On Distributed Synthesis of Dynamical Transportation Networks

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I. Introduction

Traffic congestion in large urban areas is a growing societal issue. Technological advancements in the fields of traffic sensing and network control and communication offer promising possibilities to reduce both congestion and pollution. Classical strategies [1], [2] for traffic flow control are typically based on extensive surveys aimed at identifying a model for the network and designing traffic plans that are either fixed [3] or constantly re-tuned as in SCOOT [4]. Other solutions are based on dynamic programming or model-predictive control [5], [6], and more recently on back-pressure algorithms [7], [8]. These strategies aim at stabilizing the network, but do not provide sound performance guarantees on the traffic network behavior.

Recently, positive and monotone systems are attracting increasing attention in the control community, especially in the context of distributed control of large-scale network systems. In fact, their structural properties have been found amenable for both efficient analysis and synthesis with scalable complexity [9]. Inspired by the distributed synthesis insights of [9], this paper is concerned with the problem of distributed synthesis of monotone dynamical flow networks.

Our contribution builds upon the dynamical network flow framework recently proposed in [10], [11]. We consider a continuous-time model in which links are buffers whose occupancy level dynamically changes according to mass conservation laws. Flow reaching a junction is redistributed in the outgoing links according to deterministic rules aimed at modeling both drivers' behavior and structural characteristic of the network, as well as the control exerted by the traffic manager. In particular, we explore the possibility of charging tolls and varying speed limits on the links of the network. The goal of the traffic manager is to optimize such control variables in order to improve the performance of the network, measured by a function of the state of the network at steady-state, for example, the total traffic density in the network.

The contributions of this paper are twofold. First, we formalize the problem of steady-state performance optimization in dynamic flow transportation networks. Second, we provide a scalable distributed synthesis solution in two different scenarios: if both tolls and speed limits can be decided, then the solution of a convex optimization problem yields tolls and speed limits which steer the network to the optimal steady-state. When instead speed limits are the only control variables, then a suboptimal solution for the non-convex steady-state performance optimization problem is obtained by a suitable relaxation.

Notation: A weighted directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, C)$ is a triple in which $\mathcal{V}$ is the set of nodes and $\mathcal{E}$ is the set of links, endowed with three vectors $\sigma, \tau :\in \mathcal{V}^{\mathcal{E}}$, and $C \in \mathbb{R}_+^{\mathcal{E}}$ whose entries σ_e, τ_e, and C_e denote the tail, head, and capacity of link e, respectively. For $v \in \mathcal{V}$, let $\mathcal{E}_v^+ = \{e \in \mathcal{E} : \sigma_e = v\}$ and $\mathcal{E}_v^- = \{e \in \mathcal{E} : \tau_e = v\}$. The symbols $\mathbb{R}$ and $\mathbb{R}_+$ denote the set of real and nonnegative real numbers, respectively.

II. Problem formulation

Following [10], [11], we model the topology of a transportation network as a weighted directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, C)$ whose nodes represent junctions and whose links represent segments of physical roads. On each link $e \in \mathcal{E}$ the variable $\rho_e \in [0, +\infty)$ denotes the aggregate density, or occupancy level, of traffic on it. For each $v \in \mathcal{V}$, let $\lambda_v \geq 0$ be the (uncontrollable) flow from the external world reaching v, and call the stacked version $\lambda \in \mathbb{R}_+^{\mathcal{V}}$ the vector of external inflows. We denote by $\mathcal{O} = \{v \in \mathcal{V} : \lambda_v > 0\}$ the set of origins, namely the set of nodes with nonzero inflow from the external world. Conversely, particles leave then network when they hit nodes called destinations. The set of destinations is denoted $\mathcal{D} \subseteq \mathcal{V}$. We make the following mild assumptions on the network topology.

Assumption 1: The set of destinations $\mathcal{D}$ is nonempty and $\mathcal{D} \cap \mathcal{O} = \emptyset$. Moreover, for every $v \in \mathcal{V} \setminus \mathcal{O}$ there exists at least one directed path from some origin node $o \in \mathcal{O}$ to v, and for every $v \in \mathcal{V} \setminus \mathcal{D}$ there exists at least one directed path from v to a destination node $d \in \mathcal{D}$.

