Advanced Real-Time Systems

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1 Intro to Sched Prob

Basic elements of a scheduling problem:

- a set $\mathcal{N}$ of *tasks* (aka demands, works, jobs) requiring work to be made. Since they are finite, we represent them by $\mathcal{N} = \{1, 2, \ldots, n\}$;
- a set $\mathcal{R}$ of *resources* (aka processors, machines, workers, etc.) capable to perform some work (one/many machine, heterogeneous multicore, different machines in manufacturing, etc.). Since they are finite we represent them by $\mathcal{R} = \{1, 2, \ldots, m\}$;
- a *time set* $\mathcal{T}$, over which the scheduling is performed (typically $\mathbb{N}$ or $[0, \infty)$);
- a *scheduling algorithm* $\mathcal{A}$ which produces a *schedule* S for given $\mathcal{R}$ and $\mathcal{N}$.

Characteristics of tasks:

- amount of work,
- recurrent/non-recurrent, does a task repeat over time? How often?
- on-line/off-line, do we know the parameters in advance?
- precedence constraints,
- deadlines, "the work must be completed by this instant",
- affinity, not all resources are the same,
- sequential (only one resource at time), parallel (more than one resource at time), parallelizable (one or more resources at time).

Characteristics of a resource:

- type, (coffee machine $\neq$ a CPU)
- execution rate r_k (speed), which may be time-varying or demand-varying;

- operating modes (variable speed over time, etc.)

A schedule is a function

$$S : \mathcal{R} \times \mathcal{T} \to \mathcal{N} \cup \{0\}$$

If $m = 1$ then just $S : \mathcal{T} \to \mathcal{N} \cup \{0\}$.

- If $S(k,t) = i$ then the resource k is assigned to the i-task at time t.
- If $S(k,t) = 0$ then the resource k is not assigned at time t (we say that the k-th resource is *idle* at t).
- This definition of schedule implies that at every instant t each resource is assigned to at most one task.
- Conversely, at every instant each task may be assigned any number of resources in $\mathcal{R}$.

The inverse image of i under S, $s_i \subseteq \mathcal{R} \times \mathcal{T}$, that is

$$s_i = S^{-1}(i) = \{(k,t) \in \mathcal{R} \times \mathcal{T} : S(k,t) = i\}$$

represents the resources allocated to the i-th task.

Goal of scheduling algorithm $\mathcal{A}$: find a schedule S such that:

- the constraints are met (in this case constraints have to be specified): all task deadlines are met,
- some target function is minimized/maximized (minimum makespan/delay, best "performance": requires to know how the timing affect the "performance")

Characteristics of scheduling algorithms:

- (non-)work-conserving: no idle resource if pending tasks exist;
- (non-)preemptive, I can interrupt a task while it executes;
- time-complexity: how long does it take to decide the resource assignment?

Examples of scheduling algorithms:

- First In First Out (FIFO), schedule tasks in order of arrivals;
- Round Robin (RR), divide the time in slices and assign slices in round;
- Shortest Job First (SJF) and its preemptive version Shortest Remaining Time First (SRTF);
- Earliest Deadline First (EDF), assigns priority according to the deadlines d;

- Least Laxity First (LLF), aka Least Slack Time (LST), at t assigns priority according to the smallest "laxity" $(d-t)-c'$
- Fixed Priorities (FP), tasks are prioritized.

Definition 1 *A task set $\mathcal{N}$ is* feasible *is it exists a schedule which satisfies the task constraint.*

Definition 2 *A task set $\mathcal{N}$ is* schedulable *by the scheduling algorithm $\mathcal{A}$, if $\mathcal{A}$ can produce a schedule S which does not violate any constraint of $\mathcal{N}$.*

Obviously: schedulability by any algorithm implies feasibility.

1.1 Real-Time Scheduling

Many books exist for surveys of non-real-time scheduling [26, 47]. From now on we focus on RT scheduling.

- Tasks have a an execution requirement C_i;
- Tasks are recurrent and activated *sporadically*: with a minimum interarrival (or *period*) T_i; Each activation of a task is called *job* (job $\neq$ task);
- Tasks have a *relative deadline* D_i, relative (if $D_i = T_i$ then *implicit deadline*, if $D_i \leq T_i$ then *constrained deadline*, if D_i unrelated to T_i then *arbitrary deadline*);

Also the quantity $U_i = C_i/T_i$ is called task utilization and represents the fraction of time needed by task i.

More elaborated task model do exist. Below some references.

- Models for the execution time: Stigge et al. proposed the Di-Graph RT Task Model [58], which is probably the richest task model with a tractable analysis. In the paper, which contains also a comparison with other task models such as the multiframe task model [45], the recurring RT task model [9], and the non-cyclic generalized multiframe [59], the EDF analysis is described. In reward-based scheduling [7] the longer a task executes the higher system utility is achieved. Probabilistic execution times were also considered [32] even in the case of dependence among the distributions [13].
- Models for the activation pattern: the rate-based task model specifies the number of task activations in an interval [37], in the event stream model the number of task activations are lower and upper bounded by functions [51], Velasco et al. [60] analyzed the activation pattern of control tasks activated by events.
- Models for the deadline: a deadline model in which only m deadlines have to be guaranteed among any k consecutive ones [49, 48].

Resources: single processor, multiprocessor (with m processors/cores).
A necessary condition for feasibility is:

$$\sum_{i=}^{n} U_i \leq m \tag{1}$$

From now on we focus on single processor only ($m = 1$).

Below we report a list of the most relevant works on multiprocessor RT scheduling.

- Baruah et al. [11] introduced the notion of fairness and proposed the scheduling algorithm *P-fair* which can always produce a feasible schedule, if (1) is true. Later, Anderson and Srinivasan [2] extended to sporadic tasks. P-fair requires to divide the time in quanta, hence it may have a high overhead is the cost of preemptions is not negligible.

- Other algorithms which can feasibly schedule a task set, provided that (1) is true, with fewer preemption than P-fair, are Edf with task splitting and K processors in a Group (EKG) [5], Local Largest Remaining Execution time First (LLREF) [27], and Reduction to UNiprocessor (RUN) [50].

- A simpler scheduling algorithm is certainly Global EDF (EDF). However, GEDF cannot schedule all tasks satisfying (1). Hence some schedulability tests are needed [35, 8, 14, 23]. Some variations of GEDF can significantly improve the capacity to schedule task sets. Examples are: EDF-US [57] and Earliest Deadline Zero Laxity (EDZL) [29]. If a task set with implicit deadlines with $\sum_i U_i = m$ is scheduled by GEDF, clearly some deadline will be missed. However it is possible to estimate the amount of deadline violation [31, 33], which often may be acceptable.

- Examples of global fixed priority scheduling over multiprocessor are RM-US [4] and SM-US [3].

2 Fixed Priority (FP): basics

- Tasks are sorted in decreasing priority order
  - τ_1 is the highest priority one,
  - τ_n is the lowest priority one.

Theorem 1 (Liu, Layland, 1973 [44]) *If $D_i = T_i$ then Rate Monotonic (RM) is **optimal**: if some priority assignment can schedule the task set, then RM can schedule the task set.*

Theorem 2 (Leung, Whitehead, 1982 [41]) *If $D_i \leq T_i$ then Deadline Monotonic (DM) is optimal.*

Liu and Layland [44] also proved the most popular *utilization upper bound* (checked on 24/10/2012: cited 7914 in Google Scholar).

Theorem 3 *If* $D_i = T_i$

$$\sum_{i=1}^{n} U_i \leq n(\sqrt[n]{2} - 1)$$

then $\mathcal{N}$ *is schedulable by RM.*

- The RHS is called *utilization upper bound*.
- As $n \to \infty$ the bound tends to $\log 2 \approx 0.69315$
- Is the LL bound tight? Is there any non-schedulable task set with $\sum_i U_i > U_{\mathsf{LL}}$? Draw the example.

Theorem 4 (Hyperbolic Bound [19]) *If* $D_i = T_i$ *and*

$$\prod_{i=1}^{n} (1 + U_i) \leq 2$$

then $\mathcal{N}$ *is schedulable by RM.*

- Visualization of the bound and interpretation of HB in the utilization space.
- Still some space for uncertainty. What happens in between?

