Model Reduction of Switched Affine Systems: a Method Based on Balanced Truncation and Randomized Optimization

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ABSTRACT

This paper proposes an approach to build a reduced order model for a Switched Affine (SA) system. The main idea is to transform the SA system into an equivalent Switched Linear (SL) system with state reset, and then apply balanced truncation to each mode and redefine the reset maps so as to best reproduce the free evolution of the system output. A randomized method is proposed for order selection in the case when the input is stochastic and one is interested in reproducing the output of the original SA system over a finite time-horizon. The performance of the approach is shown on a benchmark example.

Categories and Subject Descriptors

G.1.2 [**Mathematics of Computing**]: Approximation;
G.1.6 [**Mathematics of Computing**]: Optimization—*Stochastic programming*

Keywords

Model reduction; Switched systems; Randomized algorithms

1. INTRODUCTION

This paper deals with the problem of approximating a hybrid system by means of some simpler model, see e.g. [12,13,15,19,22,26] to cite a few. Hybrid systems are characterized by intertwined continuous and discrete dynamics, and are suitable for modeling complex, large scale systems, as shown in [18] where an overview on the application of hybrid models to various domains is presented. The study of hybrid systems is more challenging than that for other classes of systems, and many problems still lack an effective solution. In particular, this is the case for the design of reduced order models.

In this paper, we focus on the design of an approximate model for a switched affine system. More specifically, our goal is to obtain a simpler model of the system which can be effectively used for system verification over some finite horizon T.

HSCC'14, April 15–17, 2014, Berlin, Germany.

http://dx.doi.org/10.1145/2562059.2562131.

Verification of properties related to the hybrid system evolution, like, e.g., safety and reach/avoid properties, are typically addressed through numerical methods that scale badly with the state-space dimension, [1,2,9,11,16,20,24,27]. The aim of the approximation is then to build a model that mimics the behavior of the original system and that can be used in place of the system to scale-up numerical methods for the verification of the property of interest. When the hybrid system input is stochastic, the notion of approximate simulation introduced in [15] can be used to quantify the model performance.

The approach proposed in this paper is inspired by [19], where a balanced truncation is adopted for reducing the order of the linear dynamics governing the evolution of the continuous component of an hybrid system. The main advances with respect to [19] are

- the extension to the class of switched affine systems;
- the introduction of a novel method for defining the state reset map that provides better performance than the one adopted in [19]; and
- the introduction of a procedure to select the order of the reduced model based on a randomized approach, when the input is stochastic.

Note that, differently from most of the works on switched affine (or linear) system reduction, [23, 26], the transitions between discrete modes in the considered switched affine system class are determined by an endogenous signal that depends on the continuous state evolution, which makes the approximation problem more challenging.

The rest of the paper is organized as follows. We start with a brief review of balanced truncation for linear systems in Section 2. We then describe the considered switched affine system class (Section 3) and the proposed model reduction method (Section 4). The randomized approach to model order selection is illustrated in Section 5, whilst a numerical example showing the performance of the approach is presented in Section 6. Finally, some concluding remarks are drawn in Section 7.

2. BALANCED TRUNCATION FOR LINEAR SYSTEMS: A BRIEF REVIEW

There is a vast literature on model order reduction for linear systems (see e.g. [4, 14, 21]). In particular, balanced truncation is one of the more popular techniques, and the one adopted here for reducing the order of the continuous dynamics within each mode. The

balanced truncation method rests on the representation of the system in the balanced realization form, which is recalled next for the purpose of self-containedness.

Let $\mathscr{S}$ be a continuous-time linear time-invariant dynamic system described in state-space form through a 4-tuple of matrices $(\mathscr{A},\mathscr{B},\mathscr{C},\mathscr{D})$:

$$\mathscr{S}:\begin{pmatrix}\mathscr{A} & \mathscr{B}\\ \mathscr{C} & \mathscr{D}\end{pmatrix}.$$

Suppose that $\mathscr{S}$ is controllable, observable and asymptotically stable.

DEFINITION 1 (BALANCED SYSTEM). *System $\mathscr{S}$ is* balanced *if $\mathscr{W}_c = \mathscr{W}_o$, where*

$$\mathscr{W}_c = \int_0^\infty e^{\mathscr{A}\tau}\mathscr{B}\mathscr{B}^T e^{\mathscr{A}^T\tau}\,\mathrm{d}\tau$$
$$\mathscr{W}_o = \int_0^\infty e^{\mathscr{A}^T\tau}\mathscr{C}^T\mathscr{C} e^{\mathscr{A}\tau}\,\mathrm{d}\tau$$

are, respectively, the infinite controllability *and* observability Gramians *of $\mathscr{S}$. Furthermore, $\mathscr{S}$ is* principal-axis balanced *if $\mathscr{W}_c = \mathscr{W}_o = \Sigma$, with*

$$\Sigma = \mathrm{diag}\{\sigma_1,\sigma_2,\ldots,\sigma_n\},$$

where σ_i are the Hankel singular values of $\mathscr{S}$, listed in decreasing order.

The problem of finding the balanced realization of a system is equivalent to that of determining a balancing transformation matrix T such that

$$\begin{cases} W_c = T\mathscr{W}_c T^* \\ W_o = T^{-*}\mathscr{W}_o T^{-1}\end{cases} \quad \Rightarrow\ W_c W_o = T\,(\mathscr{W}_c\mathscr{W}_o)\,T^{-1} = \Sigma^2,$$

where T^* denotes the Hermitian adjoint of T, which, in turn, reduces to solving the following minimization problem [4]

$$\min_T \mathrm{tr}\left[T\mathscr{W}_c T^* + T^{-*}\mathscr{W}_o T^{-1}\right] = 2\,\mathrm{tr}\{\Sigma\}. \tag{1}$$

The system in the balanced state-space form is then obtained by applying the transformation matrix T, i.e.,

$$S:\begin{pmatrix}A & B\\ C & D\end{pmatrix} = \begin{pmatrix}T\mathscr{A}T^{-1} & T\mathscr{B}\\ \mathscr{C}T^{-1} & \mathscr{D}\end{pmatrix}.$$

The idea of the balanced truncation method is that in the balanced realization the state variables are ordered by decreasing importance as for their contribution to the input/output map, so that one can decompose the state vector (and the system) into two parts and neglect that with lowest importance. Formally, vector x is separated into two components

$$x = \begin{bmatrix}x_1\\ x_2\end{bmatrix},\quad S:\begin{pmatrix}A_{11} & A_{12} & B_1\\ A_{21} & A_{22} & B_2\\ C_1 & C_2 & D\end{pmatrix}.$$

with $x_1\in\mathbb{R}^{n_r}$ and $x_2\in\mathbb{R}^{n-n_r}$. Correspondingly,

$$\Sigma = \begin{bmatrix}\Sigma_1 & 0\\ 0 & \Sigma_2\end{bmatrix},$$

and if Σ_1 and Σ_2 do not contain any common element, then, the matrices A_{ii} $(i=1,2)$ are asymptotically stable [17].

