Lund University

Department of
AUTOMATIC CONTROL

Network Dynamics (FRTN30)

Exam: June 2, 2018, 8 – 13

Points and grading

All answers must include a clear motivation. The total number of points is 21. The maximum number of points for each problem is specified.

Preliminary grading system: A = total points in the exam, B = total points from the four hand-in assignments

- $A + B < 12 \implies$ grade U
- $12 \leq A + B < 17 \implies$ grade 3
- $17 \leq A + B < 21 \implies$ grade 4
- $A + B \geq 21 \implies$ grade 5

Accepted aid

Only lecture notes and a pocket calculator are allowed.

Results

The results are reported via LADOK by the 20th of June. To view your exam, please contact the exam responsible Christian Rosdahl (christian.rosdahl@control.lth.se).

Hint: Many subproblems can be solved independently of each other: If you get stuck on a subproblem, it might help to move on and go back to it later.

Good luck!

1. Consider the two unweighted graphs $\mathcal{G}_A$ and $\mathcal{G}_B$ in Figure 1.

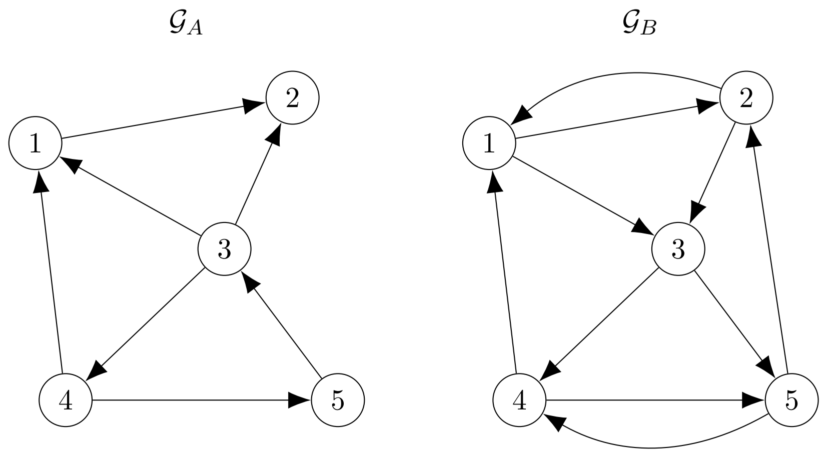


Figure 1 The graphs for Problem 1.

a. Determine the number of connected components and draw the condensation graph of both $\mathcal{G}_A$ and $\mathcal{G}_B$. (1 p)

b. Determine the invariant probability vectors π^A and π^B of $\mathcal{G}_A$ and $\mathcal{G}_B$, respectively. (1 p)

Consider now a standard discrete-time random walk on $\mathcal{G}_B$.

c. Determine the expected return time in node 1 and the one in node 5. (1 p)

d. Determine the conditional expected hitting time on the set $\mathcal{S} = \{1, 5\}$ given that the walk starts in node 2. (1 p)

e. Determine the conditional probability of hitting node 1 before node 5, given that the walk starts in node 2. (1 p)

f. Consider the standard DeGroot opinion dynamics $x(t+1) = Px(t)$ on $\mathcal{G}_B$. Determine the asymptotic opinion vector $\lim_{t\to+\infty} x(t)$ when the initial opinion vector is $x(0) = (1, 3, 2, 4, 0)$. (1 p)

g. Determine the asymptotic opinion vector for the DeGroot opinion dynamics with stubborn agents on $\mathcal{G}_B$, when the stubborn agents set is $\mathcal{S} = \{2, 4\}$ with opinions $x_2 = -2$ and $x_4 = +1$. (1 p)

Solution

a. Graph $\mathcal{G}_A$ has three connected components. Graph $\mathcal{G}_B$ has one connected component. The condensation graphs are shown in Figure 2.

b. For graph $\mathcal{G}_A$, we know that π^A will only have support on the sink-nodes in the condensation graph. Hence $\pi^A = [0 \quad 1 \quad 0 \quad 0 \quad 0]'$. Graph $\mathcal{G}_B$ is balanced, so the π^B vector will be proportional to out-degree vector w. Since all nodes have out-degree 2, $\pi^B = [1/5 \quad 1/5 \quad 1/5 \quad 1/5 \quad 1/5]'$.