3 FP exact analysis: response time

Definition 3 *The* response time R_i *of task* τ_i *is the longest time that can elapse from the activation of any job to its completion.*

$$R_i = \max_{j \geq 1} \{R_{i,j}\}$$

with $R_{i,j}$ *response time of the* j*-th job of* τ_i*.*

The idea: to compute the response time and check whether or not $R_i \leq D_i$

- if so, then all jobs of τ_1 will meet their deadline (schedulable by FP).
- if not, then some job will miss its deadline (not schedulable by FP).

Definition 4 *We define the* level-i interference $I_i(t)$ *as the maximum amount of work which can be requested by tasks with priority higher than* i *in an interval of length* t*.*

For our simple task model, it is

$$I_i(t) = \sum_{j=1}^{i-1} \left\lceil \frac{t}{T_j} \right\rceil C_j$$

and it corresponds to the scenario with all tasks activated together at 0 at the highest possible rate.

- Example of interference with different activation patterns (two alternating periods)
- The response time of the first job of τ_i is found [38, 6] as the smallest fixed point of the following equation

$$\begin{cases} R_{i,1}^{(0)} = C_i \\ R_{i,1}^{(k+1)} = C_i + I_i(R_{i,1}^{(k)}) \end{cases} \tag{2}$$

- It converges iff $\sum_{j=1}^{i-1} U_j < 1$
- Explain its rationale.
- if $R_{i,1} \leq T_i$ then $R_i = R_{i,1}$ is the longest response time among all jobs belonging to τ_i.
- otherwise ($R_{i,1} > T_i$) it is not guaranteed that the maximum job response time occurs at the first job! In such a case we have to compute the response time $R_{i,k}$ of all subsequent jobs [39].
- We care only if $D_i > T_i$ (arbitrary deadline)
- We can compute the *absolute job response time* $r_{i,j}$ as follows

$$\begin{cases} r_{i,1} = R_{i,1} \\ r_{i,j}^{(0)} = r_{i,j-1} + C_i \\ r_{i,j}^{(k+1)} = j\, C_i + I_i(r_{i,j}^{(k)}) \\ R_{i,j} = r_{i,j} - (j-1)T_i \end{cases} \tag{3}$$

  until $r_{i,j} \leq j\ T_i$.
- The interval $[0, r_{i,j^*}]$, with j^* equal to the index of job where it first is $r_{i,j} \leq j\ T_i$, is called *level-i busy period*.
- If $\sum_{j=1}^{i} U_j < 1$, j^* is finite
- If $\sum_{j=1}^{i} U_j > 1$, $\lim_j R_{i,j} = \infty$ (*level-i overload*)
- If $\sum_{j=1}^{i} U_j = 1$ and $\{T_1, \ldots, T_i\}$ rational, j^* finite because the schedule will repeat after the least common multiple of the periods

- If $\sum_{j=1}^{i} U_j = 1$ and $\{T_1, \ldots, T_i\}$ irrational, j^* infinite, but R_i can still be defined as $R_i = \sup_j R_{i,j}$
- In human cases $R_i = \max_{j \leq j^*} R_{i,j}$ and we can check $R_i \leq D_i$.
- Iterating the response time equation Eq. (2) may be too time consuming, especially if it has to be executed on-line.
- One may want to forget necessity by computing a response time upper bound.
- Suppose we have a linear upper bound of the interference $I_i(t) \leq \overline{I}_i(t) = \alpha_i t + \beta_i$. Then from (2)

$$\begin{aligned} R_i &\leq C_i + \overline{I}_i(R_i) = C_i + \alpha_i R_i + \beta_i \\ R_i - \alpha_i R_i &\leq C_i + \beta_i \\ R_i &\leq \frac{C_i + \beta_i}{1 - \alpha_i} = \overline{R}_i \end{aligned}$$

  and have the following as a just sufficient (faster) test

$$\forall i, \quad \overline{R}_i \leq D_i$$

Finding suitable α_i and β_i. By upper bounding the $\lceil x \rceil$ with $x + 1$ in $I_i(t)$ we quickly find

$$\alpha_i = \sum_{j=1}^{i-1} U_j \qquad \beta_i = \sum_{j=1}^{i-1} C_j$$

however β_i can be made [22] a bit tighter by choosing

$$\beta_i = \sum_{j=1}^{i-1} C_j (1 - U_j)$$

this bound is still valid in the arbitrary deadline case.

4 FP: scheduling points

- The drawback of the response-time test is its "black-box" nature: all task parameters are specified and we get a yes/no answer.
- In reality, it is often more desirable to have a margin on the system parameters that guarantee schedulability: *sensitivity analysis*.

An alternate exact test is the following one.

Theorem 5 (Lehoczky et al. [40]) *A constrained deadline (with $D_i \leq T_i$) task set is schedulable by FP if and only if*

$$\forall i \in \mathcal{N}, \ \exists t \in [0, D_i], \quad C_i + I_i(t) \leq t$$

Interesting, but not so practical (how do we check if it exists a point in a real interval?)

If we remember the expression of $I_i(t)$ we realize that the previous condition is equivalent to the following one, which can be better managed

$$\forall i \in \mathcal{N},\ \exists t \in \mathcal{S}_i, \quad C_i + \sum_{j=1}^{i-1} \left\lceil \frac{t}{T_j} \right\rceil C_j \leq t$$

with

$$\mathcal{S}_i = \{kT_j : k \in \mathbb{N},\ 0 < kT_j < D_i,\ j < i\} \cup \{D_i\}$$

This is the set of *scheduling points*.

However the points in $\mathcal{S}_i$ can still be many and period dependent, especially when the periods of the higher priority tasks are significantly smaller than T_i.

- Can we remove points from $\mathcal{S}_i$ to reduce the complexity?
- By removing arbitrarily points we may lose necessity.

Theorem 6 (Hyperplanes Exact Test [17]) *A constrained deadline (with $D_i \leq T_i$) task set is schedulable by FP if and only if*

$$\forall i \in \mathcal{N},\ \exists t \in \mathcal{P}_{i-1}(D_i), \quad C_i + \sum_{j=1}^{i-1} \left\lceil \frac{t}{T_j} \right\rceil C_j \leq t \tag{4}$$

with $\mathcal{P}_j(t)$ being a set recursively defined as

$$\begin{cases} \mathcal{P}_0(t) = \{t\} \\ \mathcal{P}_j(t) = \mathcal{P}_{j-1}\left(\left\lfloor \frac{t}{T_j} \right\rfloor T_j\right) \cup \mathcal{P}_{j-1}(t). \end{cases} \tag{5}$$

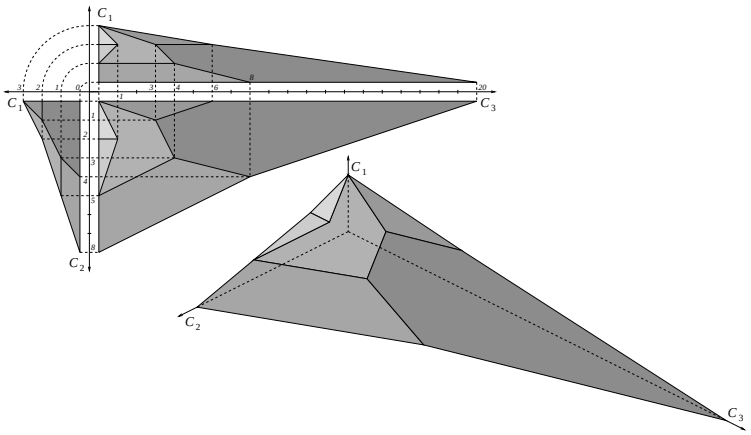
Show how $\mathcal{P}_{i-1}(D_i)$ is computed, for two tasks and U-plane.

$$\forall i \in \mathcal{N},\ \exists t \in \mathcal{P}_{i-1}(D_i), \quad C_i + \sum_{j=1}^{i-1} \left\lceil \frac{t}{T_j} \right\rceil C_j \leq t$$

If $T_1 = 3, T_2 = 8, T_3 = 20$, and $D_i = T_i$ the schedulable C_i are

5 FP: Sensitivity Analysis

- If the processor runs at speed r, then all computation times become C_i/r
- From the scheduling point condition it is not difficult [21] to find the smallest speed that guarantee FP-schedulability



$$\forall i \in \mathcal{N},\ \exists t \in \mathcal{P}_{i-1}(D_i), \quad \frac{C_i}{r} + \sum_{j=1}^{i-1} \left\lceil \frac{t}{T_j} \right\rceil \frac{C_j}{r} \leq t$$

$$\forall i \in \mathcal{N},\ \exists t \in \mathcal{P}_{i-1}(D_i), \quad r \geq \frac{C_i + \sum_{j=1}^{i-1} \left\lceil \frac{t}{T_j} \right\rceil C_j}{t}$$

$$r \geq \max_{i \in \mathcal{N}} \min_{t \in \mathcal{P}_{i-1}(D_i)} \frac{C_i + \sum_{j=1}^{i-1} \left\lceil \frac{t}{T_j} \right\rceil C_j}{t}$$

- (there is no direct way to find it using the response time)
- Example of calculation for two tasks in the plot.