A reduced order model S_r of the system can then be obtained by setting $x_2 = 0$ and eliminating its contribution, thus getting:

$$S_r:\begin{pmatrix}A_r & B_r\\ C_r & D_r\end{pmatrix} = \begin{pmatrix}A_{11} & B_1\\ C_1 & D\end{pmatrix}.$$

Alternatively, one can set $\dot{x}_2 = 0$, thus obtaining

$$S_r:\begin{pmatrix}A_r & B_r\\ C_r & D_r\end{pmatrix} = \begin{pmatrix}A_{11}-A_{12}A_{22}^{-1}A_{21} & B_1-A_{12}A_{22}^{-1}B_2\\ C_1-C_2A_{22}^{-1}A_{21} & D-C_2A_{22}^{-1}B_2\end{pmatrix}. \tag{2}$$

An estimate of the neglected state x_2 is then given by

$$\widehat{x_2} = -A_{22}^{-1}A_{21}x_1 - A_{22}^{-1}B_2u, \tag{3}$$

which corresponds to the condition $\dot{x}_2 = 0$. If Σ_1 and Σ_2 do not contain any common element, then, S_r is asymptotically stable, controllable and observable [17]. Moreover, the static gain of S_r is equal to that of the original system S.

In order to select the order of the reduced model, one can choose $\gamma\in[0,1]$ and set

$$n_r = \min\{i\in\{1,2,\ldots,n\} : \psi(i) < \gamma\},$$

where $\psi:\{1,2,\ldots n\}\to[0,1)$ is defined based on the Hankel singular values $\sigma_1\geq\sigma_2\geq\cdots\geq\sigma_n$ of system S as follows:

$$\psi(i) = 1 - \frac{\sum_{j=1}^{i}\sigma_j}{\sum_{j=1}^{n}\sigma_j}. \tag{4}$$

The bound γ can be used as a knob to control the tradeoff between the dimension of the reduced state and the quality of the approximation.

Approximation by balanced truncation preserves stability and the difference between system S and its reduced model S_r has its $\mathscr{H}_\infty$-norm bounded by the sum of the neglected Hankel singular values as follows:

$$\|S - S_r\|_{\mathscr{H}_\infty} \leq 2\,\mathrm{tr}\{\Sigma_2\}.$$

3. MODELING FRAMEWORK

We consider the class of Switched Affine (SA) systems, whose evolution is characterized through a discrete state component q_a taking values in $Q = \{1,2,\ldots,m\}$ and a continuous component $\xi_a\in\Xi_a = \mathbb{R}^n$ evolving according to an affine dynamics that depends on the operating mode q_a. Correspondingly, the output $y_a\in Y_a = \mathbb{R}^p$ is an affine function of the state and the input $u\in U = \mathbb{R}^m$ that depends on q_a as well. In formulas:

$$\begin{cases}\dot{\xi}_a(t) = \mathscr{A}_{q_a}\xi_a(t) + \mathscr{B}_{q_a}u(t) + f_{q_a}\\ y_a(t) = \mathscr{C}_{q_a}\xi_a(t) + g_{q_a}.\end{cases} \tag{5}$$

A collection of polyhedra $\{Dom_{a,i}\subseteq Y_a\times U,\, i\in Q\}$ is given, which covers the whole set $Y_a\times U$[1]. Each polyhedron $Dom_{a,i}$ is defined through a system of r_i linear inequalities:

$$Dom_{a,i} = \{(y_a,u)\in Y_a\times U : G_i^{y_a}y_a + G_i^u u \leq G_i\},$$

with $G_i^{y_a}\in\mathbb{R}^{r_i\times p}$, $G_i^u\in\mathbb{R}^{r_i\times m}$ and $G_i\in\mathbb{R}^{r_i}$.
The system evolves according to the dynamics associated with mode i as long as (ξ_a,u) is such that (y_a,u) keeps evolving within $Dom_{a,i}$ and commute to the dynamics associated with $j\in Q$ as soon as (y_a,u) exits $Dom_{a,i}$ and enters into $Dom_{a,j}$.

REMARK 1. *$Dom_{a,i}$ appears to be a function of both y_a and u. However, if $G_i^u = 0$, then, the dependence on u is not present. Furthermore, those cases when the transition condition depends on the whole state ξ_a can be reframed in our setting by including ξ_a in the output variables.*

[1] $\cup_{i\in Q} Dom_{a,i} = Y_a\times U$

REMARK 2. *Note that if $\{Dom_{a,i}, i \in Q\}$ is a polyhedral subdivision of $Y_a \times U$ (i.e., a finite collection of polyhedra on $Y_a \times U$ such that $\cup_{i\in Q} Dom_{a,i} = Y_a \times U$, each polyhedron $Dom_{a,i}$ is of dimension $p+m$, and the intersection $Dom_{a,i} \cap Dom_{a,j}$, $i \neq j$, is either empty or a common proper face of both polyhedra), then, the SA system reduces to a piecewise affine system.*

4. SYSTEM REDUCTION

In this section, we introduce a procedure for designing a reduced order model of the SA system (5) that tries to best reproduce its output y_a. The proposed procedure rests on Assumption 1 below, and is based on the following key steps:

- reformulation of the SA system as a Switched Linear (SL) system with state reset;
- model reduction of the SL system through balanced truncation of the continuous dynamics and definition of appropriate state reset maps when a mode transition occurs;
- reconstruction of the output of the SA system based on the reduced SL system.

ASSUMPTION 1. *For any $i \in Q$, matrix $\mathscr{A}_i$ is Hurwitz, $(\mathscr{A}_i, \mathscr{B}_i)$ is controllable, and $(\mathscr{A}_i, \mathscr{C}_i)$ is observable.*

4.1 Reformulation of the SA system as a SL system with state reset

We next build a SL system with state reset that is equivalent to the original SA system, in that (ξ_a, q_a) and y_a can be recovered exactly from the state and output variables of such a system.

Let $\xi \in \Xi = \Xi_a$ evolve according to a linear dynamics that depends on the operating mode $q \in Q$ as follows:

$$\begin{cases} \dot{\xi}(t) = \mathscr{A}_q \xi(t) + \mathscr{B}_q u(t) \\ y(t) = \mathscr{C}_q \xi(t) \end{cases} \tag{6}$$

where $y \in Y = Y_a$.

Set $\bar{y}_{a,q} = \mathscr{C}_q \bar{\xi}_{a,q} + g_q$, where $\bar{\xi}_{a,q} = -\mathscr{A}_q^{-1} f_q$, with $\mathscr{A}_q$ invertible by Assumption 1. A transition from mode $i \in Q$ to mode $j \in Q$ occurs as soon as $(y + \bar{y}_{a,i}, u)$ exits Dom_i and enters Dom_j, where $Dom_q = Dom_{a,q}$, $q \in Q$.