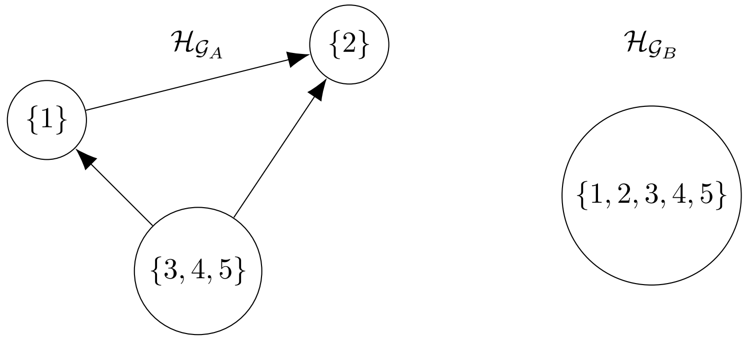


Figure 2 The condensations graphs for Problem 1.

c. The expected return time is given by

$$\mathbb{E}_1[T_1^+] = \frac{1}{\pi_1^B} = 5\,, \quad \mathbb{E}_5[T_5^+] = \frac{1}{\pi_5^B} = 5\,.$$

d.

$$\mathbb{E}_i[T_\mathcal{S}] = 0\,, \quad i \in \{1, 5\}$$

$$\begin{aligned}\mathbb{E}_2[T_\mathcal{S}] &= 1 + \frac{1}{2}\mathbb{E}_3[T_\mathcal{S}]\\ \mathbb{E}_3[T_\mathcal{S}] &= 1 + \frac{1}{2}\mathbb{E}_4[T_\mathcal{S}]\\ \mathbb{E}_4[T_\mathcal{S}] &= 1\end{aligned}$$

Solving the equation system yields $\mathbb{E}_3[T_\mathcal{S}] = \frac{3}{2}$ and $\mathbb{E}_2[T_\mathcal{S}] = \frac{7}{4}$.

e.

$$\Gamma_{1,5} = 1 \quad \Gamma_{5,5} = 0$$

$$\begin{aligned}\Gamma_{2,5} &= \frac{1}{2}\Gamma_{1,5} + \frac{1}{2}\Gamma_{3,5}\\ \Gamma_{3,5} &= \frac{1}{2}\Gamma_{4,5} + \frac{1}{2}\Gamma_{5,5}\\ \Gamma_{4,5} &= \frac{1}{2}\Gamma_{1,5} + \frac{1}{2}\Gamma_{5,5}\end{aligned}$$

Solving the system of equations yields $\Gamma_{4,5} = \frac{1}{2}$, $\Gamma_{3,5} = \frac{1}{4}$ and $\Gamma_{2,5} = \frac{5}{8}$.

f. Since the graph is aperiodic, the asymptotic opinion vector will converge to $x = \alpha\mathbb{1}$, where

$$\alpha = \sum_i \pi_i^B x_i(0) = \frac{1}{5}(1 + 3 + 2 + 4) = 2\,.$$

g. We have

$$\begin{aligned}x_5 &= \frac{1}{2}x_2 + \frac{1}{2}x_4 = -\frac{1}{2},\\ x_3 &= \frac{1}{2}x_5 + \frac{1}{2}x_4 = \frac{1}{4},\\ x_1 &= \frac{1}{2}x_2 + \frac{1}{2}x_3 = -\frac{7}{8}.\end{aligned}$$

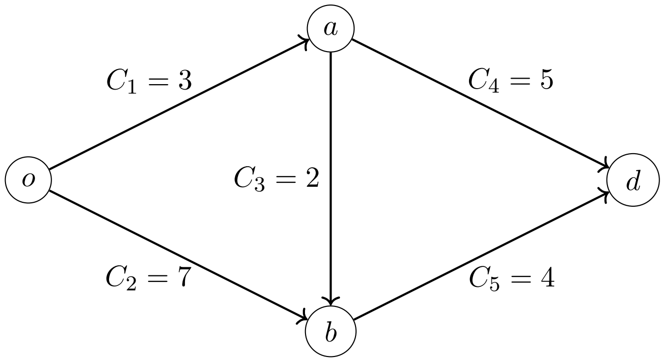


Figure 3 Link capacities for the traffic flows in Problem 2.