What is the maximum schedulable C_k?

- all tasks τ_i with $i < k$ are unaffected by C_k
- to ensure the schedulability of τ_k it must be

$$C_k \leq \max_{t \in \mathcal{P}_{k-1}(D_k)} (t - \sum_{j=1}^{k-1} \left\lceil \frac{t}{T_j} \right\rceil C_j)$$

  The RHS is the amount of *level-*$(k-1)$ *idle time* in $[0, D_k]$

- to ensure the schedulability of all tasks with lower priority $i > k$ it must be

$$C_k \leq \min_{i=k+1,\ldots,n} \max_{t \in \mathcal{P}_{i-1}(D_i)} \frac{t - (C_i + \sum_{\substack{j=1 \\ j \neq k}}^{i-1} \left\lceil \frac{t}{T_j} \right\rceil C_j)}{\left\lceil \frac{t}{T_k} \right\rceil}$$

- hence $C_k^{\max}$ is the minimum of the two RHS

Since R_k is independent of the deadline D_k, the minimum schedulable deadline is just $D_k = R_k$.

- Task periods appear in the ceiling operator. Not trivial to extract them
- More sophisticated technique needed [21].

Let $T_k^{(i)}$ denote the minimum period of τ_k such that τ_i is schedulable.

- $i < k$, meaningless
- then
$$T_k^{\min} = \max_{i \geq k} T_k^{(i)}, \tag{6}$$
- The schedulability of τ_k is not affected by T_k
  - if constrained deadline ($D_k \leq T_k$) then $T_k^{(k)} = D_k$;
  - if implicit deadline ($D_k = T_k$) then $T_k^{(k)} = R_k$.
- Computing $T_k^{(i)}$ with $i > k$ requires some efforts.

Definition 5 *Given the subset of tasks $\mathcal{M} \subseteq \mathcal{N}$, we define* level-$\mathcal{M}$ idle time *in $[0, D]$, denoted by $Y(\mathcal{M}, D)$, the amount of time in $[0, D]$ in which no task in $\mathcal{M}$ is executing, under the worst-case scenario for activations.*

Examples:

- $Y(\varnothing, D) = D$
- $Y(\{\tau_1\}, T_1) = T_1 - C_1$

  To evaluate $T_k^{(i)}$ we need $Y(\{1, \ldots, i\} \setminus \{k\}, D_i)$, because it is the time that can be consumed by τ_k, keeping τ_i schedulable.

- How do we compute $Y(\mathcal{M}, D)$?
$$Y(\mathcal{M}, D) = \max_{t \in \mathcal{P}_{|\mathcal{M}|}(D)} \left\{ t - \sum_{j \in \mathcal{M}} \left\lceil \frac{t}{T_j} \right\rceil C_j \right\}$$
- It is scaring! Let's check.
- Then the maximum number $n_k^{(i)}$ of τ_k (associated to the minimum period $T_k^{(i)}$) jobs which can preserve the schedulability of τ_i then is
$$n_k^{(i)} = \left\lfloor \frac{Y(\{1, \ldots, i\} \setminus \{k\}, D_i)}{C_k} \right\rfloor$$

- Given the maximum number of jobs $n_k^{(i)}$, what is the minimum period $T_k^{(i)}$ which preserves the schedulability of τ_i?
- $T_k^{(i)}$ is such that τ_k has $n_k^{(i)}$ jobs interfering on τ_i, and by increasing $T_k^{(i)}$ by any small amount the number of interfering jobs would increase. Hence

$$T_k^{(i)} = \frac{R_i}{n_k^{(i)}}$$

with R_i response time of τ_i with $n_k^{(i)}$ interfering jobs by τ_k

$$R_i = C_i + n_k^{(i)} C_k + \sum_{\substack{j=1 \\ j \neq k}}^{i-1} \left\lceil \frac{R_i}{T_j} \right\rceil C_j$$

Summarizing

1. For all $i > k$ we compute

$$n_k^{(i)} = \left\lfloor \frac{Y(\{1, \ldots, i\} \setminus \{k\}, D_i)}{C_k} \right\rfloor$$

$$R_i = C_i + n_k^{(i)} C_k + \sum_{\substack{j=1 \\ j \neq k}}^{i-1} \left\lceil \frac{R_i}{T_j} \right\rceil C_j$$

$$T_k^{(i)} = \frac{R_i}{n_k^{(i)}}$$

$$T_k^{\min} = \max\{D_k, T_k^{(k+1)}, T_k^{(k+2)}, \ldots, T_k^{(n)}\} \text{ if constr dl}$$

$$T_k^{\min} = \max\{R_k, T_k^{(k+1)}, T_k^{(k+2)}, \ldots, T_k^{(n)}\} \text{ if impl dl}$$

If some pessimism is acceptable, we can gain in simplicity by using sufficient tests for the sensitivity

- Using LL bound $U_{\mathsf{LL}} = n(\sqrt[n]{2} - 1)$:
  - Min speed: $r^{\min} = \frac{\sum_i U_i}{U_{\mathsf{LL}}}$
  - Max C_k: $C_k^{\max} = T_k(U_{\mathsf{LL}} - \sum_{j \neq k} U_j)$
  - Min T_k: $T_k^{\min} = \frac{C_k}{U_{\mathsf{LL}} - \sum_{j \neq k} U_j}$

6 FP: optimal design

- Schedulability tests says yes/no
- Sensitivity analysis returns the margins of variation

- What if some task parameters are left unspecified and need to be set by the designer?

Optimal design of RT systems

Given:

- a task set $\mathcal{N}$ with a set $\mathcal{X}$ of parameters that are not specified
- a performance function of $J : \mathcal{X} \to \mathbb{R}$, which returns the performance $J(x)$ of assigning the unspecified parameters equal to x

Solve the following problem

$$\max_{x \in \mathcal{X}} J(x)$$
$$s.t.\ \mathcal{N}(x) \text{ is schedulable by FP}$$

It requires to understand the geometry of the feasible region $\mathcal{X}$.

Definition 6 (Def. 1 in [10]) *A schedulability test for a scheduling policy is* sustainable *if any system deemed schedulable by the schedulability test remains schedulable when the parameters of one or more individual job[s] are changed in any, some, or all of the following ways:*

1. *decreased execution requirements;*
2. *later arrival times;*
3. *smaller jitter; and*
4. *larger relative deadlines.*

Theorem 7 (in [10]) *Exact tests of FP and EDF are sustainable.*

7 FP: optimal execution time

- $\mathcal{X} = \{C_1, \ldots, C_n\}$.
- Periods T_i and deadlines D_i are given
- Execution times have to be found so that the performance $J(C_1, \ldots, C_n)$ (function of the computation times) is maximized

An example of cost function can be

$$\max_{C_1,\ldots,C_n} \min_i J_i(C_i)$$
$$\text{s.t } \mathcal{N} \text{ is FP-schedulable}$$

with $J_i(C_i)$ task dependent performance.

- Typically $J_i(C_i)$ is non-decreasing.

