

## Lecture 6

# Linear Programming and Model Predictive Control

Linear programming and convex optimization are widely used tools in control engineering and production planning. In lecture 6 and 7 we will learn some of the basic ideas, mainly through the study of examples. First we will recall how linear programming is used for production planning in a static situation, then we will apply the same idea in presence of dynamics. Finally, Model Predictive Control is introduced as a method to control the dynamic effects.

A linear programming (LP) problem is an optimization problem of the following form:

$$\begin{aligned} &\text{Minimize} && c^T x \\ &\text{subject to} && Ax \leq b \\ &&& Fx = g \end{aligned}$$

Here  $b, c, g$  and  $x$  are vectors, while  $A$  and  $F$  are matrices. The product  $c^T x$  is a scalar number and the inequality  $Ax \leq b$  means that every entry of the vector  $b$  is at least as big as the corresponding entry in  $Ax$ .

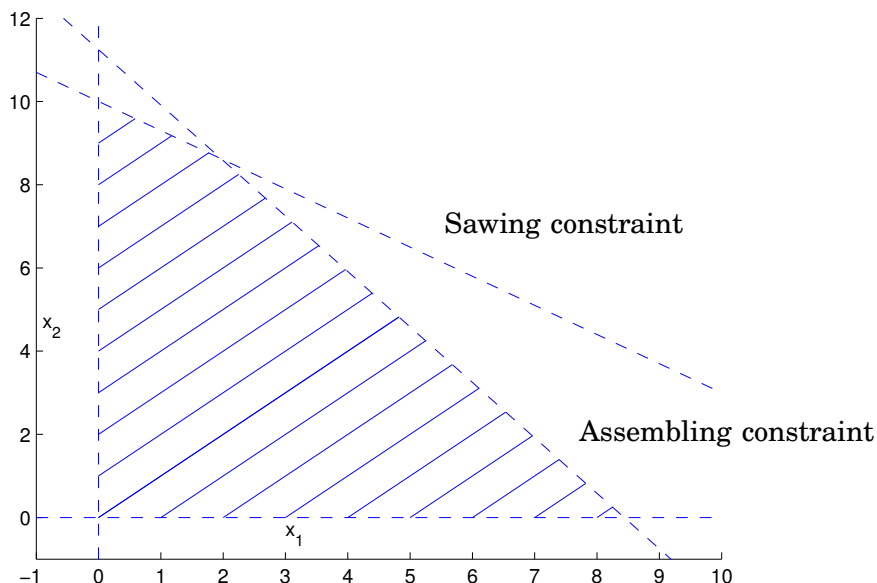
**Example 1** The linear programming problem

$$\begin{aligned} &\text{Minimize} && -x_1 - x_2 \\ &\text{subject to} && x_1 + 2x_2 \leq 1 \\ &&& 2x_1 + x_2 \leq 1 \\ &&& x_1 \geq 0 \\ &&& x_2 \geq 0 \end{aligned}$$

has the equivalent matrix formulation

$$\begin{aligned} &\text{Minimize} && (-1 \quad -1) x \\ &\text{subject to} && \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} x \preceq \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{where } x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{aligned}$$

□



**Figure 6.1** The production volumes  $x_1$  for garden furniture and  $x_2$  for sleds independently satisfy  $x_1 \geq 0$  and  $x_2 \geq 0$ . The sawing capacity constraint is  $7x_1 + 10x_2 \leq 100$  while the assembling capacity constraint is  $16x_1 + 12x_2 \leq 135$

We will now use a more extensive example to explain the use of linear programming in production planning.

**Example 2** Consider a production facility where two products are produced, **garden furniture** and **sleds**. The production volume  $x_1$  for garden furniture and the volume  $x_2$  for sleds, sold at the prices  $p_1$  and  $p_2$  respectively, generates the total income  $p_1x_1 + p_2x_2$ .