2. Consider a traffic network with link capacities

$$C_1 = 3, \quad C_2 = 7, \quad C_3 = 2, \quad C_4 = 5, \quad C_5 = 4$$

as shown in Figure 3.

a. What is the largest throughput of a feasible o-d flow? (1 p)

b. How should a total of 5 units of additional capacity be allocated to the links in order to maximize the largest throughput of a feasible o-d flow? (1 p)

c. Now assume that the following delay functions are associated to the links

$$d_1(x) = 2, \quad d_2(x) = 2 + x, \quad d_3(x) = 1, \quad d_4(x) = 4x, \quad d_5(x) = 1.$$

State the social optimum traffic assignment problem for unitary o-d flows (i.e., write down the cost function to be minimized and the constraints) and solve it by finding the socially optimal flow vector f^*. (1 p)

d. Is any of the the following flow vectors

$$f^1 = [\,\tfrac{2}{5} \quad \tfrac{3}{5} \quad 0 \quad \tfrac{2}{5} \quad \tfrac{3}{5}\,]', \quad f^2 = [\,\tfrac{3}{5} \quad \tfrac{2}{5} \quad 0 \quad \tfrac{3}{5} \quad \tfrac{2}{5}\,]',$$

a Wardrop equilibrium? Motivate your answer. (1 p)

e. Determine the price of anarchy. (.5 p)

f. Determine a vector of tolls, ω^*, such that when added to the delays, the Wardrop equilibrium coincides with the social optimum. (.5 p)

Solution

a. The four o-d cuts $\mathcal{U}_1 = \{o\}$, $\mathcal{U}_2 = \{o, a\}$, $\mathcal{U}_3 = \{o, b\}$ and $\mathcal{U}_4 = \{o, a, b\}$ have capacities

$$C_{\mathcal{U}_1} = 10, \quad C_{\mathcal{U}_2} = 14, \quad C_{\mathcal{U}_3} = 7, \quad \text{and} \quad C_{\mathcal{U}_4} = 9.$$

According to the max-flow min-cut theorem, the maximum throughput from node o to node d is the min-cut capacity, i.e., the minimum of the capacities above. The maximum flow that can be sent from o to d is thus $\tau^*_{od} = C_{\mathcal{U}_3} = 7$.

b. By adding 3 units of capacity to C_5, the min-cut capacity is increased to 10, since then $C_{\mathcal{U}_3} = 10$ and $C_{\mathcal{U}_4} = 12$. The new min-cut capacities are then $C_{\mathcal{U}_1} = 10$ and $C_{\mathcal{U}_3} = 10$. By adding two units of capacity to C_1, both these cut capacities are increased to $C_{\mathcal{U}_1} = C_{\mathcal{U}_3} = 12$, while still not exceeding any of the other cut capacities. Thus, we get

$$C_{\mathcal{U}_1} = C_{\mathcal{U}_3} = C_{\mathcal{U}_4} = 12, \quad C_{\mathcal{U}_2} = 14,$$

resulting in min-cut capacity 12. The answer is thus that three capacity units should be added to C_5 and that two capacity units should be added to C_1, so that the new capacities become

$$C_1 = 5, \quad C_2 = 7, \quad C_3 = 2, \quad C_4 = 5, \quad C_5 = 7.$$

c. The social optimum traffic assignment problem reads

$$\begin{array}{ll} \text{minimize}_f & \sum_e c_e(f_e) = \sum_e f_e d_e(f_e) = 2f_1 + (2+f_2)f_2 + f_3 + 4f_4^2 + f_5 \\ \text{subject to} & f \geq 0 \text{ and } Bf = \lambda - \mu, \end{array}$$

where B is the node-link incidence matrix, $f = (f_1, f_2, f_3, f_4, f_5)'$ is the vector of link flows, $\lambda = (1, 0, 0, 0, 0)'$ is the exogenous inflow vector and $\mu = (0, 0, 0, 0, 1)'$ is the exogenous outflow vector. This can be solved analytically by substituting the link flow variables with the path flow variables, according to

$$f_1 = z_1 + z_2, \quad f_2 = z_3, \quad f_3 = z_2, \quad f_4 = z_1, \quad f_5 = z_2 + z_3.$$