Solution is such that

- $\forall i = 1, \ldots, n-1 \quad J_i(C_i) = J_{i+1}(C_{i+1})$
- the task set is *barely FP-schedulable*: by increasing some computation time by any small amount, we make it non-schedulable.
- If exact solution is too complicated, one can always use sufficient (simpler) tests

An example with LL test

$$\max_{C_1,\ldots,C_n} \min_i J_i(C_i)$$
$$\text{s.t } \sum_i \frac{C_i}{T_i} \leq U_{\mathsf{LL}}$$

with $J_i(C_i)$ task dependent performance Solution is such that

- $\forall i = 1, \ldots, n-1 \quad J_i(C_i) = J_{i+1}(C_{i+1})$
- $\sum_i \frac{C_i}{T_i} = U_{\mathsf{LL}}$

Chung et al. [28] proposed the imprecise computation task model. In this task model, the longer a task can execute the higher accuracy can be achieved. Aydin et al [7] investigated the execution of an optional amount of computation. In their settings, the longer a job can execute, the higher reward is achieved. Hou and Kumar [36] considered the problem of maximizing the reward of tasks, which can execute chunks of code. To each chunk it is associated a reward. The problem is formulated as an LP problem with an unimodular matrix.

8 FP: optimal period

- $\mathcal{X} = \{T_1, \ldots, T_n\}$
- Typical problem in control systems: choosing the sampling periods of all controlers
- sometime it is convenient to view them as activation frequencies $f_i = \frac{1}{T_i}$, so that $\mathcal{X} = \{f_1, \ldots, f_n\}$
- Execution times C_i are specified
- Deadlines are specified either as absolute value D_i or normalized to the periods $\frac{D_i}{T_i}$.

The problem then is

$$\max_{T_1,\ldots,T_n} J(T_1,\ldots,T_n)$$
$$\text{s.t } \mathcal{N} \text{ is FP-schedulable}$$

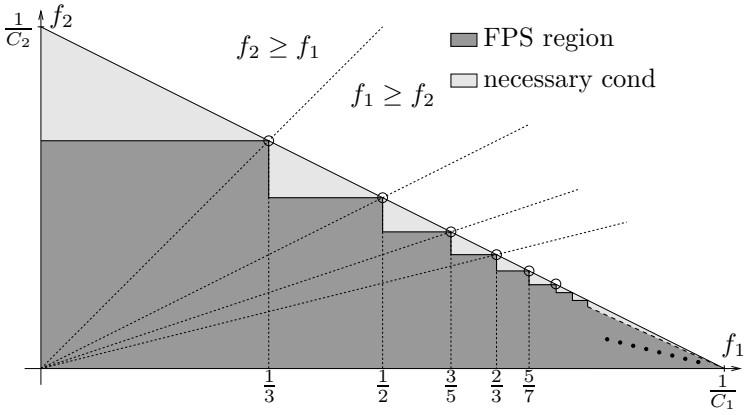
What is the geometry of the FP-schedulable periods?

Let us assume $D_i = T_i$

- Starting from a schedulable configuration, we can reduce any period, then another, ... until we reach a point with $\sum_i U_i = 1$. Hence at this final point no other period can be further reduced;
- the point that we reach changes depending on the order that we follow for reducing the periods.
- FP is sustainable.

View in the space of activation frequencies $f_i = \frac{1}{T_i}$. In the f_i space the constraint $\sum_i U_i \leq 1$ is linear.

if $n = 2$ and $C_1 = 1$, $C_2 = 2$ we have



If $\frac{\partial J}{\partial f_i} \geq 0$ all "vertices" are local optima.

Why the HB region is not contained?

1. find a good starting point over a simple constraint (for example, by using $\sum_i U_i \leq 1$)
2. using the "Min schedulable speed" find the intercept with the FP boundary
   - scaling the computation times by α is equivalent to scaling the task periods by $\frac{1}{\alpha}$
3. since $\frac{\partial J}{\partial f_i} \geq 0$, then by increasing task frequency we certainly increase the performance J

No guarantee of optimality

1. Start from an initial FP-schedulable solution $\mathbf{f}^{\mathsf{first}}$ (such as the one found previously) and set it as current solution $\mathbf{f}^{\mathsf{cur}}$
2. Use a branch and bound algorithm to enumerate, and possibly prune, all vertices $\mathbf{f}$ with better performance $J(\mathbf{f}) > J(\mathbf{f}^{\mathsf{first}})$
3. if a better solution is found, then update $\mathbf{f}^{\mathsf{cur}}$
4. when all vertices have been check $\mathbf{f}^{\mathsf{cur}}$ will be the optimum

How do we enumerate the vertices?

Theorem 8 (Proposition 2.1 in [53]) *The task set $\mathcal{N}$ is schedulable by FP **if and only if**:*

$$\forall i = 1, \ldots, n \quad \exists \mathbf{k}^{(i)} \in \mathbb{N}^{i-1}$$

such that

$$\begin{cases} C_i + \displaystyle\sum_{j=1}^{i-1} k_j^{(i)} C_j \leq T_i \\ (k_\ell^{(i)} - 1) T_\ell < C_i + \displaystyle\sum_{j=1}^{i-1} k_j^{(i)} C_j \leq k_\ell^{(i)} T_\ell \quad \ell = 1, \ldots, i-1 \end{cases}$$

Proof sketch: it is equivalent to $\forall i, \exists t \in [0, T_i] \ldots$, by setting $t = C_i + \sum_{j=1}^{i-1} k_j^{(i)} C_j$ and by setting $k_j^{(i)} = \left\lceil \frac{t}{T_j} \right\rceil$

The task set $\mathcal{N}$ is schedulable by FPS **if and only if**:

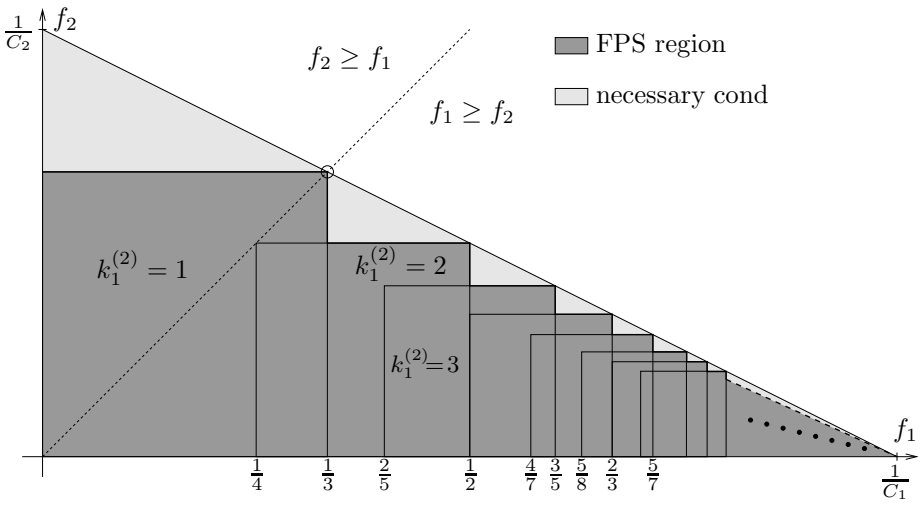
$$\forall i = 1, \ldots, n \quad \exists \mathbf{k}^{(i)} \in \mathbb{N}^{i-1}$$

such that:

$$\begin{cases} 0 \leq f_i \leq \boxed{\dfrac{1}{C_i + \sum_{j=1}^{i-1} k_j^{(i)} C_j}} \\ \dfrac{k_\ell^{(i)} - 1}{C_i + \sum_{j=1}^{i-1} n_j^{(i)} C_j} < f_\ell \leq \boxed{\dfrac{k_\ell^{(i)}}{C_i + \sum_{j=1}^{i-1} k_j^{(i)} C_j}} \quad \ell = 1, \ldots, i-1 \end{cases}$$

which are the coordinates of the vertices.

With $C_1 = 1$ and $C_2 = 2$



- A branch and bound algorithm can be used to search for the optimal task frequencies.
- More details can be found in [20].

Seto et al. [54] proposed to simply use a utilization based test

$$\sum_{i=1}^{n} \frac{C_i}{T_i} \leq U^{\mathsf{ub}}$$

to assign the periods.