When a discrete transition from mode $i \in Q$ to mode $j \in Q$ occurs at time t^-, then, ξ is reset as follows

$$\xi(t) = \xi(t^-) + \bar{\xi}_{a,i} - \bar{\xi}_{a,j}. \tag{7}$$

PROPOSITION 4.1. *Suppose that the SA and SL systems are initialized with $\xi_a(0) = \xi_{a,0}$, $q_a(0) = q_{a,0}$, and $\xi(0) = \xi_{a,0} - \bar{\xi}_{a,q_{a,0}}$, $q(0) = q_{a,0}$, respectively, and are both fed by the same input $u(t)$, $t \in [0,T]$. Then, the execution of ξ_a, q_a and y_a over $[0,T]$ can be recovered from those of ξ, q and y as follows:*

$$\begin{aligned} q_a(t) &= q(t) \\ \xi_a(t) &= \xi(t) + \bar{\xi}_{a,q(t)} \\ y_a(t) &= y(t) + \bar{y}_{a,q(t)}. \end{aligned} \tag{8}$$

PROOF. The result immediately follows by observing that $\bar{\xi}_{a,q}$ and $\bar{y}_{a,q}$ are the state and output equilibria of system (5) associated with $u = 0$. □

REMARK 3. *Note that the reset condition in* (7) *is such that variable ξ_a reconstructed from ξ according to* (8) *is continuous. Continuity of ξ_a is generally not guaranteed if ξ is approximated through a reduced order model of the SL system.*

4.2 Reduction of the SL system

A reduced order model of the SL system with reset defined before can be obtained by applying balanced truncation (2) to each single linear dynamics in (6). This is is in order to best reproduce the evolution of the output y within a fixed mode, and also the discrete transitions between modes, since they are defined through a condition involving y.

We associate to each mode $q_r \in Q$ a reduced model of order $n_{r,q} \leq n$:

$$\begin{cases} \dot{x}_{r,q_r}(t) = A_{r,q_r} x_{r,q_r}(t) + B_{r,q_r} u(t) \\ \hat{y}(t) = C_{r,q_r} x_{r,q_r}(t) + D_{r,q_r} u(t) \end{cases} \tag{9}$$

and define transitions between modes, say from mode i to mode j, by evaluating when $(\hat{y} + \bar{y}_{a,i}, u)$ exits from domain Dom_i and enters into Dom_j. As for the state reset map (7) associated with a transition from mode $i \in Q$ to mode $j \in Q$, we shall reformulate it in the following form

$$x_{r,j}(t) = L_{ji} x_{r,i}(t^-) + M_{ji} u(t^-) + N_{ji}. \tag{10}$$

where $x_{r,i}(t^-) \in \mathbb{R}^{n_{r,i}}$, $x_{r,j}(t) \in \mathbb{R}^{n_{r,j}}$, and L_{ji}, M_{ji}, N_{ji} are matrices of appropriate dimensions.

We shall present next two methods to define matrices L_{ji}, M_{ji}, N_{ji}. In both of them we shall refer to the following variables:

1. the estimate $\hat{x}_i$ of the state of the SL system dynamics associated with mode $i \in Q$ in balanced form. $\hat{x}_i$ is reconstructed from the reduced state $x_{r,i}$ according to:

$$\begin{aligned} \hat{x}_i &= \begin{bmatrix} x_{r,i} \\ -A_{i,22}^{-1} A_{i,21} x_{r,i} - A_{i,22}^{-1} B_{i,2} u \end{bmatrix} \\ &= \begin{bmatrix} I_{n_{r,i} \times n_{r,i}} \\ -A_{i,22}^{-1} A_{i,21} \end{bmatrix} x_{r,i} + \begin{bmatrix} \mathbf{0}_{n_{r,i} \times 1} \\ -A_{i,22}^{-1} B_{i,2} \end{bmatrix} u \end{aligned} \tag{11}$$

   Expression (11) can be rewritten in compact form as

$$\hat{x}_i = H_i x_{r,i} + K_i u, \tag{12}$$

   with

$$H_i = \begin{bmatrix} I_{n_{r,i} \times n_{r,i}} \\ -A_{i,22}^{-1} A_{i,21} \end{bmatrix} \qquad K_i = \begin{bmatrix} \mathbf{0}_{n_{r,i} \times 1} \\ -A_{i,22}^{-1} B_{i,2} \end{bmatrix}$$

   where $I_{n_{r,i} \times n_{r,i}}$ is an identity matrix of dimension $n_{r,i} \times n_{r,i}$, and $\mathbf{0}_{n_{r,i} \times 1}$ is a zero vector of $n_{r,i}$ elements;

2. the estimate $\hat{\xi}_i$ of the state of the SL system associated with mode $i \in Q$:

$$\hat{\xi}_i = T_i^{-1} \hat{x}_i, \tag{13}$$

   obtained from $\hat{x}_i$ through the balanced transformation matrix T_i.

We are now in a position to defined the reduced state reset maps for a transition from $i \in Q$ at time t^- to $j \in Q$ at time t.

a) reset map proposed in [19]:

We start setting

$$x_{r,j}(t) = E_{n_{r,j}} \hat{x}_j(t)$$

where $E_{n_{r,j}}$ is a matrix that extracts the first $n_{r,j}$ rows from $\hat{x}_j(t)$, being $n_{r,j}$ the dimension of $x_{r,j}$ in mode j. Now,

$$\begin{aligned} \hat{x}_j(t) = T_j \hat{\xi}_j(t) &= T_j \left(\hat{\xi}_i(t^-) + \bar{\xi}_{a,i} - \bar{\xi}_{a,j} \right) \\ &= T_j \left(T_i^{-1} \hat{x}_i(t^-) + \bar{\xi}_{a,i} - \bar{\xi}_{a,j} \right) \\ &= T_j \left(T_i^{-1} H_i x_{r,i}(t^-) + T_i^{-1} K_i u(t^-) + \bar{\xi}_{a,i} - \bar{\xi}_{a,j} \right), \end{aligned}$$

so that

$$x_{r,j}(t) = E_{n_{r,j}} T_j \left(T_i^{-1} H_i x_{r,i}(t^-) + T_i^{-1} K_i u(t^-) + \bar{\xi}_{a,i} - \bar{\xi}_{a,j} \right). \tag{14}$$

By direct comparison of this expression with (10), we get the reset matrices:

$$\begin{aligned} L_{ji} &= E_{n_{r,j}} T_j T_i^{-1} H_i \\ M_{ji} &= E_{n_{r,j}} T_j T_i^{-1} K_i \\ N_{ji} &= E_{n_{r,j}} T_j \left(\bar{\xi}_{a,i} - \bar{\xi}_{a,j} \right). \end{aligned}$$

According to a similar reasoning, the system is initialized as follows

$$\begin{aligned} q_r(0) &= q_a(0) = q_0 \\ x_{r,q_0}(0) &= E_{n_{r,q_0}} T_{q_0} \left(\xi_a(0) - \bar{\xi}_{a,q_0} \right), \end{aligned}$$

with the understanding that $(y_a(0), u(0))$ is an interior point of Dom_{a,q_0} for any admissible $u(0)$.

b) reset map best reproducing the output free evolution:

Model reduction techniques for asymptotically stable linear systems aim at finding a model that best reproduce the forced response of the system, while neglecting the free evolution. This motivates the introduction of an alternative reset map that minimizes the norm-2 error when reproducing the free evolution of the output y. More precisely, we set

$$x_{r,j} = \Psi_j \hat{\xi}_j$$

and choose Ψ_j so as to minimize

$$J = \int_0^{+\infty} \| y_{fr,j}(t) - \hat{y}_{fr,j}(t) \|^2 \, \mathrm{d}t, \tag{15}$$

where $y_{fr,j}$ and $\hat{y}_{fr,j}$ respectively denote the free evolution of the original linear dynamics (6) initialized with $\hat{\xi}_j$ and that of the reduced order dynamics (9) initialized with $x_{r,j} = \Psi_j \hat{\xi}_j$. The solution to this optimization problem can be found analytically as shown in Proposition 4.2.

