Market Driven Systems (FRTN20)

Exercise 8 - Solutions

Game Theory

Last updated: 2016

1. Let x be the number of cars going from start to A and y the number of cars going directly from A to end. Consequently, $x-y$ cars use the new road. Total travel time is

$$J(x,y) = \frac{x^2}{100} + 45(4000-x) + 45y + \frac{(4000-y)^2}{100}.$$

Minimizing with respect to x and y gives $x^* = 2250$ and $y^* = 1750$. The new travel time per car is hence

$$\frac{1}{4000} J(x^*, y^*) \approx 64.7,$$

i.e. a gain by 18 s, compared with the situation before the new road and by 15.3 minutes compared to the situation with the road, but without centralized planning.

Comment: Improving a road, or building a new road, can never increase total travel time if centralized planning is used. With decentralized planning it surpisingly can.

2. If we use the fact that

$$A_{xy} \geq \min_x A_{xy}, \ \forall x,y$$

and maximize both sides with respect to y we get

$$\max_y A_{xy} \geq \max_y \min_x A_{xy}, \ \forall x.$$

Since this holds for all x, it also holds for the x minimizing the left hand side, so

$$\min_x \max_y A_{xy} \geq \max_y \min_x A_{xy}.$$

3. Let $p := P(X = \text{black})$. The game from Y's perspective, knowing X's strategy becomes

Y: red	Y: black
-9(1-p)+5p = 14p-9	5(1-p)-p = -6p+5

This means that Y can choose between the two lines in the figure, and will choose the upper one. The optimal choice for the minimizer X is hence to use the value $p^* = 0.7$, since this minimizes the value of the upper line. This will give an outcome of 0.8, and the game hence favors the column player "I".

Since we know $\min\max = \max\min$ the outcome will be the same analysed from Xs perspective, knowing Y strategy. But to verify this let $q = P(Y = \text{black})$. This reduces the game to

X	
red	-9(1-q)+5q = 14q-9
black	5(1-q)-q = -6q+5

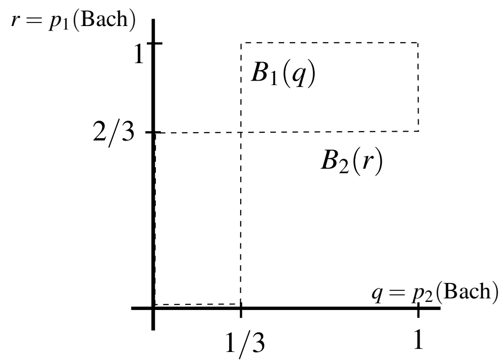
with optimal solution at $14q^* - 9 = -6q^* + 5 \Rightarrow q^* = 0.7$, also giving outcome 0.8.

4. The maximization over y is equivalent to selecting the maximal element in the vector $x^T A$. The second row $x^T A \leq \alpha 1^T$ upper bounds this maximal element by α. The bound α is minimized over x (first row). The last two lines constrain x to be a valid probability vector (positive elements, which sum up to 1).

5. Denote the strategies $r := p_1(\text{Bach})$ and $q = p_2(\text{Bach})$.

For $p_2(\text{Bach}) < 1/3$, player 1 prefers Stravinsky: $r = 0$.
For $p_2(\text{Bach}) = 1/3$, player 1 is indifferent: $r = [0,1]$.
For $p_2(\text{Bach}) > 1/3$, player 1 prefers Bach: $r = 1$

The best response functions $r^* = B_1(q)$ and $q^* = B_2(r)$ are shown in the figure. The intersection(s) are the Nash equilibrium.



6.

a. Bach and Stravinsky (cf. lecture notes).

		Player 2	
		Bach	Stravinsky
Player 1	Bach	2,1	0,0
	Stravinsky	0,0	1,2

b. Rock-paper-scissor (cf. the lecture notes).

		Player 2		
		rock	paper	scissor
	rock	0	1	-1
Player 1	paper	-1	0	1
	scissor	1	-1	0

c. Consider the game below.