- sub-optimal
- explicit solution can be found

9 Beyond the period assignment

Is the performance J function of the only task periods/frequencies?

- In control systems "periodic" means that both sampling and actuation occur at the same instant;
- In reality when a "periodic" task is scheduled over a CPU, it has a *delay* and a *jitter* in its execution, which depends on the parameters of the other tasks, too.

Add a drawing
Ideally we should:

- identify how the task periods and priorities affect the schedule $\mathcal{S}$;

- identify how the schedule affect the (control) performance;
- just perform the optimization.
- relatively recent research area
- some results do exist (will be included in the course notes)
- still space for new contributions
- How do task parameters affect the entire task schedule?
  - pattern of job start times
  - pattern of job response time
  - distribution of job response time
  - joint distribution of job response times of k consecutive jobs
- How does the task schedule affect performance?
  - can controllers be improved by the exact knowledge of sampling and actuation times?
  - how robust are controllers against variations of the task schedule?

10 Earliest Deadline First: basics

- Task model is still the same $\tau_i = (C_i, T_i, D_i)$;
- In FP, priorities are per task: all jobs of same task that have the same priority;
- In EDF, priorities are per job: jobs are prioritized according to their absolute deadline.

Theorem 9 (Liu and layland, 1973 [44]) *If a task set is feasible, then it is EDF-schedulable.*

Theorem 10 (Liu and Layland, 1973 [44]) *If $D_i = T_i$ (implicit deadline) then a task set is EDF-schedulable* ***if and only if****:*

$$\sum_{i=1}^{n} U_i \leq 1$$

- Any FP-schedulable task set is also EDF-schedulable task set.

11 EDF: demand bound function

If $D_i \neq T_i$ the schedulability condition becomes more complicated.

Theorem 11 (Lemma 3 in [12]) *The task set $\mathcal{N}$ is EDF-schedulable* ***if and only if****:*

$$\forall t \geq 0 \quad \sum_{i=1}^{n} \max\left\{0, \left\lfloor \frac{t + T_i - D_i}{T_i} \right\rfloor\right\} C_i \leq t$$

The LHS is called *demand bound function* $\mathsf{dbf}(t)$ of the task set at t.

- $\mathsf{dbf}(t)$ is the maximum amount of work of jobs with both activation and deadline in $[0, t]$.
- no per-task condition: any task task may influence others
- $\max\{0, \cdot\}$ only needed for i with $D_i > T_i$
- Obviously, checking $\forall t > 0$ is not very practical
- By observing the step shape of the dbf we can check only at the points where the steps occur

Theorem 12 (Lemma 3 in [12]) *The task set $\mathcal{N}$ is EDF-schedulable* ***if and only if*** $\sum_i U_i \leq 1$ *and:*

$$\forall t \in \mathcal{D} \quad \sum_{i=1}^{n} \max\left\{0, \left\lfloor \frac{t + T_i - D_i}{T_i} \right\rfloor\right\} C_i \leq t$$

with

$$\mathcal{D} = \{d_{i,k} : d_{i,k} = kT_i + D_i,\ i \in \mathcal{N},\ k \in \mathbb{N},\ d_{i,k} \leq D^*\}$$

and $D^* = \mathsf{lcm}(T_1, \ldots, T_n) + \max_i\{D_i\}$.

$H = \mathsf{lcm}(T_1, \ldots, T_n)$ is often called *hyperperiod* of the task set.

As suggested by Ripoll et al [52] D^* can also be set equal to the longest busy period.

An alternative approach, quite complex though, is to compute the response time of each task as proposed by Spuri [56].

- If $U < 1$, then the for large t the condition is always true
- then D^* can be computed [12] by upper bounding $\mathsf{dbt}(t)$ with a line and we find
$$D^* = \frac{U}{1 - U} \max_i\{T_i - D_i\}$$
- what happen to D^* if $\max_i\{T_i - D_i\} \leq 0$?
- the task set is obviously EDF-schedulable,

- $\max_i\{T_i - D_i\} \leq 0 \quad \Leftrightarrow \quad \forall i,\ D_i \geq T_i$
- EDF is sustainable, hence if $D_i = T_i$ is sched then $D_i \geq T_i$ also sched
- Since $U < 1$, the task set is sched

- All deadlines in $[0, D^*]$ may be too many
- Zhang and Burns [62] proposed the Quick convergence Processor-demans Analysis (QPA)

$d_{\min} \leftarrow \min\{D_i\}$
$t \leftarrow \max\{d_{i,k} : d_{i,k} \leq D^*\}$ ▷ initial assignment
while $\mathsf{dbf}(t) \leq t \wedge \mathsf{dbf}(t) > d_{\min}$ **do**
  if $\mathsf{dbf}(t) < t$ **then**
    $t \leftarrow \mathsf{dbf}(t)$
  else
    $t \leftarrow \max\{d_{i,k} : d_{i,k} < t\}$ ▷ escape from fixed points
  end if
end while
if $\mathsf{dbf}(t) \leq d_{\min}$ **then** task set EDF-schedulable
else task set not EDF-schedulable
end if

12 EDF: sufficient tests

- By replacing T_i with the more conservative value $\min\{T_i, D_i\}$, we find

$$\sum_{i=1}^{n} \frac{C_i}{\min\{T_i, D_i\}} \leq 1$$

  the ratio $\frac{C_i}{\min\{T_i, D_i\}}$ is often called *density* of the task

Devi proposed the following sufficient test

- Assuming that tasks are sorted by non-decreasing relative deadline ($D_i \leq D_2 \leq \ldots \leq D_n$)

Theorem 13 (Theorem 1 in [30]) *The task set $\mathcal{N}$ is EDF-schedulable if:*

$$\forall k = 1, \ldots, n \quad D_k \sum_{i=1}^{k} U_i + \sum_{i=1}^{k} \frac{T_i - \min\{T_i, D_i\}}{T_i} C_i \leq D_k$$

- Proved to strictly dominate the density test
- Albers et al. [1] proposed a Fully Polynomial Time Approximation Scheme.

- The i-th term in $\mathsf{dbf}(t)$ can be upper bounded by

$$\max\{0, \left\lfloor \frac{t - D_i + T_i}{T_i} \right\rfloor\} C_i \leq \mathsf{dub}_i(k,t)$$
$$= \begin{cases} \max\{0, \left\lfloor \frac{t-D_i+T_i}{T_i} \right\rfloor\} C_i & t \leq d_{i,k} = (k-1)T_i + D_i \\ U_i(t + T_i - D_i) & t > d_{i,k} \end{cases}$$

  so that

$$\frac{k}{k+1} \sum_i \mathsf{dub}_i(k,t) \leq \mathsf{dbf}(t) \leq \sum_i \mathsf{dub}_i(k,t)$$

- This enables a quite interesting result

  Theorem 14 *If $U \leq 1$ and*

$$\forall t \in \mathcal{D}(\overline{k}) = \{d_{i,k} \in \mathbb{R}^+ : d_{i,k} = (k-1)T_i + D_i, 1 \leq k \leq \overline{k}\}$$
$$\sum_{i=1}^{n} \mathsf{dub}_i(\overline{k}, t) \leq t$$

  then the task set is schedulable.
  Otherwise it is not schedulable over a CPU with speed $\frac{\overline{k}}{\overline{k}+1}$.

- In this way only $n\overline{k}$ evaluation of the dbf are needed.
- We can trade accuracy vs. complexity. As $\overline{k} \to \infty$ it becomes necessary and sufficient.
- FPTAS

Similar results can be found for FP [34].

13 EDF: sensitivity analysis

Similarly as in the FP case we can find [16] the minimum EDF-schedulable speed as follows

$$r^{\mathsf{min}} = \max_{t \in \mathcal{D}} \frac{\sum_{i=1}^{n} \max\left\{0, \left\lfloor \frac{t+T_i-D_i}{T_i} \right\rfloor\right\} C_i}{t}$$

However $D^* = H + \max_i\{D_i\}$, because we are changing the speed and then altering the linear upper bound that motivates the expression

$$D^* = \frac{U}{1-U} \max_i\{T_i - D_i\}$$

The maximum EDF-schedulable $C_k^{\max}$ can be computed in a similar way as in FP

$$C_k^{\max} = \min_{t \in \mathcal{D}, t \geq D_k} \frac{t - \sum_{\substack{i=1 \\ i \neq k}}^{n} \max\left\{0, \left\lfloor \frac{t+T_i-D_i}{T_i} \right\rfloor\right\} C_i}{\left\lfloor \frac{t+T_k-D_k}{T_k} \right\rfloor}$$

Here too, $D^* = H + \max_i\{D_i\}$.