The cost function then becomes

$$c = 2(z_1 + z_2) + (2 + z_3)z_3 + z_2 + 4z_1^2 + z_2 + z_3.$$

Furthermore, we know that the sum of the flows along all paths must be one, since we have assumed a unit throughput. Therefore, the condition $z_1+z_2+z_3 = 1$ can be used to eliminate the variable z_3, which results in the cost

$$\begin{aligned} c &= 2(z_1 + z_2) + (2 + 1 - z_1 - z_2)(1 - z_1 - z_2) + z_2 + 4z_1^2 + z_2 + (1 - z_1 - z_2) \\ &= 2 + (1 - z_1 - z_2)^2 + z_2 + 4z_1^2 + 1 - z_1. \end{aligned}$$

Requiring the gradient of $c = c(z_1, z_2)$ to be zero gives

$$\frac{\partial c}{\partial z_1} = 2(z_1 + z_2 - 1) + 8z_1 - 1 = 0, \quad \frac{\partial c}{\partial z_2} = 2(z_1 + z_2 - 1) + 1 = 0 \quad \Rightarrow$$

$$\begin{cases} 10z_1 + 2z_2 = 3 \\ 2z_1 + 2z_2 = 1 \end{cases} \Leftrightarrow \begin{cases} z_1 = 1/4 \\ z_2 = 1/4 \end{cases}.$$

Noting that $z_3 = 1 - z_1 - z_2 = 1/2$, the path flows can be substituted back in order to yield the link flows

$$f_1 = \frac{1}{2}, \quad f_2 = \frac{1}{2}, \quad f_3 = \frac{1}{4}, \quad f_4 = \frac{1}{4}, \quad f_5 = \frac{3}{4}.$$

d. According to the definition, a flow is a Wardrop equilibrium if the flow z_i along path z_i is non-zero only if the delay along this path is smaller than or equal to the delay along all other paths. For the flow vector f^2, we have delays along the paths $p_1 = (o, a, d)$, $p_2 = (o, a, b, d)$ and $p_3 = (o, b, d)$ given by

$$\begin{aligned}
\text{delay}(p_1) &= d_1(f_1) + d_4(f_4) = 2 + 4 \cdot \frac{3}{5} = \frac{22}{5}, \\
\text{delay}(p_2) &= d_1(f_1) + d_3(f_3) + d_5(f_5) = 2 + 1 + 1 = 4, \\
\text{delay}(p_3) &= d_2(f_2) + d_5(f_5) = 2 + \frac{2}{5} + 1 = \frac{17}{5}.
\end{aligned}$$

We can, for example, see that the delay on path p_1 is larger than the delay on path p_2. For the flow vector to be a Wardrop equilibrium, this should imply that $z_1 = f_4 = 0$, which is not the case. Thus, f^2 cannot be a Wardrop equilibrium. On the other hand, for the flow vector f^1, we have delays along the paths given by

$$\begin{aligned}
\text{delay}(p_1) &= d_1(f_1) + d_4(f_4) = 2 + 4 \cdot \frac{2}{5} = \frac{18}{5}, \\
\text{delay}(p_2) &= d_1(f_1) + d_3(f_3) + d_5(f_5) = 2 + 1 + 1 = 4, \\
\text{delay}(p_3) &= d_2(f_2) + d_5(f_5) = 2 + \frac{3}{5} + 1 = \frac{18}{5}.
\end{aligned}$$

Here we see that the only paths which have non-zero flows are the ones with shortest delay, i.e., p_1 and p_3. On the path p_2, we have the path flow $z_2 = f_3 = 0$. Thus, f^1 is a Wardrop equilibrium.

e. The price of anarchy is the ratio between the average delay at the Wardrop equilbrium and the average delay at the social optimum. In this case we have

$$\text{PoA} = \frac{18/5}{7/2} = \frac{36}{35}\,.$$

f. The Wardrop equilibrium will coincide with the social optimum if we to the delay functions $d_e(x)$ add tolls ω_e^* that are given by

$$\omega_e^* = c_e'(f_e^*) - d_e(f_e^*) = f_e^* d_e'(f_e^*),$$

where f^* is the flow vector that solves the social optimum traffic assignment problem. Computing this gives

$$\omega_1^* = 0, \quad \omega_2^* = f_2^* = \frac{1}{2}, \quad \omega_3^* = 0, \quad \omega_4^* = 4f_4^* = 1, \quad \omega_5^* = 0.$$