The production of both products involves two main steps **sawing** and **assembling**. Limitations in the sawing capacity gives rise to the constraint

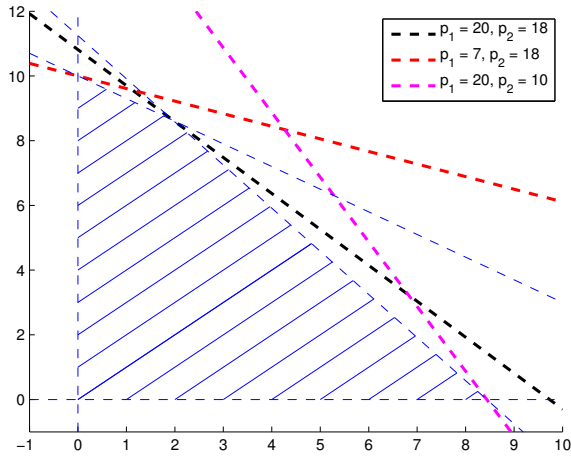
$$7x_1 + 10x_2 \leq 100 \quad (\text{Sawing})$$

while the corresponding constraint in assembling capacity is

$$16x_1 + 12x_2 \leq 135 \quad (\text{Assembling})$$

Given the prices  $p_1$  and  $p_2$ , an optimal production allocation can be found by solving the LP problem

$$\begin{aligned} &\text{Maximize} && p_1x_1 + p_2x_2 \\ &\text{subject to} && 7x_1 + 10x_2 \leq 100 \\ &&& 16x_1 + 12x_2 \leq 135 \\ &&& x_1 \geq 0 \\ &&& x_2 \geq 0 \end{aligned}$$



**Figure 6.2** Three different market price cases are considered. The first ( $p_1 = 20, p_2 = 18$ ) results in an optimal allocation where both sawing and assembling capacity are fully utilized. For the second ( $p_1 = 7, p_2 = 18$ ), the optimal allocation is to only produce sleds, leaving some assembling capacity unused, while for the third ( $p_1 = 20, p_2 = 10$ ), the optimal allocation is to only produce garden furniture.

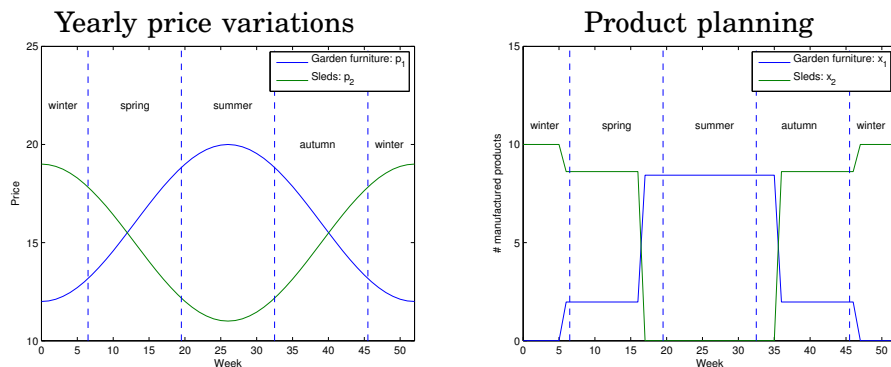
**Case 1,  $p_1 = 20, p_2 = 18$ :** In this case, sleds and furniture have a similar price, so optimality is achieved for the allocation where both sawing and assembling capacity are fully utilized.

**Case 2,  $p_1 = 7, p_2 = 18$ :** When sleds can be sold at considerably higher price than garden furniture, the optimal production allocation is to only produce sleds, in spite of the fact that the assembling capacity then becomes underutilized.

**Case 3,  $p_1 = 20, p_2 = 10$ :** When garden furniture is considerably higher valued, the optimal production allocation is to only produce furniture, in spite of the fact that the sawing capacity becomes underutilized.

Taking into account that prices varies with the season, we also get an optimal production allocation that varies over the year. See Figure 6.3.

□



**Figure 6.3** The prices of sleds and garden furniture vary over the year as shown in the left diagram. As a result, also the optimal product allocation changes. The right plot shows how the solution to the linear program varies. In the summer, only garden furniture is produced and in the winter only sleds. Both in the spring and in the autumn, there is a period when both items are produced in parallel.

So far, we were assuming that production allocations can be changed instantaneously and that transition costs can be neglected. We will now extend the study to take transition dynamics into account, first for our example, then for a more general case.

