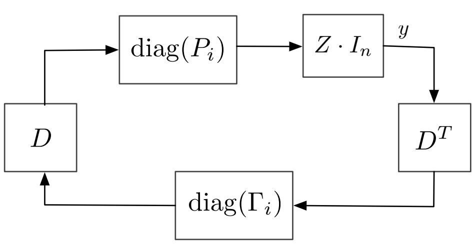

Fig. 2. A synchronisation setup with dynamical interconnection matrix.

Definition 3.2: The interconnection in Figure 1 is said to reach synchronisation if

$$|y_i(t) - y_j(t)| \to 0 \quad \text{as} \quad t \to \infty$$

for all $i, j \in \{1, 2, \ldots, n\}$ and $e, f \in \mathbf{L}_2$.

In other words, $\lim_{t\to\infty} y(t)$ lies in the subspace spanned by 1_n, i.e. $\operatorname{span}\{1_n\}$. This means the output y_i of each of the agent P_i synchronises to the same trajectory defined by the imaginary-axis poles of Z.

Remark 3.3: If $Z(s) = 1$, one recovers the standard setup of feedback interconnection, whereby synchronisation in the definition above corresponds to feedback stability. By defining $Z(s) := \frac{1}{s}$, one recovers the standard consensus problem where all y_i's are to asymptotically converge to the same constant value. By contrast, if $Z(s) := \frac{\omega_0}{s^2+\omega_0^2}$ and synchronisation takes place, then each y_i will converge to a sinusoid of frequency ω_0 and the same phase/magnitude. Another example is $Z(s) := \frac{1}{s^2}$, where the system outputs synchronise to a ramp function.

IV. Robust synchronisation analysis

This section introduces a unified framework within which to analyse the problem of synchronisation using integral quadratic constraints (IQCs) [3]. To this end, some results from robustness of closed-loop interconnections are needed and provided next.

A. Feedback robustness

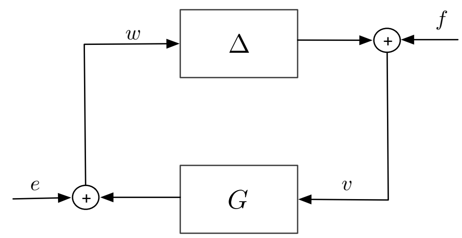

Fig. 3. Standard feedback configuration.

Definition 4.1: Given $\epsilon \geq 0$, $\Delta : \operatorname{dom}(\Delta) \subset \mathbf{H}^n_{2\epsilon}(j\mathcal{Q}) \to \mathbf{H}^m_{2\epsilon}(j\mathcal{Q})$ and $G : \operatorname{dom}(G) \subset \mathbf{H}^m_{2\epsilon}(j\mathcal{Q}) \to \mathbf{H}^n_{2\epsilon}(j\mathcal{Q})$, the feedback interconnection of Δ and G in Figure 3, denoted $[\Delta, G]$:

$$\begin{cases} v = \Delta w + f \\ w = Gv + e \end{cases} \tag{1}$$

is said to be $\mathbf{H}_{2\epsilon}$-*stable* if the map $(v, w) \mapsto (f, e)$ has a bounded inverse on $\mathbf{H}^{2n}_{2\epsilon}$.

Given an $\mathbf{H}_{2\epsilon}$-stable $[\Delta, G]$, define the generalised robustness margin with the ambient space taken to be $\mathbf{H}_{2\epsilon}(j\mathcal{Q})$ by

$$b^\epsilon_{\Delta,G} := \inf_{v\in\mathscr{G}_\epsilon(\Delta), w\in\mathscr{G}'_\epsilon(G)} \frac{\|v+w\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}}{\|v\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}}. \tag{2}$$

Furthermore, given two systems $\Delta_1 : \operatorname{dom}(\Delta_1) \subset \mathbf{H}^n_{2\epsilon}(j\mathcal{Q}) \to \mathbf{H}^m_{2\epsilon}(j\mathcal{Q})$ and $\Delta_2 : \operatorname{dom}(\Delta_2) \subset \mathbf{H}^n_{2\epsilon}(j\mathcal{Q}) \to \mathbf{H}^m_{2\epsilon}(j\mathcal{Q})$, define the generalised gap metric as follows:

$$\begin{aligned} \delta^\epsilon(\Delta_1, \Delta_2) &:= \|\Pi_{\mathscr{G}_\epsilon(\Delta_1)} - \Pi_{\mathscr{G}_\epsilon(\Delta_2)}\|_{\mathcal{C}_\epsilon(j\mathcal{Q})} \\ &= \max\left\{\vec{\delta}(\Delta_1, \Delta_2), \vec{\delta}(\Delta_2, \Delta_1)\right\}, \end{aligned} \tag{3}$$

where the directed gap

$$\begin{aligned} \vec{\delta}^\epsilon(\Delta_k, \Delta_l) &:= \gamma\left(\mathbf{\Pi}_{(\mathscr{G}_\epsilon(\Delta_l))^\perp}\mathbf{\Pi}_{\mathscr{G}_\epsilon(\Delta_k)}\right) \\ &= \sup_{x_k\in\mathscr{G}_\epsilon(\Delta_k)} \inf_{x_l\in\mathscr{G}_\epsilon(\Delta_l)} \frac{\|x_k - x_l\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}}{\|x_k\|_{\mathcal{C}_\epsilon(j\mathcal{Q})}}. \end{aligned} \tag{4}$$

See [5] for the original definitions of the robustness margin and gap metric with respect to the ambient space $\mathbf{H}_2$.

Proposition 4.2: Suppose $[\Delta_1, G]$ is $\mathbf{H}_{2\epsilon}$-stable with $b^\epsilon_{\Delta_1,G} > \delta^\epsilon(\Delta_1, \Delta_2)$, then $[\Delta_2, G]$ is $\mathbf{H}_{2\epsilon}$-stable.

Proof: The claim can be established following the arguments in [5, Thm. 3] or [11, Prop. III.1], where the result is proven with respect to the ambient space $\mathbf{H}_2$. ■

B. IQC conditions for synchronisation

Throughout, $j\mathcal{Q} = \{jq_1, jq_2, \ldots, jq_K\}$ is taken to be the finite set of poles on $j\mathbb{R}$ of Z and given a linear operator X, the shorthand notation ZX is used to denote $(Z \cdot I_n)X$. First, a blended IQC/gap metric result on the generalised $\mathbf{H}_{2\epsilon}$ feedback stability is given below.

Theorem 4.3: Given $\epsilon \geq 0$, the feedback interconnection of $\Delta : \mathrm{dom}\,(\Delta) \subset \mathbf{H}^n_{2\epsilon}(j\mathcal{Q}) \to \mathbf{H}^m_{2\epsilon}(j\mathcal{Q})$ and $G : \mathrm{dom}\,(G) \subset \mathbf{H}^m_{2\epsilon}(j\mathcal{Q}) \to \mathbf{H}^n_{2\epsilon}(j\mathcal{Q})$ in Figure 3 is $\mathbf{H}_{2\epsilon}$-stable if there exist a homotopy $\tau \in [0,1] \mapsto G_\tau$ with $G_1 = G$ that is continuous in the gap metric $\delta^\epsilon(\cdot,\cdot)$ and a Hermitian $\Pi \in \mathbf{C}_\epsilon(j\mathcal{Q})^{(n+m)\times(n+m)}$ such that:

(i) $[\Delta, G_0]$ is $\mathbf{H}_{2\epsilon}$-stable;
(ii) $\langle v, \Pi v\rangle_{\mathcal{C}_\epsilon(j\mathcal{Q})} \geq 0$ for all $v \in \mathscr{G}_\epsilon(\Delta)$;
(iii) there exists a $\gamma > 0$ for which $\langle w, \Pi w\rangle_{\mathcal{C}_\epsilon(j\mathcal{Q})} \leq -\gamma\|w\|^2_{\mathcal{C}_\epsilon(j\mathcal{Q})}$ for all $w \in \mathscr{G}'_\epsilon(G_\tau)$ and $\tau \in [0,1]$.

Proof: Following the first part of the proof for [1, Thm. 4.4], it can be shown that there exists an $\eta > 0$ for which

$$\|v+w\|^2_{\mathcal{C}_\epsilon(j\mathcal{Q})} \geq \eta^2\|w\|^2_{\mathcal{C}_\epsilon(j\mathcal{Q})} \quad \forall v \in \mathscr{G}_\epsilon(\Delta), w \in \mathscr{G}'_\epsilon(G_\tau), \tau \in [0,1]. \tag{5}$$