The density on link e evolves according to the following mass conservation law

$$\dot{\rho}_e = f_e^{\mathrm{in}}(\rho) - f_e^{\mathrm{out}}(\rho)$$

where $f_e^{\mathrm{in}}(\rho)$ and $f_e^{\mathrm{out}}(\rho)$ denote the inflow into and the outflow from link e, respectively, and are functions of the density on the links of the network. We shall consider *distributed policies*, such that the inflow and outflow f_e^{in} and f_e^{out} are a function of the local densities around link e only. In particular, we assume that for every link $e \in \mathcal{E}$, $f_e^{\mathrm{out}} = h_e f_e(\rho_e)$, where $f_e(\cdot)$, called the *flow function at link* e, is a concave function such that $f_e(0) = 0$, $f_e' > 0$ and $\lim_{\rho_e \to \infty} f_e(\rho_e) = C_e$. The value C_e, the maximum outflow

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from edge e, is called the *capacity* of the edge. The variables $\{h_e\}_{e\in\mathcal{E}}$, $h_e \in [0,1]$ are control variables which can be used to artificially reduce the capacity on a link, thus enforcing a certain speed limit on the links of the network.

Remark 1: By assuming that the outflow on a link only depends on the density of the link itself, we implicitly assume that the network always remains, or is forced to remain, in *freeflow*. In the context of supply-and-demand models for transportation systems [12], this means that supply is always higher than demand.

We assume that for a link e such that $v = \sigma_e$,

$$f_e^{\mathrm{in}}(\rho) = G_e^v(\rho)\left(\lambda_v + \sum_{j\in\mathcal{E}_v^-} f_j^{\mathrm{out}}(\rho_j)\right),$$

where $\lambda_v + \sum_{j\in\mathcal{E}_v^-} f_j^{\mathrm{out}}(\rho_j)$ is the total flow through node v, and $G_e^v(\rho)$, called *routing policy*, tells how the flow through node v is split into the subsequent links. In particular, we impose the following model, where $\alpha_e \in \mathbb{R}$ and $\beta_e \in \mathbb{R}_+$,

$$G_e^v(\rho) = \frac{e^{-\beta_e\rho_e-\alpha_e}}{\sum_{j\in\mathcal{E}_v^+} e^{-\beta_j\rho_j-\alpha_j}}, \forall e \in \mathcal{E}_v^+, v \in \mathcal{V}. \tag{1}$$

This routing policy models both the behavior of drivers and possible actions of traffic managers. The parameter α_e can be interpreted as the willingness of a particle to turn into e when isolated in the network. The lower α_e, the more particles want to turn into e. It represents the aggregate effect of two different quantities, which are preference of drivers derived from historical memory, such as fastest path in business as usual, and tolls on links, as charging tolls upon usage of a certain link e increases the corresponding α_e. The parameter β_e can be interpreted instead as the willingness to react to the state of the network, as the lower β_e, the more the drivers will follow the preferred path. In the extreme cases, for $\beta_e = 0$ the state of the network does not affect the choice of drivers, while if $\beta_e = +\infty$ there is no preferred path and drivers simply turn into the less congested link.

Putting together the previous definitions, we obtain the following controlled dynamical system for all $e \in \mathcal{E}$

$$\begin{aligned}\dot\rho_e &= \Phi_e(\rho,\alpha,h)\\ &= \frac{e^{-\beta_e\rho_e-\alpha_e}}{\sum_{k\in\mathcal{E}_{\sigma_e}^+} e^{-\beta_k\rho_k-\alpha_k}}\left(\lambda_{\sigma_e} + \sum_{j\in\mathcal{E}_{\sigma_e}^-} h_j f_j(\rho_j)\right) - h_e f_e(\rho_e).\end{aligned} \tag{2}$$

The routing policy (1) is a particular case of *distributed monotone* routing policies [10], [11], which implies that the transportation network is a monotone system in the sense of Hirsch [13], [14]. The following result is proved in [10].