**Example 3** Suppose that we want to extend the production capacity by hiring extra personnel. The extra capacity will not be fully available from day one. Instead we assume the following transition dynamics:

$$\begin{aligned} x_3(t+1) &= 0.7x_3(t) + 30u_3(t) && \text{(learning dynamics for sawing)} \\ x_4(t+1) &= 0.7x_4(t) + 40.5u_4(t) && \text{(learning dynamics for assembling)} \end{aligned}$$

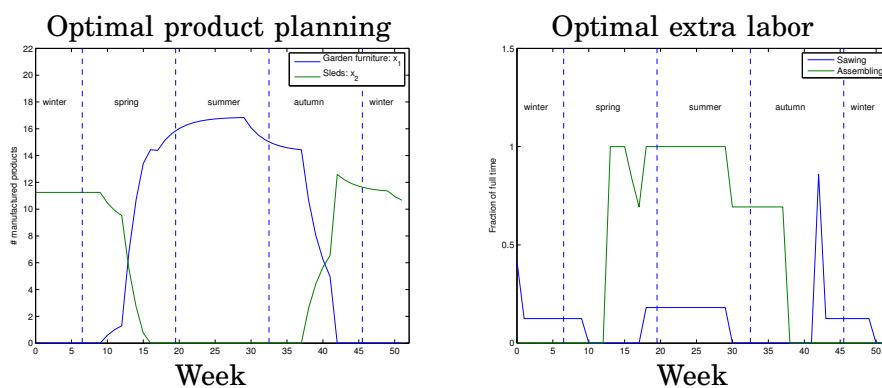
where  $u_3 \in [0, 1]$ ,  $u_4 \in [0, 1]$  is the fraction of full time employment. Starting from  $x_3(0) = 0$ , this means that the extra sawing capacity with one additional full-time employee becomes

$$\begin{aligned} x_3(1) &= 30 && \text{on day } t = 0 \\ x_3(3) &= (1 + 0.7 + 0.7^2)30 = 65.7 && \text{on day } t = 3 \\ \lim_{t \rightarrow \infty} x_3(t) &= (1 + 0.7 + 0.7^2 + \dots)30 = \frac{30}{1 - 0.7} = 100 && \text{asymptotically} \end{aligned}$$

This gives the following capacity increase for the production facility:

$$\begin{aligned} 7x_1(t) + 10x_2(t) &\leq 100 + x_3(t) && \text{(Sawing with extra personnel)} \\ 16x_1(t) + 12x_2(t) &\leq 135 + x_4(t) && \text{(Assembling with extra personnel)} \end{aligned}$$

Suppose that the weekly cost of hiring personnel is  $p_3 = 100$  for sawing and  $p_4 = 100$  for assembly. Then maximization of the total profit over one year gives rise to the following variations in production allocations:



**Figure 6.4** The plots indicate that extra personnel for assembly is generally more profitable than for sawing, but only in the summer. This is due to a combination of two facts, the high number 40.5 in the learning dynamics for assembly and the access capacity of sawing that was previously left unused in the summer, but which can now be exploited to build more garden furniture.

The plots have been computed by solving the following linear program:

$$\begin{aligned}
 \max \quad & \sum_{t=0}^{51} [p_1(t)x_1(t) + p_2(t)x_2(t) - p_3u_3(t) - p_4u_4(t)] \\
 \text{subject to} \quad & 7x_1(t) + 10x_2(t) \leq 100 + x_3(t) \\
 & 16x_1(t) + 12x_2(t) \leq 135 + x_4(t) \\
 & x_3(t+1) = 0.7x_3(t) + 30u_3(t) \\
 & x_4(t+1) = 0.7x_4(t) + 40.5u_4(t) \quad t = 0, \dots, 52 \\
 & 0 \leq u_3(t) \leq 1 \\
 & 0 \leq u_4(t) \leq 1 \\
 & x_3(0) = x_3^0, x_4(0) = x_4^0
 \end{aligned}$$

□

The example above is a special case of the following more general linear program:

$$\begin{aligned}
 \text{Maximize} \quad & \sum_{t=1}^N [p(t)^T x(t) - q(t)^T u(t)] \\
 \text{subject to} \quad & x(t+1) = Ax(t) + Bu(t), \quad x(0) = x^0 \\
 & Cx(t) \preceq d \\
 & 0 \leq u \leq 1
 \end{aligned}$$

It should be noted that  $x$  can be eliminated from the equations using the relationship

$$x(t) = A^t x^0 + \sum_{s=0}^{t-1} A^{t-s-1} B u(s)$$

Then we get a linear program in the variables  $u(0), \dots, u(t-1)$  only.