		Player 2	
		l	r
Player 1	u	0,0	3,-1
	d	0,0	1,1

Assume first that 1 starts. Choosing u will result in $(0,0)$, while choosing d will result in $(1,1)$ (given that 2 is rational). Hence, if 1 starts, the outcome will be $(1,1)$. If, instead, 2 starts, the optimal strategies will be (l,u) or (l,d), both resulting in the outcome $(0,0)$.

Consequently, both 1 and 2 benefit from 1 being the leader.

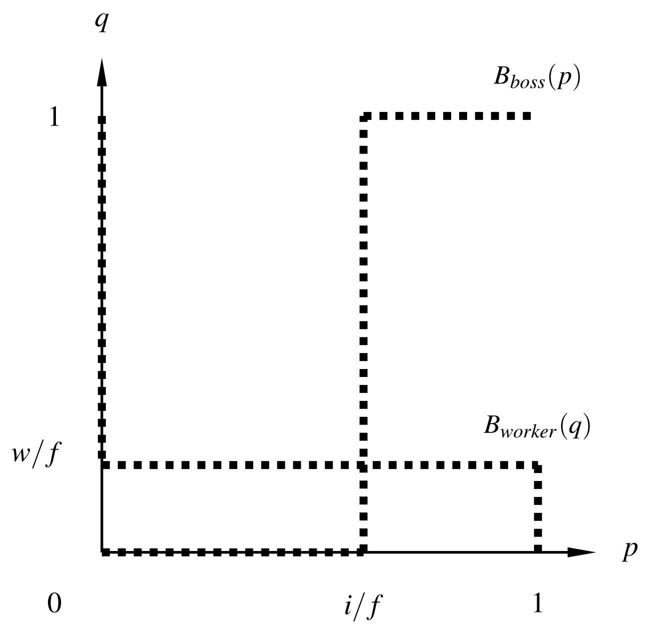
7. Denote by $p = P(\text{worker shirking})$ and $q = P(\text{boss inspecting})$. For the worker, the game then reduces to

worker	work	-w
	shirk	-fq

Similarly, for the boss the game reduces to

boss	
not inspect	inspect
g(1-p)	g(1-p)+fp-i

The corresponding best response curves are shown in the figure below.



The Nash equilibrium occurs in the intersection

$$(p^*, q^*) = \left(\frac{i}{f}, \frac{w}{f}\right).$$

For the game instance $w = 1, f = 5, i = 3, g = 1$ this corresponds to the worker working $3/5$ of the time and the boss inspecting $1/5$ of the time. The expected outcomes are then

$$U_{\text{worker}} = -w = -1$$
$$U_{\text{boss}} = g(1 - p^*) = \frac{2}{5}.$$

8.

a. The expected outcome for the boss is now always decreased by inspecting, in fact

$$U_{\text{boss}}(p,q) = g(1-p)(1-q) + (g-i)(1-p)q - ipq = -iq - gp + g$$

which is maximized by $q^* = 0$. The situation for the worker is unchanged, and from the best response curve we see that his best response will be $p^* = 1$, i.e. shirking.

b. The boss declares his strategy, defined by q, whereupon the worker chooses p. The Stackelberg strategy of the boss is then

$$q^* = \arg\max_q U_{\text{boss}}(q, B_{\text{worker}}(q)) = \arg\max_q (g - gB_{\text{worker}}(q) - iq).$$

From the figure we can see that the best response from the worker is

$$p^* = B_{\text{worker}}(q) = \begin{cases} 1 & , q < \frac{w}{f} \\ [0,1] & , q = \frac{w}{f} \\ 0 & , q > \frac{w}{f} \end{cases}$$

The best response function for $q = w/f$ is the entire interval, which complicates things somewhat. However by using any $q > w/f$ one will achieve the response $p^* = 0$. This means that the any inspection rate above w/f will induce work being the optimal response.