3. Consider a strategic form game with player set $\{1, 2, 3\}$, action space $\mathcal{A} = \{-1, +1\}$, and utilities

$$u_1(x) = x_1 x_2 \,, \qquad u_2(x) = x_2(x_1 + x_3) \,, \qquad u_3(x) = x_3 x_2 \,.$$

a. Determine the best response functions $\mathcal{B}_1(x_2, x_3)$, $\mathcal{B}_2(x_1, x_3)$, and $\mathcal{B}_3(x_1, x_2)$. (.5 p)

b. Determine the set of Nash equilibria. (.5 p)

c. Draw the graph whose node set is the configuration space of the game and whose links correspond to the possible transitions of the asynchronous best response dynamics labeled by the corresponding rates. (.5 p)

d. Determine the conditional probability that the asynchronous best response dynamics $X(t)$ gets absorbed in configuration $(1, 1, 1)$ given that it starts from configuration $(-1, 1, -1)$. (.5 p)

e. Is this a potential game? Motivate your answer. (.5 p)

f. For the noisy best response dynamics with noise parameter $1/\eta = 1$, determine the stationary probability $\pi_{(1,1,1)}$ to be in configuration $(1, 1, 1)$. (.5 p)

Now consider a game with the same player set and action space and utilities

$$u_1(x) = x_1 x_2 \,, \qquad u_2(x) = -x_2(x_1 + x_3) \,, \qquad u_3(x) = x_3 x_2 \,.$$

g. Determine the best response functions $\mathcal{B}_1(x_2, x_3)$, $\mathcal{B}_2(x_1, x_3)$, and $\mathcal{B}_3(x_1, x_2)$. (.5 p)

h. Determine the set of Nash equilibria. (.5 p)

i. Is this a potential game? Motivate your answer. (.5 p)

j. For the asynchronous best response dynamics, determine the stationary probability $\pi_{(1,1,1)}$ to be in configuration $(1, 1, 1)$. (.5 p)

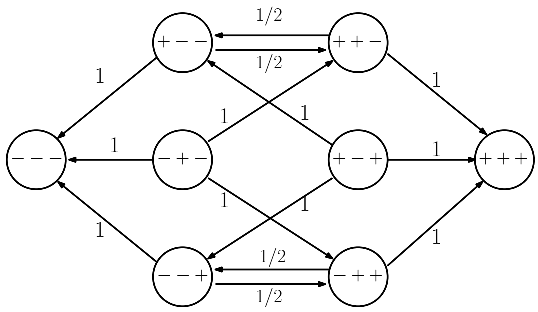
Solution

a.

$$\mathcal{B}_1(x_2, x_3) = \mathcal{B}_3(x_1, x_2) = x_2 \,, \qquad \mathcal{B}_2(x_1, x_3) = \begin{cases} -1 & \text{if } \; x_1 = x_3 = -1 \\ \{-1, +1\} & \text{if } \; x_1 = -x_3 \\ +1 & \text{if } \; x_1 = x_3 = +1 \,. \end{cases}$$

b. At a Nash equilibrium, players 1 and 3 both copy the action of player 2. In turn player 2 plays the action that both player 1 and 3 play. Hence, there are two Nash equilibria, the configurations $(-1, -1, -1)$ and $(+1, +1, +1)$.

c.



d. Because of symmetries, the absorbing probabilities satisfy

$$\alpha = \Gamma^{(+,+,+)}_{(-,+,-)} = 1-\Gamma^{(+,+,+)}_{(+,-,+)} \qquad \beta = \Gamma^{(+,+,+)}_{(+,-,-)} = \Gamma^{(+,+,+)}_{(-,-,+)} = 1-\Gamma^{(+,+,+)}_{(+,+,-)} = 1-\Gamma^{(+,+,+)}_{(-,+,+)}\,.$$

Conditioning on the first step when starting from configurations $(+,-,-)$ and $(-,+,-)$, respectively, we get

$$\beta = \frac{1}{3}(1-\beta)\,, \qquad \alpha = \frac{2}{3}(1-\beta)\,.$$

Then, $\beta = 1/4$ and $\alpha = 1/2$. Hence the probability that, when starting from configuration $(-,+,-)$, the system gets absorbed in $(+,+,+)$ is $\Gamma^{(+,+,+)}_{(-,+,-)} = \alpha = 1/2$.