Since the feedback interconnection $[\Delta, G_\tau]$ is $\mathbf{H}_{2\epsilon}$-stable for $\tau = 0$ by hypothesis, inequality (5) implies that the corresponding robust stability margin $b^\epsilon_{\Delta,G_0} \geq \eta > 0$; see (2). By continuity in the gap metric, there exists an $\zeta > 0$ such that $\delta^\epsilon(G_h, G_{h+\tau}) < \eta$ for all $\tau \in [0,\zeta]$ and $h \in [0, 1-\zeta]$. Application of Proposition 4.2 then leads to the feedback interconnection of Δ and G_τ being $\mathbf{H}_{2\epsilon}$-stable for $\tau \in [0,\zeta]$. By (5), it follows again that $b^\epsilon_{\Delta,G_\zeta} \geq \eta > 0$. Repetitively applying the aforementioned arguments yields $\mathbf{H}_{2\epsilon}$-stability of the feedback interconnection $[\Delta, G_\tau]$ for $\tau \in [\zeta, 2\zeta], [2\zeta, 3\zeta], \ldots$ in succession, and eventually for $\tau = 1$, as required. ■

The unified IQC/gap metric based result on synchronisation is in order.

Theorem 4.4: Consider the feedback configuration in Figure 1, where Z is a proper rational scalar transfer function with poles in $j\mathcal{Q}$, $P := \bigoplus_{i=1}^n P_i : P_i \in \mathbf{F}; P_i(jq) \neq 0\, \forall jq \in j\mathcal{Q}$, and $\Gamma \in \mathbf{F}^{n\times n}$ satisfies Assumption 3.1. Suppose P and Γ have no poles in $j\mathcal{Q}$. Let $\tau \in [0,1] \mapsto \mathcal{P}_\tau \in \mathbf{F}^{n\times n}$ be a homotopy that is continuous in the gap metric $\delta^0(\cdot,\cdot)$ such that $[Z\mathcal{P}_0, \Gamma]$ reaches synchronisation and $\mathcal{P}_1 = P$. Then the feedback $[ZP, \Gamma]$ reaches synchronisation if there exists a Hermitian $\Pi \in \mathbf{C}^{2n\times 2n}$ such that for all $\omega \in \mathbb{R} \setminus \mathcal{Q} = (q_1,\infty) \cup (q_2, q_1) \cup \ldots \cup (q_K, q_{K-1}) \cup (-\infty, q_K)$,

(i) $\begin{bmatrix} V(j\omega) \\ U(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} V(j\omega) \\ U(j\omega) \end{bmatrix} \geq 0;$

(ii) $\begin{bmatrix} Z(j\omega)Y_\tau(j\omega) \\ X_\tau(j\omega) \end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} Z(j\omega)Y_\tau(j\omega) \\ X_\tau(j\omega) \end{bmatrix} \leq -\gamma I;$
$\forall \tau \in [0,1]$ and some $\gamma > 0$,

where $\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} Y_\tau \\ X_\tau \end{smallmatrix}\right]$ are, respectively, strong right graph representations for Γ and $\mathcal{P}_\tau$, which satisfy $\mathscr{G}_\epsilon(\Gamma) = \left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right] \mathbf{H}_{2\epsilon}$ and $\mathscr{G}'_\epsilon(\mathcal{P}_\tau) = \left[\begin{smallmatrix} Y_\tau \\ X_\tau \end{smallmatrix}\right] \mathbf{H}_{2\epsilon}$ for all $\epsilon \geq 0$.

Proof: First note that by hypothesis, $[Z\mathcal{P}_0, \Gamma]$ is $\mathbf{H}_{2\epsilon}$-stable for all $\epsilon > 0$. Let $\Psi := 2\Pi + \gamma I$. The quadratic inequalities above can thus be restated as

$$\begin{bmatrix} Z(j\omega)Y_\tau(j\omega) \\ X_\tau(j\omega) \end{bmatrix}^* \Psi(j\omega) \begin{bmatrix} Z(j\omega)Y_\tau(j\omega) \\ X_\tau(j\omega) \end{bmatrix} \leq -\gamma I;$$
$$\begin{bmatrix} V(j\omega) \\ U(j\omega) \end{bmatrix}^* \Psi(j\omega) \begin{bmatrix} V(j\omega) \\ U(j\omega) \end{bmatrix} \geq \gamma I,$$

for all $\omega \in \mathbb{R} \setminus \mathcal{Q}$ and $\tau \in [0,1]$. Given $\epsilon > 0$, define $\bar{\Psi}_\epsilon \in \mathbf{C}_\epsilon(j\mathcal{Q})$ by $\bar{\Psi}_\epsilon(s) := \Psi(j\omega)$ for $s = \sigma + j\omega \in \mathcal{C}_\epsilon(j\mathcal{Q})$. Since Z, X_τ, Y_τ, U, and V are analytic on $\mathbb{C}_+$, it follows that there exists a sufficiently small $\epsilon^* > 0$ such that