PROPOSITION 4.2. *Matrix Ψ_j minimizing* (15) *for any $\hat{\xi}_j$ is given by*

$$\Psi_j = \mathscr{W}_{r,o,j}^{-1} \mathscr{W}_{\times,j}.$$

where

$$\mathscr{W}_{r,o,j} = \int_0^{+\infty} (e^{A_{r,j}t})^T C_{r,j}^T C_{r,j} e^{A_{r,j}t} \, \mathrm{d}t \tag{16}$$

$$\mathscr{W}_{\times,j} = \int_0^{+\infty} (e^{A_j t})^T C_j^T C_{r,j} e^{A_{r,j}t} \, \mathrm{d}t \tag{17}$$

and invertibility of the infinite observability Gramian $\mathscr{W}_{r,o,j}$ is guaranteed by the observability of the reduced order model (9) *with $q = j$.*

PROOF. The cost function J can be written as

$$\begin{aligned} J =& \int_0^{+\infty} (C_j e^{A_j t} \hat{\xi}_j - C_{r,j} e^{A_{r,j}t} x_{r,j})^T (C_j e^{A_j t} \hat{\xi}_j - C_{r,j} e^{A_{r,j}t} x_{r,j}) \, \mathrm{d}t \\ =& x_{r,j}^T \mathscr{W}_{r,o,j} x_{r,j} - 2 x_{r,j} \mathscr{W}_{\times,j} \hat{\xi}_j + \hat{\xi}_j^T \mathscr{W}_{o,j} \hat{\xi}, \end{aligned}$$

where we set

$$\mathscr{W}_{o,j} = \int_0^{+\infty} (e^{A_j t})^T C_j^T C_j e^{A_j t} \, \mathrm{d}t.$$

Then, the minimum of J as a function of $x_{r,j}$ satisfies

$$\frac{\partial J}{\partial x_{r,j}} = 2 \mathscr{W}_{r,o,j} x_{r,j} - 2 \mathscr{W}_{\times,q'} \hat{\xi}_j = 0$$

yielding the reset map

$$x_{r,j} = \mathscr{W}_{r,o,j}^{-1} \mathscr{W}_{\times,j} \hat{\xi}_j.$$

□

Note that the quantity (16) is the solution of the Lyapunov equation

$$A_{r,j} \mathscr{W}_{r,o,j} + \mathscr{W}_{r,o,j} A_{r,j}^T + C_{r,j}^T C_{r,j} = 0,$$

while quantity (17) is the solution of the Sylvester equation

$$A_{r,j}^T \mathscr{W}_{\times,j} + \mathscr{W}_{\times,j} A_j + C_{r,j}^T C_j = 0.$$

Given Ψ_j, the following derivations

$$\begin{aligned} x_{r,j}(t) =& \Psi_j \hat{\xi}_j(t) = \Psi_j \left(\hat{\xi}_i(t^-) + \bar{\xi}_{a,i} - \bar{\xi}_{a,j} \right) = \\ =& \Psi_j \left(T_i^{-1} \hat{x}_i(t^-) + \bar{\xi}_{a,i} - \bar{\xi}_{a,j} \right) \\ =& \Psi_j \left(T_i^{-1} H_i x_{r,i}(t^-) + T_i^{-1} K_i u(t^-) + \bar{\xi}_{a,i} - \bar{\xi}_{a,j} \right) \end{aligned} \tag{18}$$

using the reset map (7) and equations (13) and (12) lead to the following definition of the matrices in the reset map (10):

$$\begin{aligned} L_{ji} &= \Psi_j T_i^{-1} H_i, \\ M_{ji} &= \Psi_j T_i^{-1} K_i, \\ N_{ji} &= \Psi_j \left(\bar{\xi}_{a,i} - \bar{\xi}_{a,j} \right). \end{aligned}$$

As for the system initialization, we set

$$\begin{aligned} q_r(0) &= q_a(0) = q_0 \\ x_{r,q_0}(0) &= \Psi_j \left(\xi_a(0) - \bar{\xi}_{a,q_0} \right). \end{aligned}$$

A different reset map that accounts for the switching nature of the system can be obtained by considering a finite horizon $[0, \tau]$ for the minimization of the free evolution error:

$$J = \int_0^{\tau} \| y_{fr,j}(t) - \hat{y}_{fr,j}(t) \|^2 \, \mathrm{d}t.$$

The resulting optimal $\Psi_j^{(\tau)}$ can be computed through the following expression

$$\Psi_j^{(\tau)} = \mathscr{W}_{r,o,j}^{-1}(\tau) \mathscr{W}_{\times,j}(\tau),$$

with

$$\begin{aligned} \mathscr{W}_{r,o,j}(\tau) &= \int_0^{\tau} (e^{A_{r,j}t})^T C_{r,j}^T C_{r,j} e^{A_{r,j}t} \, \mathrm{d}t \\ \mathscr{W}_{\times,j}(\tau) &= \int_0^{\tau} (e^{A_j t})^T C_j^T C_{r,j} e^{A_{r,j}t} \, \mathrm{d}t, \end{aligned}$$

the proof being analogous to that in the infinite horizon case. The above finite horizon quantities can be computed as

$$\begin{aligned} \mathscr{W}_{r,o,j}(\tau) &= \mathscr{W}_{r,o,j} - \int_{\tau}^{+\infty} (e^{A_{r,j}t})^T C_{r,j}^T C_{r,j} e^{A_{r,j}t} \, \mathrm{d}t = \mathscr{W}_{r,o,j} - \mathscr{W}_{r,o,j}^{(\tau,\infty)}, \\ \mathscr{W}_{\times,j}(\tau) &= \mathscr{W}_{\times,j} - \int_{\tau}^{\infty} (e^{A_j t})^T C_j^T C_{r,j} e^{A_{r,j}t} \, \mathrm{d}t = \mathscr{W}_{\times,j} - W_{\times,j}^{(\tau,\infty)}, \end{aligned}$$

where the quantities $\mathscr{W}_{r,o,j}^{(\tau,\infty)}$ and $\mathscr{W}_{\times,j}^{(\tau,\infty)}$ can be obtained respectively as the solution of the Lyapunov and Sylvester equations

$$\begin{aligned} A_{r,j} \mathscr{W}_{r,o,j}^{(\tau,\infty)} + \mathscr{W}_{r,o,j}^{(\tau,\infty)} A_{r,j}^T + \left(e^{A_{r,j}\tau} \right)^T C_{r,j}^T C_{r,j} e^{A_{r,j}\tau} = 0, \\ A_{r,j}^T \mathscr{W}_{\times,j}^{(\tau,\infty)} + \mathscr{W}_{\times,j}^{(\tau,\infty)} A_j + \left(e^{A_{r,j}\tau} \right)^T C_{r,j}^T C_j e^{A_j \tau} = 0, \end{aligned}$$

which are identical to the previous ones except for the fact that C_j and $C_{r,j}$ are replaced by $C_j e^{A_j \tau}$ and $C_{r,j} e^{A_{r,j} \tau}$, respectively. Note that well-posedness of the above equations is guaranteed by the fact that A_j and $A_{r,j}$ are Hurwitz.