e. This is a network coordination game on a simple line graph with 3 nodes, hence it is a potential game with potential function

$$\Phi(x_1, x_2, x_3) = |x_1 + x_2| + |x_2 + x_3|$$

coinciding with twice the number of undirected links connecting nodes with the same action. Another potential function is given by $\Phi(x_1, x_2, x_3) = x_2(x_1+x_3)$.

f.

$$\pi_{(1,1,1)} = \frac{e^{\Phi(1,1,1)}}{\sum_x e^{\Phi(x)}} = \frac{e^4}{2e^4 + 4e^2 + 2} = \frac{1}{2(1+e^{-2})^2}$$

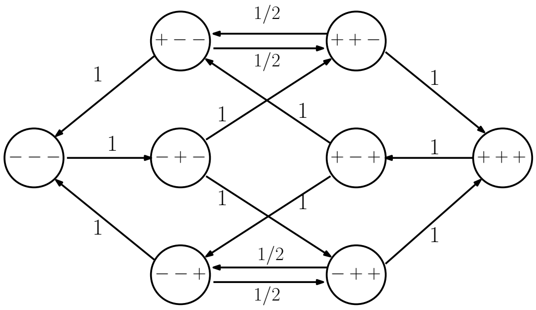
g.

$$\mathcal{B}_1(x_2, x_3) = \mathcal{B}_3(x_1, x_2) = x_2\,, \qquad \mathcal{B}_2(x_1, x_3) = \begin{cases} -1 & \text{if} \quad x_1 = x_3 = +1 \\ \{-1,+1\} & \text{if} \quad x_1 = -x_3 \\ +1 & \text{if} \quad x_1 = x_3 = -1\,. \end{cases}$$

h. At a Nash equilibrium, players 1 and 3 both should copy the action of player 2, while player 2 should play the opposite of the action that both player 1 and 3 play. Hence, there are no Nash equilibria.

i. The game is not a potential one because it admits no Nash equilibria.

j.



Because of symmetries, the stationary probabilities satisfy

$$a = \overline{\pi}_{(+,+,+)} = \overline{\pi}_{(-,-,-)}\,, \qquad b = \overline{\pi}_{(+,-,+)} = \overline{\pi}_{(-,+,-)}\,,$$

$$c = \overline{\pi}_{(-,+,+)} = \overline{\pi}_{(+,+,-)} = \overline{\pi}_{(-,-,+)} = \overline{\pi}_{(+,-,-)}\,.$$

Observe that the maximum degree is $\omega^* = 2$. Then, mass conservation for the Q chain in nodes $(-,-,-)$, $(-,+,-)$, and $(+,-,-)$, respectively, give us

$$a = \frac{1}{2}a + \frac{1}{2}c + \frac{1}{2}c\,, \qquad b = \frac{1}{2}a\,, \qquad c = \frac{1}{4}c + \frac{1}{4}c + \frac{1}{2}b\,,$$

i.e., $c = b = \frac{1}{2}a$ and from $4c + 2a + 2b = 1$ we get $a = 1/5$ and $b = c = 1/10$. Hence,

$$\overline{\pi}_{(+,+,+)} = \frac{1}{5}\,.$$

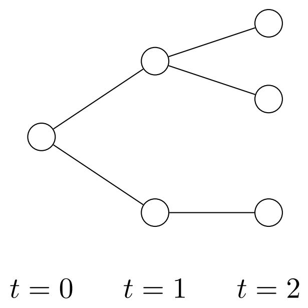


Figure 4 The first two generations of the Galton-Watson process for Problem 4.