Proposition 1 (Th.1, [10]): Let $\mathcal{G} = (\mathcal{V},\mathcal{E},C)$ be a network satisfying Assumption 1 with inflow λ and routing policies as in (1), with fixed $\{\alpha_e\}_{e\in\mathcal{E}}$ and $\{h_e\}_{e\in\mathcal{E}}$. Let $\rho(t)$ denote the solution of (2) with initial condition ρ°. There exist limit point $\bar\rho(\alpha,h)$, $\bar\rho_e(\alpha,h) \in [0,+\infty]$, and limit flow $\bar f(\alpha,h)$, $\bar f_e(\alpha,h) \in [0,C_e]$, independent of ρ°, such that

$$\lim_{t\to\infty}\rho(t) = \bar\rho(\alpha,h), \quad \lim_{t\to\infty} f^{\mathrm{out}}(t) = \bar f(\alpha,h).$$

For every choice of $\{\alpha_e\}_{e\in\mathcal{E}}$ and $\{h_e\}_{e\in\mathcal{E}}$ we have a unique limit point as per Proposition 1. We are interested in the performance of such a limit point, measured by the a convex function $\Psi : \mathbb{R}_+^{\mathcal{E}} \to \mathbb{R}_+$ increasing in each argument. A standard example is $\Psi(\rho) = \sum_{e\in\mathcal{E}}\rho_e$, the total volume of traffic in the network. Formally, we aim at solving the following optimization problem

$$\begin{aligned}\min_{\rho,\alpha,h} \quad & \Psi(\rho) & \\ \text{s.t.} \quad & \rho_e \geq 0, & e\in\mathcal{E}\\ & \alpha_e \in \mathbb{R}, & e\in\mathcal{E}\\ & 0 \leq h_e \leq 1, & e\in\mathcal{E}\\ & \Phi_e(\rho,h,\alpha) = 0, & e\in\mathcal{E}\end{aligned} \tag{3}$$

A necessary condition for (3) can be derived as follows. A cut of the network is a subset $\mathcal{U}$ of nodes not containing any destination. We call the quantity $C_{\mathcal{U}} = \sum_{e:\sigma_e\in\mathcal{U},\tau_e\notin\mathcal{U}} C_e$ the capacity of the cut, and $\lambda_{\mathcal{U}} = \sum_{v\in\mathcal{U}}\lambda_v$ the inflow in the cut. The celebrated max-flow min-cut theorem [15] states that there exists an equilibrium flow in the network if and only if $\min_{\mathcal{U}}\{C_{\mathcal{U}} - \lambda_{\mathcal{U}}\} > 0$. Then, a necessary condition for the problem (3) to be feasible is that this inequality is satisfied.

Remark 2: More in general, the control variables $\{\alpha_e\}_{e\in\mathcal{E}}$ and $\{h_e\}_{e\in\mathcal{E}}$ could be designed as functions of the state of the network, with local information pattern. While such a state-dependent control would help improving the performance of the traffic network in the transient towards the equilibrium, we leave its design for future research.

III. Equilibrium selection with full knowledge of the network

In the next two sections we address two different scenarios, corresponding to the possibility, or not, of using tolls. In what follows, we assume that full knowledge of the parameters of the network is available.

A. Equilibrium selection using tolls

The following result states that if a traffic manager can charge tolls on each link, and if she has knowledge of the parameters $\{\beta_e\}_{e\in\mathcal{E}}$, then the tolls can be used to solve exactly the problem (3). To this aim, we need to solve the following optimization problem

$$\begin{aligned}\min_{\rho,\mu} \quad & \Psi(\rho) & \\ \text{s.t.} \quad & \rho_e \geq 0, & e\in\mathcal{E}\\ & 0 \leq \mu_e \leq f_e(\rho_e), & e\in\mathcal{E}\\ & \lambda_v + \textstyle\sum_{j\in\mathcal{E}_v^-}\mu_j \leq \sum_{e\in\mathcal{E}_v^+}\mu_e, & v\in\mathcal{V}\end{aligned} \tag{4}$$

Theorem 1: Let $\mathcal{G} = (\mathcal{V},\mathcal{E})$ be a network satisfying Assumption 1 with inflow λ. Let $\rho(t)$ be the solution of the controlled dynamical system (2) with initial condition $\rho(0) = \rho^\circ$. Assume that $\min_{\mathcal{U}}(C_{\mathcal{U}} - \lambda_{\mathcal{U}}) > 0$, and let (ρ^*,μ^*) be the solution of (4). Then setting

$$\alpha_e^* = -\log\mu_e^* - \beta_e\rho_e^*, \quad h_e^* = 1, \quad \forall e\in\mathcal{E} \tag{5}$$

yields $\bar\rho(\alpha^*,h^*) = \rho^*$ and (ρ^*,α^*,h^*) solves (3).