The matrices in the reset map (10) and the system initialization are given by:

$$L_{ji} = \Psi_j^{(\tau)} T_i^{-1} H_i,$$
$$M_{ji} = \Psi_j^{(\tau)} T_i^{-1} K_i,$$
$$N_{ji} = \Psi_j^{(\tau)} \left(\bar{\xi}_{a,i} - \bar{\xi}_{a,j} \right)$$

and

$$q_r(0) = q_a(0) = q_0$$
$$x_{r,q_0}(0) = \Psi_j^{(\tau)} \left(\xi_a(0) - \bar{\xi}_{a,q_0} \right).$$

The choice for τ depends on the settling times of the different mode dynamics. A sensible choice is suggested in the numerical example of Section 6.

4.3 Reconstruction of the SA system output

The output of the SA system is reconstructed based on (8) using the output $\hat{y}$ of the SL reduced system as an estimate of the output y of the SL system:

$$\hat{y}_a(t) = \hat{y}(t) + \bar{y}_{a,q_r(t)}.$$

5. A RANDOMIZED METHOD FOR MODEL ORDER SELECTION

In this section, a randomized method is described for selecting the order of the reduced order model of the SA system when the input u is stochastic and the goal is verifying a finite horizon property that depends on the behavior of the SA system output y_a along the time horizon T.

The proposed method involves feeding the reduced model and the system with some realizations of the stochastic input. This in practice means that either the distribution of the input is known, or some of its realizations are available as historical time series.

As discussed in Section 2, a sensible way of choosing the order of the reduced model for a linear system is setting a threshold value for the ψ function in (4) and then define the order accordingly. By following the same logic as in [19], a function $\psi_q : \{1,2,\dots n\} \to [0,1)$ can then be considered for each mode $q \in Q$

$$\psi_q(i) = 1 - \frac{\sum_{j=1}^{i} \sigma_{j,q}}{\sum_{j=1}^{n} \sigma_{j,q}},$$

where $\sigma_{1,q} \geq \sigma_{2,q} \geq \cdots \geq \sigma_{n,q}$ are the Hankel singular values of the SL system dynamics (6) in mode q, and the order of the model (9) defining the reduced SL system can be set according to

$$n_{r,q} = \min\{i \in \{1,2,\dots,n\} : \psi_q(i) < \gamma\},$$

for each $q \in Q$.

Our goal is now to introduce a method for choosing an appropriate value for γ.

To this purpose, we denote by $\hat{y}_a^\gamma$ the estimate of y_a obtained through the reduced SL system with parameter γ, and by Γ the (finite) set of threshold values for γ, those that result in a different choice for $\{n_{r,q},\ q \in Q\}$.

In order to choose an appropriate order for the reduced dynamics associated to each mode, we quantify the approximation error through some function $d_T(\cdot,\cdot)$ that maps each pair of trajectories $y_a(t)$, $t \in T$, and $\hat{y}_a^\gamma(t)$, $t \in T$, into a positive real number $d_T(y_a, \hat{y}_a^\gamma)$ that represents the extent to which the output y_a of the SA system differs from its estimate $\hat{y}_a^\gamma$ along the time horizon T. Obviously, if we set $\gamma = 0$, then, no reduction is performed and $d_T(y_a, \hat{y}_a^\gamma) = 0$ since $\hat{y}_a^\gamma(t) = y_a(t)$, $t \in T$.

Note that $d_T(y_a, \hat{y}_a^\gamma)$ is a random quantity since it depends on the realization of the stochastic input $u(t)$ and the (possibly) stochastic initialization $\xi_a(0)$ of the SA system.

According to the notion of approximate simulation in [3, 10, 15], we assess the approximation quality of the reduced order model with parameter γ through the maximal value $\rho_\gamma^\star$ taken by $d_T(y_a, \hat{y}_a^\gamma)$ over all realizations of the stochastic input and initial state except for a set of probability at most $\varepsilon \in (0,1)$. An 'optimal' value for γ can then be chosen by inspecting the values of $\rho_\gamma^\star$ as a function of γ and selecting the appropriate compromise between quality of the approximation and tractability of the resulting reduced order model.

For each $\gamma \in \Gamma \subset [0,1]$, the approximation quality $\rho_\gamma^\star$ of the reduced order model with parameter γ is the solution to the following chance-constrained optimization problem:

$$CCP_\gamma : \min_{\rho} \rho \tag{19}$$
$$\text{subject to: } \mathbb{P}\{d_T(y_a, \hat{y}_a^\gamma) \leq \rho\} \geq 1 - \varepsilon.$$

REMARK 4 (CHOICE OF $d_T(y_a, \hat{y}_a^\gamma)$). *As argued in [3], the directional Hausdorff distance*

$$d_T(y_a, \hat{y}_a^\gamma) = \sup_{t \in T} \inf_{\tau \in T} \|y_a(t) - \hat{y}_a^\gamma(\tau)\| \tag{20}$$

is a sensible choice for $d_T(y_a, \hat{y}_a^\gamma)$ when performing probabilistic verification such as, e.g., estimating of the probability that y_a will enter some set within the time horizon T. For the verification of more complex reachability properties, such as that of reaching some set only after passing through some region within a given finite time interval, however, this choice for $d_T(y_a, \hat{y}_a^\gamma)$ is not adequate since the timing information is lost, and one can opt for

$$d_T(y_a, \hat{y}_a^\gamma) = \sup_{t \in T} \|y_a(t) - \hat{y}_a^\gamma(t)\|.$$

Irrespectively of the choice for $d_T(y_a, \hat{y}_a^\gamma)$, solving the chance-constrained problem (19) is known to be difficult, [25], since it involves determining, among all sets of realizations of the stochastic input and initial state that have a probability $1 - \varepsilon$, the one that provides the best (lowest) value for $d_T(y_a, \hat{y}_a^\gamma)$. We then head for an approximate solution where instead of considering all the possible realizations for the stochastic uncertainty, we consider only a finite number N of them called "scenarios", extracted at random according to their probability distribution, and treat them as if they were the only admissible uncertainty instances. This leads to the formulation of Algorithm 1, where the chance-constrained solution is determined using some empirical violation parameter $\eta \in (0, \varepsilon)$.