14 EDF: space of comp times

Theorem 15 (Lemma 3 in [12]) *$\mathcal{N}$ is EDF-schedulable **if and only if** $\sum_i U_i \leq 1$ and:*

$$\forall t \in \mathcal{D} \quad \sum_{i=1}^{n} \max\left\{0, \left\lfloor \frac{t + T_i - D_i}{T_i} \right\rfloor\right\} C_i \leq t$$

with

$$\mathcal{D} = \{d_{i,k} : d_{i,k} = kT_i + D_i,\ i \in \mathcal{N},\ k \in \mathbb{N},\ d_{i,k} \leq D^*\}$$

and $D^* = \mathsf{lcm}(T_1, \ldots, T_n) + \max_i\{D_i\}$.

$H = \mathsf{lcm}(T_1, \ldots, T_n)$ is often called the *hyperperiod*.

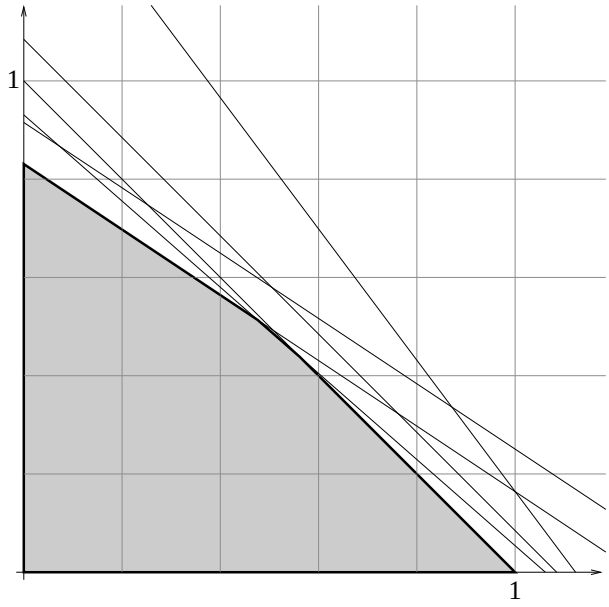
- Since we are investigating the space of C_i, no smaller D^* can be considered.
- For given T_i and D_i, the space of EDF-schedulable C_i is convex!

Let us assume $T_1 = 4, D_1 = 5$ and $T_2 = 6, D_2 = 5$.

$$\mathcal{D} = \{5, 9, 11, 13, 17\}$$

Hence equations are

$$\begin{bmatrix} 4 & 6 \\ 8 & 6 \\ 8 & 12 \\ 12 & 12 \\ 16 & 18 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \leq \begin{bmatrix} 5 \\ 9 \\ 11 \\ 13 \\ 17 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$



Hence only the equations

$$\begin{bmatrix} 4 & 6 \\ 16 & 18 \\ 1 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} U_1 \\ U_2 \end{bmatrix} \leq \begin{bmatrix} 5 \\ 17 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

are needed to be checked, corresponding to $\mathcal{D} = \{5, 17\}$

- Finding a general method to reduce $\mathcal{D}$ without losing sufficiency would be an interesting contribution.
- Being the exact region convex the optimal assignment of computation times C_i does not present big difficulties.

15 EDF: space of deadlines, periods

- Example: $n = 2$, $T_1 = T_2$, $U_1 + U_2 \leq 1$
- We can reduce D_1 to C_1, but then $D_2 = C_1 + C_2$, or
- $D_2 = C_2$, but then $D_1 = C_1 + C_2$
- convex comb of these two points are not EDF-sched
- Let us assume $T_1 = T_2 = \ldots = T_n$, and $\sum_i U_1 \leq 1$.

- For any permutation $p : \mathcal{N} \to \mathcal{N}$, a "vertex" has coordinates

$$D_{p(i)} = \sum_{j=1}^{i} C_{p(j)}$$

Being the periods and deadlines into the set $\mathcal{D}$ and into the $\lfloor \cdot \rfloor$, dbf condition is unfit to show the space of feasible periods/deadline.

Theorem 16 (Theorem 1 in[18], Theorem 2 in [15]) *The task set $\mathcal{N}$ is EDF-schedulable* ***if and only if****:*

$$\forall \mathbf{k} \in \mathbb{N}^n \setminus \{\mathbf{0}\} \quad \exists i \in I_{\mathbf{k}} \quad \sum_{j=1}^{n} C_j k_j \leq (k_i - 1)T_i + D_i$$

where

$$I_{\mathbf{k}} = \{j : k_j \neq 0\}$$

is the set of non-zero indexes in $\mathbf{k}$*.*

$$\forall \mathbf{k} \in \mathbb{N}^n \setminus \{\mathbf{0}\} \quad \exists i : k_i \neq 0 \quad \sum_{j=1}^{n} C_j k_j \leq (k_i - 1)T_i + D_i$$

can be seen as the problem of finding a cover to $\mathbb{N}^n \setminus \{\mathbf{0}\}$ with halfspaces of equations

$$(C_i - T_i)k_i + \sum_{j \neq i} C_j k_j \leq -T_i + D_i$$

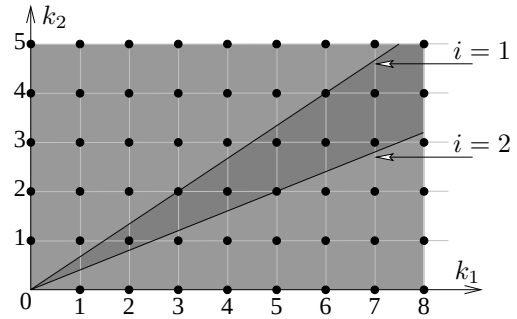
considering $(k_1, \ldots, k_n)$ as variables.

- Remember: $\exists i : k_i \neq 0$, not just $\exists i \in \mathcal{N}$
- If $U = 1$ linearly dependent.

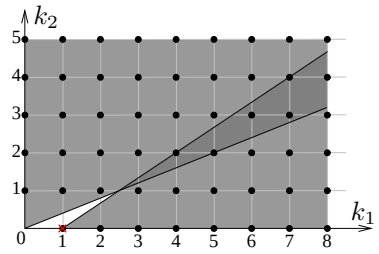
If $T_1 = D_1 = 4$, $C_1 = 2$, and $T_2 = D_2 = 8$, $C_2 = 3$, then

$$\begin{aligned} -2k_1 + 3k_2 &\leq 0 \\ 2k_1 - 5k_2 &\leq 0 \end{aligned}$$

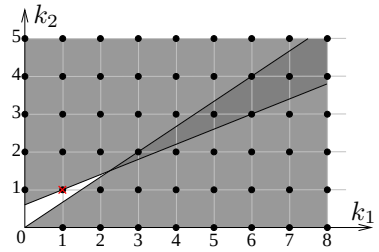
have to cover $\mathbb{N}^2 \setminus \{(0,0)\}$



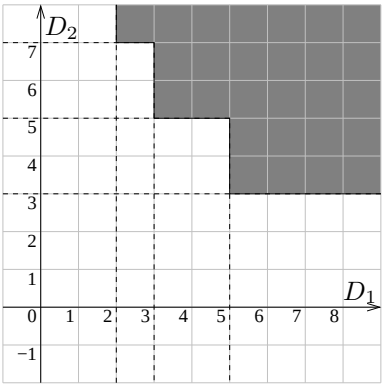
- If $U > 1$ cannot cover, ever.
- Deadline D_i appears at the RHS of the i-th inequality only
- Changing D_i means to translate boundary of the i-th halfspace