$$\begin{bmatrix} Z(s)Y_\tau(s) \\ X_\tau(s) \end{bmatrix}^* \bar{\Psi}_\epsilon(s) \begin{bmatrix} Z(s)Y_\tau(s) \\ X_\tau(s) \end{bmatrix} \leq -\frac{\gamma}{2} I;$$
$$\begin{bmatrix} V(s) \\ U(s) \end{bmatrix}^* \bar{\Psi}_\epsilon(s) \begin{bmatrix} V(s) \\ U(s) \end{bmatrix} \geq \frac{\gamma}{2} I$$

for all $s \in \mathcal{C}_\epsilon(j\mathcal{Q})$, $\tau \in [0,1]$, and $0 < \epsilon \leq \epsilon^*$. These imply that $\langle v, \bar{\Psi}_\epsilon v\rangle_{\mathcal{C}_\epsilon(j\mathcal{Q})} \geq 0$ for all $v \in \mathscr{G}_\epsilon(\Gamma)$ and $\langle w, \bar{\Psi}_\epsilon w\rangle_{\mathcal{C}_\epsilon(j\mathcal{Q})} \leq -\frac{\gamma}{2}\|w\|^2_{\mathcal{C}_\epsilon(j\mathcal{Q})}$ for all $w \in \mathscr{G}'_\epsilon(Z\mathcal{P}_\tau)$ and $\tau \in [0,1]$. Also from analyticity of X_τ and Y_τ on $\mathbb{C}_+$, continuity of $\tau \mapsto \mathcal{P}_\tau$ in $\delta^\epsilon(\cdot,\cdot)$ follows from that in $\delta^0(\cdot,\cdot)$, since the integral contour $\mathcal{C}_0(j\mathcal{Q})$ can be continuously deformed into $\mathcal{C}_\epsilon(j\mathcal{Q})$. By Theorem 4.3, it thus follows that the feedback configuration $[ZP, \Gamma] := [Z\mathcal{P}_1, \Gamma]$ is $\mathbf{H}_{2\epsilon}$-stable for all $0 < \epsilon \leq \epsilon^*$. In turn, this implies that

$$Z(s)P(s)\left(I - \Gamma(s)Z(s)P(s)\right)^{-1} = P(s)\left(\frac{1}{Z(s)}I - \Gamma(s)P(s)\right)^{-1} \tag{6}$$

has no poles on $\bar{\mathbb{C}}_+ \setminus j\mathcal{Q}$, i.e. $\det(\frac{1}{Z(s)}I - \Gamma(s)P(s))$ has no zeros on $\bar{\mathbb{C}}_+ \setminus j\mathcal{Q}$. Moreover, by Assumption 3.1, $\det(\frac{1}{Z(s)}I - \Gamma(s)P(s))$ has a zero at every $s \in j\mathcal{Q}$ corresponding to the null space $\mathcal{N}$ satisfying $P(s)\mathcal{N} \subset \mathrm{span}\{1_n\}$, and the multiplicity of the zero is the same as that of the pole s of Z. As such, (6) has a pole at every $s \in j\mathcal{Q}$ of the same multiplicity as Z. This implies that the synchronisation subspace defined by the imaginary-axis poles of Z is asymptotically stable, as required. ■

V. Numerical Simulations

In our numerical simulations we consider a scenario of perturbed consensus with uncertain but bounded communication delays. Consider $n = 10$ agents deployed on a circle, each connected with the leftmost and rightmost neighbors, whose dynamics are described by $h_i(s) = Z(s)P_i(s)$, $i = 1, \ldots, 10$, where $Z(s) = \frac{1}{s}$ is the nominal model and $P_i(s)$ is an unstable multiplicative perturbation. In particular, $P_i(s) = \frac{1}{s-\lambda_i}$, $i = 1, \ldots, 10$, where the poles the perturbations are chosen randomly according to $\lambda_i \in \mathcal{U}[\lambda_{\mathrm{nom}} - \delta_\lambda, \lambda_{\mathrm{nom}} + \delta_\lambda]$. For the simulation, we set $\lambda_{\mathrm{nom}} = 1$ and $\delta_\lambda = 0.2$.