Notably, if the number N of extractions is appropriately chosen, the obtained estimate of $\rho_\gamma^\star$ is chance-constrained feasible, uniformly with respect to $\gamma \in \Gamma$, with a-priori specified (high) probability. This result is based on the "scenario theory", [7], which was first introduced for solving uncertain convex programs via randomization [5] and then extended to chance-constrained optimization problems in [6].

PROPOSITION 5.1. *Select a confidence parameter $\beta \in (0,1)$ and an empirical violation parameter $\eta \in (0, \varepsilon)$. If N is such that*

$$\sum_{i=0}^{\lfloor \eta N \rfloor} \binom{N}{i} \varepsilon^i (1-\varepsilon)^{N-i} \leq \frac{\beta}{|\Gamma|}, \tag{21}$$

Algorithm 1 Randomized solution

1: extract N realizations of the stochastic input $u^{(i)}(t)$, $t \in T$, $i = 1,2,\ldots,N$, and N samples of the initial condition $\xi_a(0)^{(i)}$, $i = 1,2,\ldots,N$, and let $k = \lfloor \eta N \rfloor$;

2: for all $\gamma \in \Gamma$ do

2.1: determine the N realizations of the output signals $y_a^{(i)}(t)$ and $\hat{y}_a^{\gamma,(i)}(t)$, $t \in T$, $i = 1,2,\ldots,N$, when the SL system and the reduced order model with parameter γ are fed by the extracted uncertainty instances;

2.2: compute

$$\hat{\rho}^{(i)} := d_T(y_a^{(i)}, \hat{y}_a^{\gamma,(i)}), i = 1,2,\ldots,N;$$

and determine the indices $\{h_1, h_2, \ldots h_k\} \subset \{1,2,\ldots,N\}$ of the k largest values of $\{\hat{\rho}^{(i)}, i = 1,2,\ldots,N\}$

2.3: set

$$\hat{\rho}_\gamma^\star = \max_{i \in \{1,2,\ldots,N\} \setminus \{h_1,h_2,\ldots,h_k\}} \hat{\rho}^{(i)}.$$

then, the solution $\hat{\rho}_\gamma^\star$, $\gamma \in \Gamma$, to Algorithm 1 satisfies

$$\mathbb{P}\{d_T(y_a, \hat{y}_a^\gamma) \le \hat{\rho}_\gamma^\star\} \ge 1 - \varepsilon, \ \forall \gamma \in \Gamma, \tag{22}$$

with probability at least $1 - \beta$. □

If we discard the confidence parameter β for a moment, this proposition states that for any $\gamma \in \Gamma$, the randomized solution $\hat{\rho}_\gamma^\star$ obtained through Algorithm 1 is feasible for the chance-constrained problem (19). As η tends to ε, $\hat{\rho}_\gamma^\star$ approaches the desired optimal chance constrained solution $\rho_\gamma^\star$. In turn, the computational effort grows unbounded since N scales as $\frac{1}{\varepsilon - \eta}$, [6], therefore, the value for η depends in practice from the available computational resources.

As for the confidence parameter β, one should note that $\hat{\rho}_\gamma^\star$ is a random quantity that depends on the randomly extracted input realizations and initial conditions. It may happen that the extracted samples are not representative enough, in which case the size of the violation set will be larger than ε. Parameter β controls the probability that this happens and the final result holds with probability $1 - \beta$. N satisfying (21) depend logarithmically on $|\Gamma|/\beta$, [6], so that β can be chosen as small as 10^{-10} (and, hence, $1 - \beta \simeq 1$) without growing significantly N.

PROOF (PROPOSITION 5.1). Note that the chance-constrained problem (19) needs to be solved for a finite number $|\Gamma|$ of values for γ. The application of Theorem 2.1 in [6] to the randomized solution obtained with Algorithm 1 for each given $\bar{\gamma} \in \Gamma$, provides the following guarantees on the solution $\hat{\rho}_{\bar{\gamma}}^\star$:

$$\mathbb{P}\{d_T(y_a, \hat{y}_a^{\bar{\gamma}}) \le \hat{\rho}_{\bar{\gamma}}^\star\} \ge 1 - \varepsilon, \text{ with probability at least } 1 - \frac{\beta}{|\Gamma|}.$$

As a result, guarantee (22) involving all $\gamma \in \Gamma$ holds except for a set whose probability can be upper bounded by $\sum_{i=1}^{|\Gamma|} \frac{\beta}{|\Gamma|} = \beta$, thus proving the thesis. □

Notice that the guarantees provided by Proposition 5.1 are valid irrespectively of the underlying probability distribution of the input, which may even not be known explicitly, e.g., when feeding Algorithm 1 with historical time series as realizations of the stochastic input u.

6. A NUMERICAL EXAMPLE

In this section we present a numerical example to show the performance of the proposed approach for model reduction. The example is inspired by a benchmark for hybrid system verification presented in [8].

6.1 Model description

The example deals with the heating of a number of rooms in a house. Each room has one single heater, but there is some constraint on the number of "active" heaters that can possibly be on at the same time. The temperature in each room depends on the temperature of the adjacent rooms, on the outside temperature, and on whether a heater is on in the room or not. The heater is controlled by a typical thermostat, i.e., it is switched on if the temperature is below a certain threshold, and off if it is beyond another (higher) threshold. Differently from the original benchmark in [8], we model also the dynamic of the heaters.

When the temperature in a room, say room i, falls below a certain level, its heater may become active (and eventually be switched on) if a heater was active in one of the adjacent rooms, say room j, provided that the temperature in room j is significantly higher than that in room i. In this case, we shall say for brevity that the heater is "moved" from room i to room j. The underlying *rationale* of the control policy is that, even if all the rooms have their own heater, the number of heaters that can be on at the same time must be limited, so as to exploit also the heat exchange among the rooms in order to maintain some minimum temperature in all rooms.

Let T_i be the temperature in room i, T_{ext} the outside temperature, and h_i a boolean variable that is 1 when the heater is on in room i, and 0 otherwise.