4. Consider a Galton-Watson branching process with offspring distribution

$$p_0 = 0.2, \qquad p_1 = 0.3, \qquad p_2 = 0.5, \qquad p_k = 0\,, \quad k \geq 3\,.$$

a. Determine the extinction probability. (1 p)

Now, assume that, up to the second generation, the graph generated by such branching process looks like the one in Figure 4. Conditioned on this event,

b. what is the conditional probability of survival for the branching process? (.5 p)

c. what is the conditional probability that the diameter of the graph generated thereafter by the branching process will be at least 5? (.5 p)

Solution

a. The probability that the branching process extinguishes is given by

$$p_0 + p_1\theta + p_2\theta^2 = \theta\,,$$

which can be rewritten as

$$0.2 - 0.7\theta + 0.5\theta^2 = 0$$

or

$$0.4 - 1.4\theta + \theta^2 = 0\,.$$

This equation has two roots, $\theta_1 = 0.4$ and $\theta_2 = 1$, hence the probably that each branch will extinguish is 0.4.

b. The conditional probability that all three branches will extinguish is $0.4^3 = 0.064$ and hence the conditional probability of survival is $1 - 0.064 = 0.936$.

c. If the graph should get a diameter of at least 5, at least one of the leaves must get a child. The probably that none of the leaves gets a child is p_0^3, so the probability that the diameter of the graph will be at least 5 is $1 - p_0^3 = 0.992$.

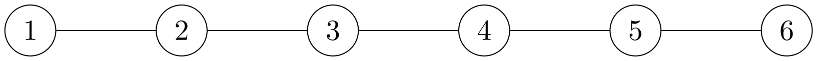


Figure 5 The graph for Problem 5.

5. Consider the SIR epidemics on a simple line graph with six nodes as in Figure 5. The state $X_i(t)$ of each node belongs to the set $\{0, 1, 2\}$, where 0 stands for susceptible, 1 stands for infected, and 2 for recovered. Let the interaction frequency parameter be $\beta = 1$, the recovery rate be unitary, and let the initial configuration be $X(0) = (1, 0, 0, 0, 0, 0)$, i.e., node 1 is infected and all other nodes are susceptible.

a. What are the absorbing configurations that are reachable from the given $X(0)$? (.5 p)

b. What is the probability that node 2 gets infected before node 1 recovers? (.5 p)

c. What is the probability that all six nodes eventually are recovered? (.5 p)

d. What is the probability that the final number of recovered nodes is 4? (.5 p)

Solution

a. Since all infected nodes eventually will get recovered, the transition $1 \to 2$ will eventually happen given that the node is in state 1. Thus, an absorbing configuration cannot contain any node in state 1. Moreover, if none of the nodes are in state 1 (infected), none of the nodes will change its state further. Therefore, all nodes must be in state 0 or 2 in an absorbing configuration. The first node is initially infected, so it must be recovered (in state 2) in the absorbing configuration. Furthermore, for a node to be recovered, it must have been infected by a neighbor, which means that at least one neighbor must be recovered (in state 2) for each recovered node in an absorbing state. All possible absorbing states are thereby $X = (2, 0, 0, 0, 0, 0)$, $X = (2, 2, 0, 0, 0, 0)$, $X = (2, 2, 2, 0, 0, 0)$, $X = (2, 2, 2, 2, 0, 0)$, $X = (2, 2, 2, 2, 2, 0)$ and $X = (2, 2, 2, 2, 2, 2)$.

b. Node 1 will get recovered when a rate-1 Poisson clock ticks. Node 2 will get infected if another independent rate-1 Poisson clock ticks before node 1 has recovered. Thus, we have two independent rate-1 Poisson clocks and if the first one ticks before the second one, node 2 gets infected, while if the second one ticks before the first one, node 2 does not get infected. Since the Poisson clocks are equivalent and independent, this implies that the probability for each of the two outcomes must be $1/2$.

c. That a node eventually is recovered is equivalent with that it at some point gets infected. We know from the previous subproblem that node 2 gets infected with probability $1/2$. Given that this has happened, repeating the same reasoning as in the previous subproblem (but with nodes 3 and 2 instead of 2 and 1), gives the conditional probability $1/2$ that node 3 gets infected. The probability that nodes 2 and 3 get infected (and thus eventually recovered) is thus $(1/2)^2$. Repeating the same reasoning for the following nodes gives that all six nodes eventually get infected and thus eventually get recovered with probability $(1/2)^5$.

d. For the final number of recovered nodes to be 4, it must hold that the first four nodes get infected and that node 4 recovers before node 5 has been infected. The probability that node 4 gets infected is $(1/2)^3$. Given that this has happened, the conditional probability that node 5 does not get infected before node 4 has recovered is $1/2$. Thus, the total probability for these events to happen is $(1/2)^4$.