The interconnection operator takes the form

$$\Gamma(s) = \Gamma_s(s)I + DE(s)D^T$$

where $\Gamma_s(s) = \frac{25s}{s+5}$, so that, as required, $\Gamma(0)\mathbf{1} = 0$, I is an

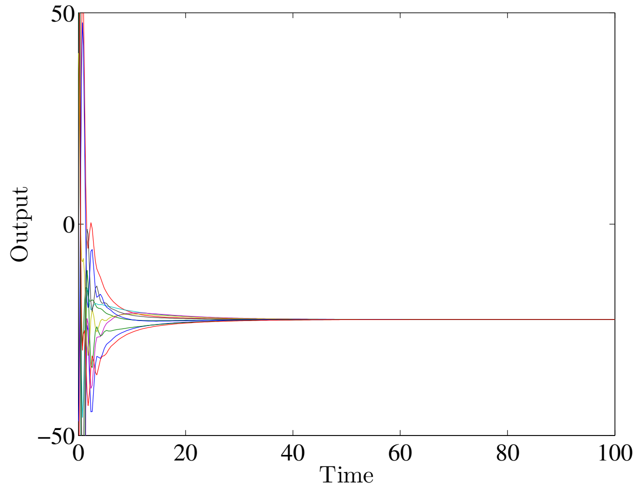

Fig. 4. A typical trajectory of the system described in Section V. The perturbed agents reach consensus despite delayed communications.

identity matrix, D is the incidence matrix of the graph, and $E(s)$ is a diagonal transfer function whose (e,e)-th element, corresponding to link $e=(i,j)$, for some (i,j), is $E_{ee}(s) = e^{-sd_e}$, d_e being the delay on channel e. We consider fixed communication delays, and we set $d_e = 0.25$.

Since the graph is a circle, let the nodes be numbered in such a way that the i-th node communicates with $i-1$ and $i+1$ (modulo n). Also, number the edges as follows: $e=1=(1,2)$, $e=2=(2,3)$, $\ldots$, $e=n=(n,1)$. Then the incidence matrix of the graph is

$$D = \begin{bmatrix} 1 & 0 & & & 0 & -1 \\ -1 & 1 & \ddots & & & 0 \\ 0 & -1 & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & \ddots & \ddots & \ddots & \\ \vdots & \vdots & & \ddots & \ddots & 0 & 0 \\ 0 & 0 & & & \ddots & 1 & 0 \\ 0 & 0 & & & & -1 & 1 \end{bmatrix}$$

A typical trajectory of the system is depicted in Fig. 4, in which it is shown that the system reaches consensus on a non-zero value.

In order to establish synchronization of the proposed systems, we employ Theorem 4.4 by looking for $\Pi \in \mathbf{C}^{20\times 20}$ such that

- it holds true
$$\begin{bmatrix} Z(j\omega)\otimes_{i=1}^{10} Y_\tau^i(j\omega) \\ \otimes_{i=1}^{10} X_\tau^i(j\omega)\end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} Z(j\omega)\otimes_{i=1}^{10} Y_\tau^i(j\omega) \\ \otimes_{i=1}^{10} X_\tau^i(j\omega)\end{bmatrix} \le -\gamma, \qquad \forall \tau\in[0,1], \quad \forall \omega > 0$$
where $\begin{bmatrix} Y_\tau^i \\ X_\tau^i \end{bmatrix}$ is a strong right representation for $\mathcal{P}_\tau^i(s) = \frac{1}{s+(1-\tau)\lambda^0-\tau\lambda_i}$, for $i=1,\ldots,10$, and where the network in which to each P_i we substitute $\mathcal{P}_0^i = \frac{1}{s+\lambda_0}$, $\lambda_0 > 0$, reaches synchronization. For this example, it suffices to consider
$$\begin{cases} Y_\tau^i(s) = \frac{1}{s+1} \\ X_\tau^i(s) = \frac{s-(1-\tau)\lambda^0-\tau\lambda_i}{s+1} \end{cases} \qquad i=1,\ldots,10$$
- it holds true,
$$\begin{bmatrix} V(j\omega) \\ U(j\omega)\end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} V(j\omega) \\ U(j\omega)\end{bmatrix} \ge 0, \qquad \forall \omega > 0$$

where $\left[\begin{smallmatrix} V \\ U \end{smallmatrix}\right]$ is a strong right representation for Γ. Notice that, for $\lambda^0 = 1$ the network achieves synchronization, as it can be checked using the techniques proposed in [2].