The heat transfer coefficient between room i and room j is k_{ij}, and the one between room i and the external environment is $k_{e,i}$. We assume that the heat exchange is symmetric, i.e., $k_{ij} = k_{ji}$. We say that rooms i and j are adjacent if $k_{ij} > 0$. The volume of the room is V_i, and the wall surface between room i and room j is $S_{r,ij}$, while that between room i and the environment is $S_{e,i}$. Air density and heat capacity are $\rho_a = 1.225\,\text{kg/m}^3$ and $c = 1005\,\text{J/(kg K)}$, respectively. Letting $\phi_i = \rho_a c V_i$, we can formulate the following dynamic model for room i and its heater:

$$\begin{aligned} \phi_i \dot{T}_i &= \sum_{j \ne i} S_{r,ij} k_{ij} \left(T_j - T_i\right) + S_{e_i} k_{e,i} \left(T_{\text{ext}} - T_i\right) + \kappa_i \theta_i \\ \tau_{h,i} \dot{\theta}_i &= -\theta_i + h_i \cdot p_i - \chi_i T_{\text{ext}} \end{aligned} \tag{23}$$

which is an affine system, with T_i representing the temperature in the i-th room, κ_i representing the maximum heat flow rate that the heater can provide, while $p_i \in \{0,1\}$ is a binary variable indicating if the heater is active in room i. The heater dynamics is represented by a first-order system with a time constant $\tau_{h,i}$. If we neglect the term $-\chi_i T_{\text{ext}}$ in the heater dynamics and set $h_i = p_i = 1$, the heater state variable θ_i will tend to 1 so that the heater will provide its maximum heat flow rate κ_i to the room when it is active and on. The term $-\chi_i T_{\text{ext}}$ is introduced to account for the influence of the external temperature on the effectiveness of the heating system.

6.2 The switching control policy

There is a *room policy*, which decides whether or not to switch on the heater of a single room, and a *building policy* which decides how to "move" the heaters that can be switched on.

As for the room policy, each room has a thermostat that switches the heater on if the measured temperature is below a certain threshold, and off when the temperature reaches a higher temperature. For each room we define thresholds on_i and off_i: the heater in room i is on if $T_i \leq on_i$ and off if $T_i \geq off_i$.

On the other hand, the building policy can be defined as follows. A heater is moved from room j to an adjacent room i if the following holds

- room i has no active heater;
- room j has an active heater;
- temperature $T_i \leq get_i$;
- the difference $T_j - T_i \geq dif_i$.

Notice that the control policy may have non-deterministic behaviors, since a room j may have more than one room, e.g., rooms i_1 and i_2, that is adjacent, and it may happen that conditions for the building policy to move the heater to room i_1 and to room i_2 are satisfied at the same time. To avoid non-deterministic choices in the policy, each room is identified by some integer index, and, in the previously mentioned situation, the heater is always moved to the room with higher index.

Apparently enough, the switching nature of the system originates from the control policy. The complexity of the considered system significantly increases with the number of rooms, thus making the problem particularly suitable for reduction when dealing with realistic cases.

6.3 The considered system

In the following we consider four adjacent rooms as represented in Figure 1, having each its own heater, but with the constraint that only three heaters can be active at the same time, i.e., $\sum_{i=1}^{4} p_i = 3$.

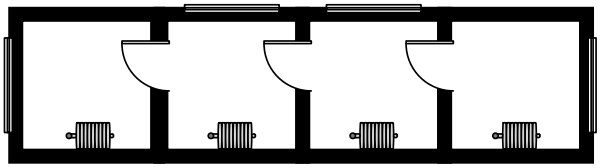

Figure 1: Scheme of the four rooms.

The rooms have different heat transfer coefficients among them, but identical geometric characteristics. The considered parameters are reported in Table 1.

Parameters			
k_{12}	$2\,\mathrm{W/(m^2K)}$	$S_{r,ij}$	$12\,\mathrm{m^2}$
k_{23}	$5\,\mathrm{W/(m^2K)}$	$S_{e,i}$	$24\,\mathrm{m^2}$
k_{34}	$2\,\mathrm{W/(m^2K)}$	V_i	$48\,\mathrm{m^3}$
$k_{e,i}$	$1\,\mathrm{W/(m^2K)}$	χ_i	10^{-5}

Table 1: Four rooms parameters.

The outside temperature is modeled as a sinusoidal source of period 24 hours with an offset of 4°C, affected by a band-limited Gaussian noise with zero mean and variance 4.

We assume that the initial conditions are deterministic and given by

$$T(0) = \begin{bmatrix} 20 \\ 20 \\ 20 \\ 20 \end{bmatrix}, \quad \theta(0) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad h(0) = p(0) = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

where T is the vector of the 4 rooms temperatures, θ is the vector of the heaters states, h and p are the vectors denoting, respectively, the on/off status and the active/inactive status of the heaters. Obviously, $p(0)$ satisfies the condition that only 3 over the 4 heaters are active. As for the (switching) control policy parameters, we use

$$off = \begin{bmatrix} 21 \\ 21 \\ 21 \\ 21 \end{bmatrix}, \quad on = \begin{bmatrix} 20 \\ 20 \\ 20 \\ 20 \end{bmatrix}, \quad get = \begin{bmatrix} 19 \\ 19 \\ 19 \\ 19 \end{bmatrix}, \quad dif = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}. \quad (24)$$

According to the described policy, model (23) can be represented as a SA system with continuous state $\xi_a = \begin{bmatrix} T' & \theta' \end{bmatrix}'$, input $u = T_{\text{ext}}$, and output $y_a = T$:

$$\begin{aligned} \dot{\xi}_a &= \mathscr{A}\,\xi_a + \mathscr{B}\,u + f_{q_a} \\ y_a &= \mathscr{C}\,\xi_a. \end{aligned} \quad (25)$$

As for the mode q_a, it is identified by the value of h and p, which determine the affine term entering the dynamics of ξ_a. The polyhedral sets Dom_{a,q_a} are determined by the building and room control policies through the threshold values (24) as described in Section 6.2.

Notice that in this example only the affine term f_{q_a} depends on the discrete mode $q_a \in Q$, while the state-space matrices $(\mathscr{A}, \mathscr{B}, \mathscr{C})$ are constant.

As for the choice of the order of the reduced model, the standard approach used in balanced truncation techniques [19] and resting on classical Hankel Singular Values (HSV) analysis can be applied so as to identify to what extent reducing the system dynamics in each single mode. This analysis is independent of the discrete mode. More importantly, it does not consider the impact of the choice of the order on the switched system approximation, which involves also mode transitions.

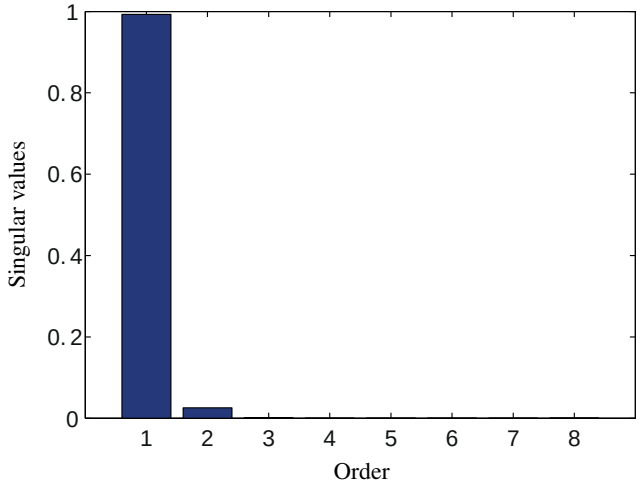

Figure 2: Hankel Singular Values sorted by decreasing magnitude.