## Model Predictive Control

The discussion in the previous section was based on the idealized assumption that prices as well as learning models are known in advance. Such planning can be very useful, but it is often highly desirable to update the plans when new data becomes available. *Model Predictive Control (MPC)* is a technique developed for this purpose. The main idea will be described for optimal control problems of the following form:

$$\begin{aligned}
 \text{Minimize} \quad & \sum_{t=0}^N \ell(x(t), u(t)) \\
 \text{subject to} \quad & x(t+1) = f(x(t), u(t)) \quad , \quad x(0) = x_0 \\
 & x(t) \in X \quad , \quad u(t) \in U \\
 & \text{for } t = 0, \dots, N
 \end{aligned}$$

Comparing to our previous example, the state constraint  $x(t) \in X$  would correspond to constraints on sawing and assembly capacity. The functions  $\ell$  and  $f$  may in general be nonlinear, but this could make the optimization more difficult.

The Model Predictive Control scheme is defined by as follows:

1. At time  $t$ , collect current values of all state variables  $x(t)$ .
2. Based on the collected measurements and the given model for dynamics, solve the optimization problem to determine the decision variables  $u(t, t), u(t, t + 1), \dots, u(t, t + N)$  for the  $N + 1$  upcoming time instances  $t, t + 1, \dots, t + N$ .
3. Implement  $u(t, t)$ .
4. Wait one time step before updating  $t$  and returning to 1.

Notice that  $u(t, t + 1), \dots, u(t, t + N)$  are predictions of future inputs based on information available at time  $t$ . Only  $u(t, t)$  is implemented. The remaining predictions are discarded when new measurements are collected at time  $t + 1$ .

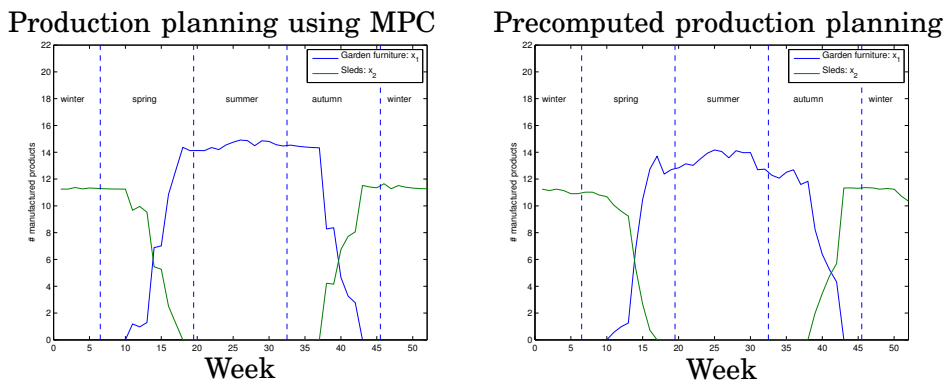
**Example 4** For the same example as before, the MPC iteration looks as follows:

1. In week  $t$ , get data for extra employment capacity  $(x_3(t), x_4(t))$  and price predictions  $(p_1, p_2)$  for the coming weeks
2. Solve the linear programming problem in the end of Example 3 based on the collected data to get  $u_3(t, t), \dots, u_4(t, t + N)$ .
3. Hire extra personnel for the following week according to the obtained solution  $(u_3(t, t), u_4(t, t))$ .
4. In the end of next week repeat the procedure, i.e. go to 1.

We will now apply this procedure for the case that the learning dynamics have the form

$$\begin{aligned} x_3(t + 1) &= 0.75x_3(t) + 30u_3(t) + v_3(t) \\ x_4(t + 1) &= 0.65x_4(t) + 40.5u_4(t) + v_4(t) \end{aligned}$$

where  $v_3(t)$  and  $v_4(t)$  are uniformly distributed random numbers in the intervals  $[-0.3x_3(t), 0]$  and  $[-0.3x_4(t), 0]$  respectively. The product prices  $p_1(t)$  and  $p_2(t)$  are additively affected by uniformly distributed random noise in  $[-1, 1]$ . See Figure 6.5 below.



**Figure 6.5** The MPC scheme (left) for this example gives 8.6% better yield than dynamic production planning based on the nominal model (right)!

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As illustrated in the example, the use of Model Predictive Control can lead to significant improvements. Some of the benefits are

- Good constraint handling
- Good support for intuition
- Complex systems and large data sets can be handled

However, there are also difficulties:

- Calculation times may be large
- System model needed
- State measurements needed. (Estimates can be used, but the effects of poor estimates are hard to analyze.)

In spite of the difficulties, the MPC technique is continuously spreading into new application domains.