Let $\Pi(j\omega)$ be

$$\begin{bmatrix} -\Gamma(j\omega)^* \\ I \end{bmatrix} (-\kappa(j\omega)) \begin{bmatrix} -\Gamma(j\omega) & I \end{bmatrix}$$

where $\kappa(j\omega)$ is a non-negative integral function on the imaginary axis. Since $\Gamma = UV^{-1}$, clearly we have $-\Gamma V + U = 0$ and therefore

$$\begin{bmatrix} V(j\omega) \\ U(j\omega)\end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} V(j\omega) \\ U(j\omega)\end{bmatrix} \ge 0, \qquad \forall \omega > 0.$$

Using this Π, the condition

$$\begin{bmatrix} Z(j\omega)\otimes_{i=1}^{10} Y_\tau^i(j\omega) \\ \otimes_{i=1}^{10} X_\tau^i(j\omega)\end{bmatrix}^* \Pi(j\omega) \begin{bmatrix} Z(j\omega)\otimes_{i=1}^{10} Y_\tau^i(j\omega) \\ \otimes_{i=1}^{10} X_\tau^i(j\omega)\end{bmatrix} \le -\gamma I, \qquad \forall \tau\in[0,1], \quad \forall \omega > 0$$

boils down to

$$-\kappa(j\omega) M(j\omega)^* M(j\omega) \le -\gamma I, \qquad \forall \tau\in[0,1], \quad \forall \omega > 0$$

for $M(j\omega) = \otimes_{i=1}^{10} X_\tau^i(j\omega) - \Gamma(j\omega) Z(j\omega) \otimes_{i=1}^{10} Y_\tau^i(j\omega)$. Since $\kappa(j\omega)$ is an arbitrary non-negative integral function on the imaginary axis, the above-mentioned condition holds as long as the maximum eigenvalue of $-M(j\omega)^* M(j\omega)$ is bounded away from zero all $\tau\in[0,1]$ and all $\omega > 0$.

Figure 5 shows the result of the numerical study. In particular, in the figure we plot, for $\omega \in \mathbb{G}_\omega$,

$$r(\omega) = \max_{\tau\in\mathbb{G}_\tau} \lambda_{\max}(-M(j\omega)^* M(j\omega))$$

$\lambda_{\max}(A)$ is the maximum eigenvalue of the square matrix A, and $\mathbb{G}_\omega$ and $\mathbb{G}_\tau$ are grids at which the frequency responses are evaluated. In particular, $\mathbb{G}_\tau = [0, 0.01, \ldots, 1]$ and $\mathbb{G}_\omega$ contains $[0.01, \ldots, 0.09, 1, \ldots, 999, 1000]$ and the opposites with respect to zero frequency (the latter being included for sake of completeness). Since clearly the maximum eigenvalue of $-M(j\omega)^* M(j\omega)$ is bounded away from zero for all τ (in the considered grid) and all $\omega > 0$ (in the considered grid), we can conclude that the network reaches consensus by employing Theorem 4.4.

VI. Conclusions

The paper presents a unified framework for analysing synchronisation of multi-agent networks in which the agents and the dynamical interconnection operator are allowed to be open-loop unstable. It encompasses numerous results, such as those based on Nyquist criterion, as shown along the lines in [1]. An interesting future research direction involves the

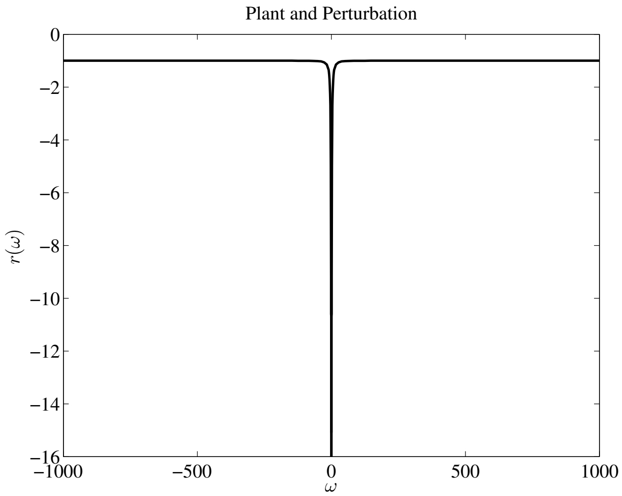

Fig. 5. Numerical check of inequality ii), Theorem 4.4, for the numerical example.

study of cooperative formation control within the developed framework.

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