The outcome for the boss will then be $g - iq$ which can be made arbitrarly close to

$$g - iq^* = g - iw/f > 0$$

Due to the fact that the boss needs to declare his inspection rate in advance, an equilibrium has arisen in (work, inspect w/f). Note that this situation is more advantageous for the boss than the outcome 0.

9.

a. The probability density function of each random variable $X_i \in X = \{X_1, \ldots, X_n\}$ is $f_{X_i}(x) = 1$ and their distribution functions are given by

$$F_{X_i}(x) = \text{Prob}(X_i \leq x) = \int_0^x f_{X_i}(t)dt = x.$$

Let Y_k be the k$^{\text{th}}$ largest element in X. I.e., Y_1 is the largest, and so on. The distribution function for Y_1 is easily calculated using the observeration that $Y_1 < x$ means that all n variables are below x. The probability for this is

$$F_{Y_1}(x) = \prod_{i=1}^{n} F_{X_i}(x) = x^n,$$

and the density function for the Y_1 can then be calculated from $f_{Y_1} = \frac{dF_{Y_1}}{dx} = nx^{n-1}$.
The distribution function for Y_k is

$$F_{Y_k}(x) = \text{Prob}(Y_k \leq x) = P(\text{at most k-1 values are above } x)$$

Splitting the event "at most k values above x" into the disjoint events "at most k-1 values above x " and "exactly k values above x" we see that

$$F_{Y_{k+1}}(x) = F_{Y_k}(x) + \binom{n}{k} x^{n-k}(1-x)^k.$$

This recursion gives all distribution functions and density functions, for example

$$f_{Y_2} = \frac{dF_{Y_2}}{dx} = \frac{d}{dx}(x^n + nx^{n-1}(1-x)) = n(n-1)x^{n-2}(1-x)$$

b. We seek

$$m_k = E(Y_K) = \int_0^1 x f_{Y_k}(x) dx = \int_0^1 x \frac{d}{dx} F_{Y_k}(x) dx$$

where the last equaility is a consequence of the fundamental theorem of calculus. Combining this with the recursive expression for Y_{k+1} yields

$$m_{k+1} = m_k + \binom{n}{k} \int_0^1 x \frac{d}{dx} \left(x^{n-k}(1-x)^k \right) dx$$

where (by partial integration)

$$\int_0^1 x \frac{d}{dx} \left(x^{n-k}(1-x)^k \right) dx = -\frac{k!(n-k)!}{(n+1)!}.$$

Hence

$$m_{k+1} = m_k - \frac{n!}{k!(n-k)!} \cdot \frac{k!(n-k)!}{(n+1)!} = m_k - \frac{1}{n+1}.$$

It is straight forward to compute

$$m_1 = \int_0^1 x \frac{d}{dx} x^n dx = n \int_0^1 x^n dx = \frac{n}{n+1} [x^{n+1}]_0^1 = \frac{n}{n+1}$$

which yields the general result

$$m_k = 1 - \frac{k}{n+1}.$$

10. To confirm that it is a Nash equilibrium, we assume everyone is bidding $(n-1)/n$ times their valuation and show that it is not beneficial for a single bidder to deviate from this strategy.

The revenue of a bidder with valuation V and bid X is

$$U(X) = (V - X) \cdot P(X > \text{all other bids}) = (V - X) \cdot P\left(\frac{n}{n-1} X > \text{all other valuations} \right)$$
$$= (V - X) \left(\frac{n}{n-1} X \right)^{n-1}, \quad \text{if } X < (n-1)/n, \quad \text{else } (V - X)$$

The function is maximized when

$$0 = \frac{dU(X^*)}{dX} \Rightarrow X^* = \frac{n-1}{n} V$$

which shows that the the claim in the problem text is true.

From the definition it is straight forward to find the probability of winning the auction

$$P_i(V_i) = V_i^{n-1}.$$

A result from the lecture gives the expected payoff S_i through

$$\frac{dS_i}{dV_i} = V_i^{n-1} \Rightarrow S_i(V_i) = \frac{V_i^n}{n}.$$