Figure 2 shows the HSV of system (25) sorted by decreasing magnitude. On the basis of the HSV, it seems that most of the dynamics can be caught by reducing the continuous dynamics of the SA system to a first-order one. Indeed, computing the distance (4) used in [19] results in $\psi(1) \cdot 100 = 2.64\%$.

As anticipated, this evaluation of the quality of the reduced model does not account for the impact of mode transitions, thus care has to be taken when applying it to the context of SA systems. In fact, classical balanced truncation techniques are typically based on the assumption that the free evolution of the system can be neglected since it asymptotically vanishes in an asymptotically stable linear system, fact that notoriously does not hold true when dealing with hybrid behaviors.

6.4 Proposed model reduction method

We apply now the proposed model reduction method to the considered system, including the randomized method for order selection based on the directional Hausdorff distance (20). In particular, referring to the chance constrained optimization problem (19), we choose $\varepsilon = 0.1$, $\beta = 10^{-6}$. Thus, setting $\eta = 0.05$, and solving the implicit formula (21), the number of experiments to be performed for each possible threshold value for γ is $N = 778$, corresponding to a number $\lfloor \eta N \rfloor = 38$ of realizations to be removed, as described in Algorithm 1.

The randomized order selection is performed with the reset maps (14) proposed in [19], map (18) proposed here for the first time, both in its finite and infinite horizon versions. As for the choice of the finite horizon, the time constant τ_h of the heater is chosen.

Figure 3 shows a realization of the temperatures obtained with the original model and with the reduced models of order 5 implementing the three reset maps.

Notice that there is a discrete map $m_\gamma : \Gamma \to \{1, 2, \ldots, n\}$ between the threshold values of γ and the corresponding order n_r of the reduced order model. In formulas

$$n_r = \underset{i=\{1,2,\ldots,n\}}{\arg\min} \left\{ d_T(y_a, \hat{y}_a^\gamma) \leq \hat{\rho}_\gamma^\star \right\}.$$

For the sake of clarity, it is more convenient to express the estimate of $\rho_\gamma^\star$ as a function of the reduced order n_r. The values for $\hat{\rho}_\gamma^\star$ obtained with the different reset methods are presented in Figure 4 as a function of n_r.

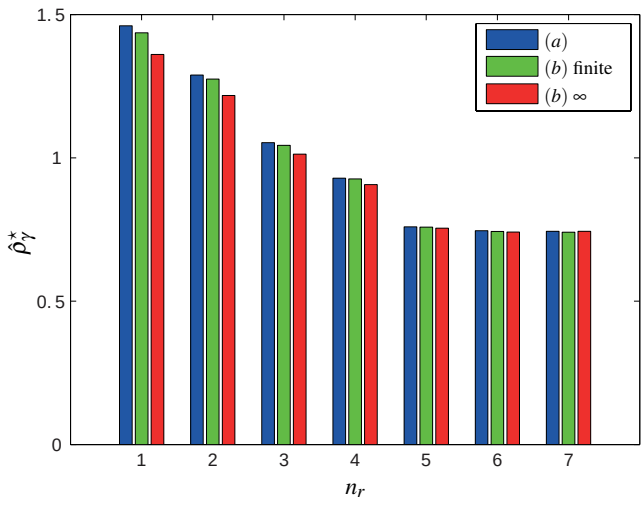

Figure 4: Performance of different reduced models as a function of the order n_r and of the adopted reset map.

6.5 Discussion

Two facts can be noticed by analyzing the results presented in Figure 4. First of all, the reset map affects the value of the directional Hausdorff distance, and the novel reset maps exhibit a better performance for any order n_r chosen for the reduction.

Furthermore, the outcome of our analysis through the randomized approach is quite different from that based on the HSV only (see Figure 2). In fact, reducing the system to a first-order approximation results in quite bad performance when the goal of the approximation is the analysis of reachability properties for which the directional Hausdorff distance is a suitable accuracy measure. In addition, such a drastic reduction yields discontinuities in the state reset that may possibly produce chattering behaviors. On the other hand, from the randomized based analysis it appears that one can push the reduction up to a fifth order without degrading significantly the accuracy of the model.

7. CONCLUSIONS

In this work, we presented a novel approach to model reduction of switched affine systems using balanced truncation for reducing the continuous affine dynamics. The main novel ingredients of the approach are:

- the introduction of suitable state reset maps that serve the purpose of making the reduced model best reproduce the free evolution of the original system; and
- the integration in the reduced order model design of a randomized procedure for model order selection.

The considered class of switched systems is characterized by an endogenous switching signal, in that the transitions between modes are determined by the evolution of the continuous state component. The method can be applied also to the case when transitions are determined by some exogenous switching signal, possibly probabilistic as in the case of Markov jump linear systems, [28]. In the case when the switching signal is subject to some dwell time τ_D and the approximated dynamics has a settling time smaller than τ_D, then, the approximation error introduced by the state reset will be negligible.

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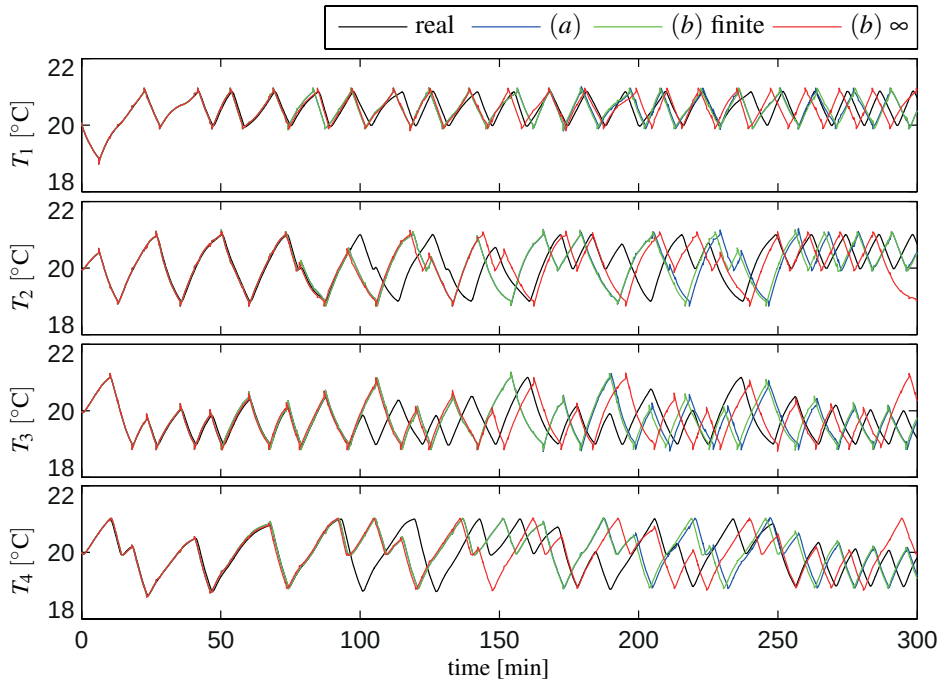

Figure 3: Comparison of the temperatures evolution obtained with the original model and with the reduced ones implementing the considered reset maps.

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