$D_1^{\min} = 2 \quad (= C_1)$



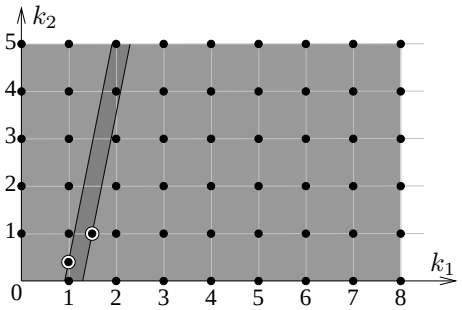
$D_2 = 5$



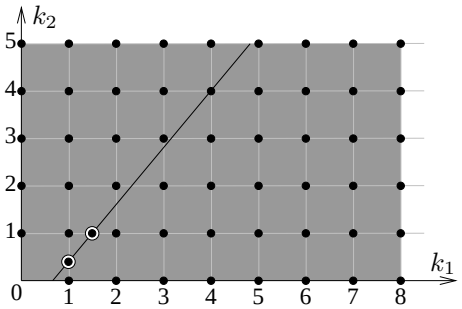
- The i-th period has an effect only on the i-th equation
- however T_i appears in the coefficients and at the RHS
- changing T_i is a rotation of the i-th halfspace
- the points in common between two equations with T_i' and T_i'' are

$$(C_i - T_i')k_i + \sum_{j \neq i} C_j k_j = -T_i' + D_i$$
$$(C_i - T_i'')k_i + \sum_{j \neq i} C_j k_j = -T_i'' + D_i$$
$$k_i = 1$$
$$\sum_{j \neq i} C_j k_j = D_i - C_i$$

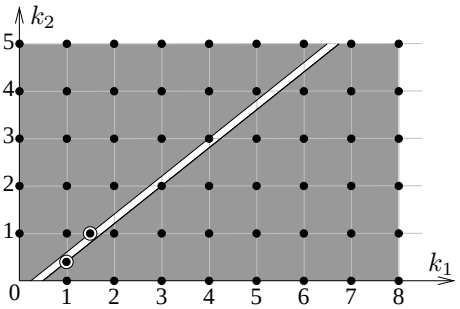
- As T_1 changes, the 1st boundary rotates around
$$k_1 = 1, \quad k_2 = \frac{D_1 - C_1}{C_2}$$
- As T_2 changes, the 2nd boundary rotates around
$$k_1 = \frac{D_2 - C_2}{C_1}, \quad k_2 = 1$$
- $C_1 = 2$, $D_1 = 4$, and $C_2 = 5$, $D_2 = 8$
- changing T_1 rotates around $(1, 0.4)$
- changing T_2 rotates around $(1.5, 1)$



- $T_1 = 27$
- $T_2 = {}^{27}/_{5}$



- $T_1 = 8$
- $T_2 = {}^{20}/_{3}$



- $T_1 = 6$
- $T_2 = {}^{15}/_{2}$

By making some more restrictive hypothesis, we can derive the following sufficient EDF-test

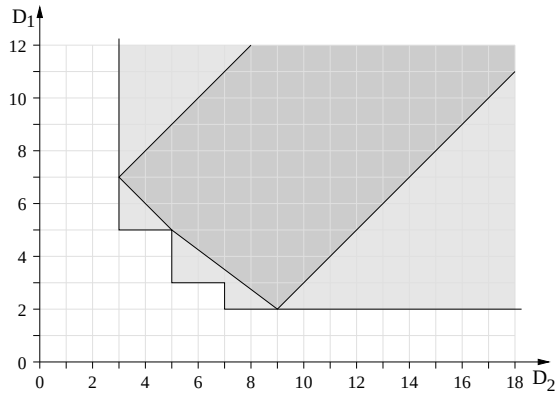
Theorem 17 (Chantem et al. [25]) *A task set is EDF-schedulable if:*

$$\begin{cases} \forall i, \quad D_i \leq T_i + D_{\min} \\ \sum_{i=1}^{n} U_i \, D_i \geq \sum_{i=1}^{n} C_i - D_{\min} \, (1 - \sum_{i=1}^{n} U_i). \end{cases}$$

with $D_{\min} = \min_i D_i$.

This condition has the advantage of being convex in the space of deadlines.

If $C_1 = 2$, $T_1 = 4$ and $C_2 = 4$, $T_2 = 7$, we have



- $\mathcal{X} = \{T_1, D_1, \ldots, T_n, D_n\}$
- Typical problem in control systems:

- given the execution times $\{C_1, \ldots, C_n\}$
- choose the sampling periods and deadlines (=delays) of all controllers
- such that a control cost (LQG?) is minimized

$$\min_{T_1, D_1, \ldots, T_n, D_n} J(T_1, D_1, \ldots, T_n, D_n)$$
$$\text{s.t } \mathcal{N} \text{ is EDF-schedulable}$$

We could try [61] to find optimal T_i and then D_i

1. Suppose $C_1 = C_2 = 1$
2. Suppose that the optimal T_i are $T_1 = 2.1$, $T_2 = 1.91$
3. How much can we reduce the deadlines?
   - very little.
4. However by choosing non-optimal periods $T_1 = T_2 = 2$, we can actually reduce one of the two deadlines by 1, possibly making the cost J smaller

- To best of my knowledge it is still unsolved
- (due to bizarre shape of the feasible region)
- solving this problem optimally would be a decent contribution

16 Hierarchical Scheduling: intro

- Applications often need to be isolated, otherwise a misbehaviour on one (such as taking longer than expected) can cause misbehaviors on other applications
- Example of high priority task executing for longer
- The execution of each application is controlled by a mechanism (aka server, resource reservation), which prevents the application from running more than planned
- Such a mechanism provides the abstraction of a *virtual resource*: a resource which is not always fully available
- How can we guarantee the timing constraints in presence of such a mechanism?

17 System model

- We model an application by a set of n tasks (C_i, T_i, D_i)
  - C_i computation time
  - T_i period
  - D_i deadline
- All tasks are scheduled by some scheduling algorithm (FP, EDF, etc.) **over a the virtual resource**. Such a scheduling alg. is often called *local scheduler* (local to the virtual resource)
- How to we check schedulability over virtual resources?
- The model of a virtual resource must capture the "not-fully-available" characteristic.

Definition 7 (Compare with [46, 42, 55, 43]) *We define the* supply bound function $\mathsf{sbf}(t)$ *of a virtual resource as the minimum amount of execution time available in any interval of length t.*

- Let us assume a virtual resource that provides execution time (to the application) in
$$\bigcup_{k\in\mathbb{Z}} [4k, 4k+1]$$
- What is its $\mathsf{sbf}(t)$?
- Remember: "any interval of length t" not necessarily $[0, t]$
- Let us now assume that the virtual resource is implemented by a periodic server with period (=deadline) P, time budget Q
- What is its $\mathsf{sbf}(t)$
- Remember: scenario of minimum possible supply must be assumed
$$\mathsf{sbf}(t) = \begin{cases} 0 & t \in [0, P-Q] \\ (k-1)Q & t \in (kP-Q, (k+1)P-2Q] \\ t-(k+1)(P-Q) & \text{otherwise} \end{cases}$$
with $k = \left\lceil \frac{t-(P-Q)}{P} \right\rceil$.
- This example explains why this is often called *hierarchical scheduling* since it is about to schedule a task set within another task
- Let us assume a virtual resource that provides execution time (to the application) in
$$[2,3] \cup [5,7] \cup [10,12] \quad \text{with period } 12$$
- Remember: the interval with the minimum supply can start at different points for different t

18 Schedulability conditions in hierarchical scheduling

Theorem 18 (FP-schedulability [42]) *A constrained deadline (with $D_i \leq T_i$) task set is FP-schedulable over a virtual resource with* $\mathsf{sbf}(t)$, ***if and only if***

$$\forall i \in \mathcal{N},\ \exists t \in \mathcal{P}_{i-1}(D_i), \quad C_i + \sum_{j=1}^{i-1} \left\lceil \frac{t}{T_j} \right\rceil C_j \leq \mathsf{sbf}(t)$$

Theorem 19 (EDF-schedulability [55]) *The task set $\mathcal{N}$ is EDF-schedulable over a virtual resource with* $\mathsf{sbf}(t)$, ***if and only if:***

$$\forall t \geq 0 \quad \sum_{i=1}^{n} \max\left\{0, \left\lfloor \frac{t + T_i - D_i}{T_i} \right\rfloor\right\} C_i \leq \mathsf{sbt}(t)$$

- Testing with the exact $\mathsf{sbf}(t)$, for example

$$\mathsf{sbf}(t) = \begin{cases} 0 & t \in [0, P-Q] \\ (k-1)Q & t \in (kP-Q, (k+1)P-2Q] \\ t-(k+1)(P-Q) & \text{otherwise} \end{cases}$$

  with $k = \left\lceil \frac{t-(P-Q)}{P} \right\rceil$,
  may be too complicated.