Proof: The theorem is proved in two steps. First one can show that if (ρ^*, μ^*) is the solution of (4), then $\sum_{e\in\mathcal{E}} \rho_e^* \leq \sum_{e\in\mathcal{E}} \rho_e$ for any feasible point (ρ, α, h) of (3), and moreover that if $\min_{\mathcal{U}}(C_{\mathcal{U}} - \lambda_{\mathcal{U}}) > 0$, then $\rho_e^* < \infty$ and $\mu_e^* < C_e$ for all $e \in \mathcal{E}$. Then, as pointed out in [11], it is possible to induce a certain equilibrium making use of tolls. Indeed, it can be proved that, setting $\{\alpha_e^*\}_{e\in\mathcal{E}}$ and $\{h_e^*\}_{e\in\mathcal{E}}$ as in (5), the solution of (2) converges to ρ^*. Since this is a feasible point for (3), (ρ^*, α^*, h^*) solves (3).

Remark 3: Differently from (3), the optimization problem (4) is convex and can be easily solved in a distributed way using known techniques [16]. ∎

B. Equilibrium selection without tolls

We assume in this section that tolls cannot be charged, so $\{\alpha_e\}_{e\in\mathcal{E}}$ are fixed and represent the drivers' preferences, and the equilibrium selection process relies on the control variables $\{h_e\}_{e\in\mathcal{E}}$ only. Exploiting the structure of $G_e^v(\rho)$ the following proposition can be proven.

Proposition 2: If $\{G_e^v(\cdot)\}_{e\in\mathcal{E}_v^+, v\in\mathcal{V}}$ is of the form (1), then (3) is equivalent to

$$\begin{array}{lll}
\min\limits_{\rho,z} & \Psi(\rho) & \\
\text{s.t.} & \rho_e \geq 0, & e\in\mathcal{E} \\
& z_e \in \mathbb{R}, & e\in\mathcal{E} \\
& z_e - \log f_e(\rho_e) \leq 0, & e\in\mathcal{E} \\
& z_e + \beta_e\rho_e + \alpha_e = z_j + \beta_j\rho_j + \alpha_j, & e,j\in\mathcal{E}_v^+, v\in\mathcal{V} \\
& \lambda_v + \sum_{e\in\mathcal{E}_v^-} e^{z_e} \leq \sum_{e\in\mathcal{E}_v^+} e^{z_e}, & v\in\mathcal{V}
\end{array} \tag{6}$$

with the change of variables $e^{z_e} = h_e f_e(\rho_e)$ for all $e\in\mathcal{E}$.

The constraints in the optimization problem (6) are all convex, except $\lambda_v + \sum_{e\in\mathcal{E}_v^+} e^{z_e} \leq \sum_{e\in\mathcal{E}_v^-} e^{z_e}$ for all $v\in\mathcal{V}$. Hence, the problem is not readily solvable as in the case when tolls are available.

We propose an heuristic approach called Sigmoidal Programming iteration, based on the convex-concave procedure [17]. First, we rewrite the constraints in (6) by introducing the variables $\{\mu_e\}_{e\in\mathcal{E}}$ and setting $\mu_e = h_e f_e(\rho_e) = e^{z_e}$, for all $e\in\mathcal{E}$. In this way, we obtain the following problem, equivalent to (6)

$$\begin{array}{lll}
\min\limits_{\rho,z,\mu} & \Psi(\rho) & \\
\text{s.t.} & \rho_e \geq 0, & e\in\mathcal{E} \\
& z_e \in \mathbb{R}, & e\in\mathcal{E} \\
& \mu_e \geq 0, & e\in\mathcal{E} \\
& z_e - \log f_e(\rho_e) \leq 0, & e\in\mathcal{E} \\
& z_e + \beta_e\rho_e + \alpha_e = z_j + \beta_j\rho_j + \alpha_j, & e,j\in\mathcal{E}_v^+, v\in\mathcal{V} \\
& \lambda_v + \sum_{e\in\mathcal{E}_v^-} e^{z_e} \leq \sum_{e\in\mathcal{E}_v^+} e^{z_e}, & v\in\mathcal{V} \\
& \mu_e = e^{z_e}, & e\in\mathcal{E}
\end{array} \tag{7}$$

Given an estimate $\bar{z}_e$ of z_e, we obtain a convexification of the last set of equality constraints by linearizing around $\bar{z}_e$, i.e., considering the new constraint $\mu_e = e^{\bar{z}_e}(1 + z_e - \bar{z}_e)$. Solving the problem with this new constraints yields a new estimate of z_e, which is then used for a new iteration step. This is formalized as follows:

- basic step: set $z_e^{(0)} = 0$ for all $e\in\mathcal{E}$;
- iterative step: for $k = 1, 2, \ldots,$ find $(\hat{\rho}, \hat{z}, \hat{\mu})$ that solve the convex problem

$$\begin{array}{lll}
\min\limits_{\rho,z,\mu} & \Psi(\rho) & \\
\text{s.t.} & \rho_e \geq 0, & e\in\mathcal{E} \\
& \mu_e \geq 0, & e\in\mathcal{E} \\
& z_e \in \mathbb{R}, & e\in\mathcal{E} \\
& z_e - \log f_e(\rho_e) \leq 0, & e\in\mathcal{E} \\
& z_e + \beta_e\rho_e + \alpha_e = z_j + \beta_j\rho_j + \alpha_j, & e,j\in\mathcal{E}_v^+, \\
& & v\in\mathcal{V} \\
& \lambda_v + \sum_{e\in\mathcal{E}_v^-} \mu_e \leq \sum_{e\in\mathcal{E}_v^+} \mu_e, & v\in\mathcal{V} \\
& \mu_e = e^{z^{(k-1)}}(1 + z_e - z_e^{(k-1)}), & e\in\mathcal{E}
\end{array} \tag{8}$$

  and set $\rho^{(k)} = \hat{\rho}$, $z^{(k)} = \hat{z}$, and $\mu^{(k)} = \hat{\mu}$.
- Stop if some stopping criterion is satisfied.

If n is the number of iterations, we consider $\rho^* = \rho^{(n)}$, $z^* = z^{(n)}$, and $\mu^* = \mu^{(n)}$ to be the solution to our problem, from which one can obtain the control variables setting $h_e^* = \frac{e^{z_e^*}}{f_e(\rho_e^*)}$ for all $e\in\mathcal{E}$.

Notice that since $e^{\bar{z}}(1 + z - \bar{z}) \leq e^z$ for all $z\in\mathbb{R}$, we have, for all $e\in\mathcal{E}$, $\mu_e \leq e^{z_e}$, and thus, for all $v\in\mathcal{V}$, $\lambda_v + \sum_{e\in\mathcal{E}_v^-} e^{z_e} \leq \sum_{e\in\mathcal{E}_v^+} \mu_e \leq \sum_{e\in\mathcal{E}_v^+} e^{z_e}$. As such, the feasibility set of (8) is a subset of that of (6), thus even though the obtained solution is suboptimal at each step, it always belongs to the original feasibility set. Moreover, (8) is clearly a convex optimization problem, thus a solution can be found, at any iteration step, by means of known techniques [16]. Finally, notice that while the procedure converges for a large number of iterations [17], no guarantee can be offered that the algorithm stops close to a global minimum.

IV. Conclusion

In this work we provide a first step towards distributed equilibrium selection in transportation networks. Future research includes and is not limited to extension of this approach to the back-pressure model provided in [18] and the supply-and-demand model [12], the design of density dependent controllers to improve the performance during the transient, and the design of adaptive controllers which do not require full knowledge of the parameters of the network.

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