- It is safe (sufficient) to make the test by using a **lower** bound to $\mathsf{sbf}(t)$

Given a supply bound function $\mathsf{sbf}(t)$, it can be lower bounded [46] by any of the following functions

$$\mathsf{lsbf}(t) = \max\{0, \alpha(t-\Delta)\}$$

with

$$\alpha \leq \lim_{t\to\infty} \frac{\mathsf{sbf}(t)}{t}$$

and

$$\Delta = \sup_{t\geq 0} \left\{ t - \frac{\mathsf{sbf}(t)}{\alpha} \right\}$$

Typically we take $\alpha = \ldots$.

- α is often called *bandwidth* of the virtual resource;
- Δ is often called *delay* of the virtual resource.
  - $\Delta \geq \sup\{t : \mathsf{sbf}(t) = 0\}$ always.

1. What is the $\mathsf{lsbf}(t)$ of a periodic (Q, P) server?

$$\alpha = \frac{Q}{P}, \qquad \Delta = 2(P - Q)$$

2. What is the $\mathsf{lsbf}(t)$ of

$$[1, 2] \cup [3, 6] \quad \text{with period } 6$$

$$\alpha = \frac{5}{12}, \qquad \Delta = 1.5 \ (> \text{longest idle time})$$

1. Can we schedule by FP the two (implicit deadline) tasks $C_1 = 2$, $T_1 = 7$ and $C_2 = 2$, $T_2 = 15$, over a virtual resource implemented by a periodic server with period $P = 4$ and budget $Q = 2$?
2. Is it FP-schedulable over the virtual resource, if it is abstracted by the lsbf?
3. Is it EDF-schedulable over the virtual resource, if abstracted by the sbf?
4. Is it EDF-schedulable over the virtual resource, if abstracted by the lsbf?

19 Designing a virtual resource

- Often the task set is given
- We just have to design the virtual platform parameters such that the application is schedulable, and
- the minimum amount of (real, physical) resource is consumed (the bandwidth α is minimal).

For any pair (t, w), $t, w \geq 0$, let us define the region of feasible parameters as

$$\begin{aligned} F_{\alpha,\Delta}(t, w) &= \{(\alpha, \Delta) \in \mathbb{R}^2 : \mathsf{lsbf}(t) \geq w, \alpha \geq 0\} \\ &= \{(\alpha, \Delta) \in \mathbb{R}^2 : \alpha(t - \Delta) \geq w, \alpha \geq 0\} \end{aligned}$$

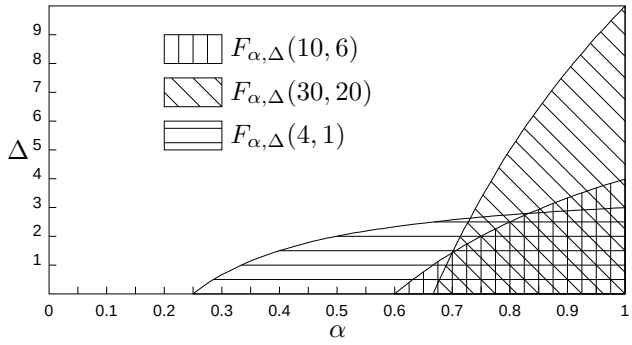
If local scheduler is EDF, then the problem is:

$$\begin{aligned} &\mathsf{minimize}\ J(\alpha, \Delta) \\ &\quad \mathsf{s.t.}\ (\alpha, \Delta) \in \bigcap_{t \in \mathcal{D}} F_{\alpha,\Delta}(t, \mathsf{dbf}(t)) \end{aligned}$$

If local scheduler is FP, then the problem is:

$$\begin{aligned} &\mathsf{minimize}\ J(\alpha, \Delta) \\ &\quad \mathsf{s.t.}\ (\alpha, \Delta) \in \bigcap_{i \in \mathcal{N}} \bigcup_{t \in \mathcal{P}_{i-1}(D_i)} F_{\alpha,\Delta}(t, C_i + I_i(t)) \end{aligned}$$

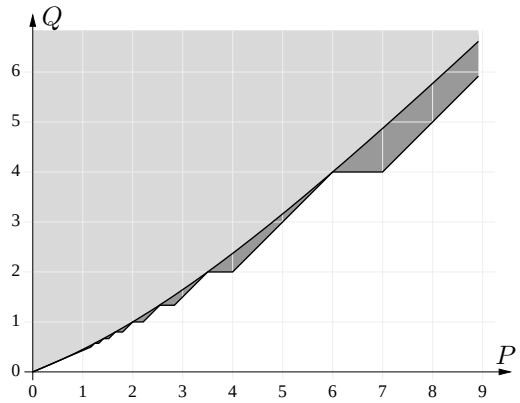
- $F_{\alpha,\Delta}(t,w)$ is convex



- If local scheduler is EDF the feasible region is convex.
- What is the cost function that it is reasonable to minimize?
- $J = \alpha$
  - good motivation (as little bandwidth as possible)
  - however, it may lead to impractical implementations
  - $J = \alpha$ implies that $\Delta^{\text{opt}} = 0$
  - $\Delta = 0$ can never be physically achieved. For example, if a periodic server is used, then
$$P = \frac{\Delta}{2(1-\alpha)}$$
- To prevent $\Delta = 0$ cases, we can consider to minimize the really consumed bandwidth
$$J = \underbrace{\alpha}_{\text{useful}} + \underbrace{\frac{c}{P}}_{\text{wasted}} = \alpha + 2c\frac{1-\alpha}{\Delta}$$
with c equal to some overhead. It is quasiconvex [24], hence good for optimization.
- lsbf is a lower bound, hence the previous analysis is not optimal. Exact analysis should consider the exact shape of the sbf

$$F_{P,Q}(t,w) = \{(P,Q) \in \mathbb{R}^2 : \mathsf{sbf}_{(Q,\,P)\text{ server}}(t) \geq w\}$$

In the figure below $F_{P,Q}(10,4)$



If local scheduler is EDF, then the problem is:

$$\begin{aligned} &\mathsf{minimize} \frac{Q}{P} + \frac{c}{P} \\ &\quad \mathsf{s.t.}\ (P,Q) \in \bigcap_{t \in \mathcal{D}} F_{P,Q}(t, \mathsf{dbf}(t)) \end{aligned}$$

If local scheduler is FP, then the problem is:

$$\begin{aligned} &\mathsf{minimize} \frac{Q}{P} + \frac{c}{P} \\ &\quad \mathsf{s.t.}\ (P,Q) \in \bigcap_{i \in \mathcal{N}} \bigcup_{t \in \mathcal{P}_{i-1}(D_i)} F_{P,Q}(t, C_i + I_i(t)) \end{aligned}$$

Level sets of the cost function are lines going through $P = 0$ and $Q = -c$.

20 Conclusions

- Hierarchical scheduling is well motivated by the need to compose application which are designed and guaranteed in isolation
- The optimal selection of the parameters of the virtual resource was discussed
  - EDF over lsbf easy because convex
  - FP over lsbf complicated, but approachable
  - EDF over sbf (exact supply) messy
  - FP over sbf messy mess
- With the simplest possible task model
- FP over uniprocessor resource
  - schedulability analysis

  - sensitivity analysis
  - optimal design of the task set
- EDF over uniprocessor resource
  - schedulability analysis
  - sensitivity analysis
  - optimal design of the task set
- FP+EDF over uniprocessor virtual resource
  - schedulability analysis
  - optimal design of the virtual resource
- If you want to publish in this area, you can complicate the task model, the scheduling algorithm, or the resource model according to your taste (possibly getting closer to the reality).

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